Journal of Convex Analysis
Volume 5 (1998), No. 1, 119–132

Characterizations of Generalized Monotone Nonsmooth Continuous Maps using Approximate Jacobians

V. Jeyakumar*
Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia.
e-mail: jeya@maths.unsw.edu.au

D. T. Luc
Institute for Mathematics, Hanoi, Vietnam.
e-mail: dtluc@thevinh.ac.vn

S. Schaible
Graduate School of Management, University of California, Riverside, California, USA.
e-mail: schaible@ucrac1.ucr.edu

Received October 28, 1996

This paper presents necessary and/or sufficient conditions for a continuous map to be monotone, quasi-monotone or pseudomonotone. The results are given in terms of approximate Jacobian matrices which reduce to convexificators for a real-valued map. The results extend corresponding results obtained using Clarke generalized Jacobian matrices.

Keywords: Monotone maps, quasimonotone maps, pseudomonotone maps, approximate Jacobians

1. Introduction

Monotonicity of vector-valued maps plays a crucial role in the study of Complementarity Problems, Variational Inequality Problems and Equilibrium Problems as convexity of real-valued maps does in Mathematical Programming. Karamardian [18] showed that the existence of a solution of a nonlinear complementarity problem holds under generalized monotonicity, namely pseudomonotonicity. Recently, generalized monotonicity properties have been extensively studied and applied in the context of Variational Inequality Problems and Equilibrium Problems; see [1, 2, 4, 5, 8, 9, 10, 19, 20, 23, 24, 25, 26] and other references therein. More recently, various characterizations of generalized monotonicity were given for locally Lipschitz maps [22] using the Clarke generalized Jacobian [3].

The purpose of this paper is to present characterizations of generalized monotonicity of nonsmooth continuous maps which are not necessarily locally Lipschitz. The approach is based on the new concept of approximate Jacobian introduced in [15] for continuous

*The work of this author was carried out while he was visiting the A. Gary Anderson Graduate School of Management, University of California, Riverside.

ISSN 0944-6532 / $ 2.50

vector-valued maps. For a locally Lipschitz map, the Clarke generalized Jacobian can be chosen as an approximate Jacobian. When the map is real-valued, the notion of approximate Jacobian coincides with the notion of convexificator [14]; see also [6, 7]. The results rely on the mean-value theorem established for a continuous map that admits approximate Jacobians.

The outline of the paper is as follows. In Section 2 we introduce the notion of approximate Jacobian and establish a form of mean value condition for vector-valued continuous maps. In Section 3 we present necessary and/or sufficient conditions for a continuous map to be monotone. Sections 3 and 4 provide characterizations of quasimonotone maps and pseudomonotone maps in terms of approximate Jacobian matrices.

2. Preliminaries

This section contains notation, definitions and preliminaries which will be used throughout the paper. Let S be an open convex subset of $\mathbb{R}^n$ and $F : S \to \mathbb{R}^m$ a continuous map $F = (f_1, \dots .f_m)$. For each $v \in \mathbb{R}^m$ the composite function, $(vF) : S \to \mathbb{R}$ is defined by

$$(vF)(x) = \langle v,\ F(x)\rangle = \sum_{i=1}^{m} v_i f_i(x).$$

The lower Dini directional derivative and upper Dini directional derivative of vF at x in the direction $u \in \mathbb{R}^n$ are defined by

$$(vF)^-(x,u) := \liminf_{t \downarrow 0} \frac{(vF)(x+tu) - (vF)(x)}{t}$$

$$(vF)^+(x,u) := \limsup_{t \downarrow 0} \frac{(vF)(x+tu) - (vF)(x)}{t}.$$

We denote by $L(\mathbb{R}^n, \mathbb{R}^m)$ the space of all $(n \times m)$ matrices. The *convex hull* and the *closed convex hull* of a set $A \subset \mathbb{R}^n$ are denoted by $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ respectively.

Let us now define for continuous maps the notion of approximate Jacobian.

Definition 2.1 ([15]). The map $F : S \to \mathbb{R}^m$ admits an **approximate Jacobian**, $\partial^* F(x)$ at $x \in S$ if $\partial^* F(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ is closed and for each $v \in \mathbb{R}^m$

$$(vF)^-(x,u) \le \sup_{M \in \partial^* F(x)} \langle Mv, u\rangle \ \ \forall u \in \mathbb{R}^n. \tag{2.1}$$

An element M of $\partial^* F(x)$ is called an **approximate Jacobian matrix** of F at x. Note that condition (2.1) is equivalent to the condition

$$(vF)^+(x,u) \ge \inf_{M \in \partial^* F(x)} \langle Mv, u\rangle \ \ \forall u \in \mathbb{R}^n. \tag{2.2}$$

In the case $m = 1$, the inequality (2.1) (or (2.2)) is equivalent to the condition

$$F^-(x,u) \le \sup_{x^* \in \partial^* F(x)} \langle x^*, u\rangle \ \text{ and } \ F^+(x,u) \ge \inf_{x^* \in \partial^* F(x)} \langle x^*, u\rangle \forall u \in \mathbb{R}^n. \tag{2.3}$$

Thus, the set $\partial^* F(x)$ is a convexificator of F at x [6, 13, 14]. Note that condition (2.3) is equivalent to the condition that for each $\alpha \in \mathbb{R}$

$$(\alpha F)^-(x,u) \leq \sup_{x^* \in \partial^* F(x)} \langle \alpha x^*, u\rangle \ \ \forall u \in \mathbb{R}^n. \tag{2.4}$$

This can be seen by applying to (2.4) $\alpha = 1$ and $\alpha = -1$. Observe that condition (2.4) can be written as

$$(\alpha F)^+(x,u) \geq \inf_{x^* \in \partial^* F(x)} \langle \alpha x^*, u\rangle \ \ \forall u \in \mathbb{R}^n. \tag{2.5}$$

For applications of convexificators see [7, 13, 14, 16]. Clearly, a map admits many approximate Jacobians in general. Trivially, for $F : S \to \mathbb{R}^m$ the whole space $L(\mathbb{R}^n, \mathbb{R}^m)$ serves as an approximate Jacobian for F at any point in S. If $F : S \to \mathbb{R}^m$ is continuously differentiable at x, then any closed set containing $\nabla F(x)^T$ is an approximate Jacobian of F at x, where $\nabla F(x)$ is the $m \times n$ Jacobian matrix of partial derivatives of F at x. Suppose that F is locally Lipschitz at $x \in S$. Then the Clarke generalized Jacobian [3]

$$\partial_C F(x) = \mathrm{co}\{\lim_{n\to\infty} \nabla F(x_n)^T : x_n \to x, \ \{x_n\} \subset K\}$$

is an approximate Jacobian of F at x [15]. Here, K is a dense set of points in S on which F is differentiable. For a numerical example, consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = (|x| - |y|, x + y).$$

Then it can be verified that

$$\partial^* F(0) = \left\{ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

is an approximate Jacobian of F at 0. On the other hand, the Clarke generalized Jacobian

$$\partial_C F(0) = \mathrm{co}\left(\left\{ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \right).$$

which is also an approximate Jacobian at 0 and contains $\partial^* F(0)$. From an application point of view, the "smaller" an approximate Jacobian is, the better.

Definition 2.2. The map $F : S \to \mathbb{R}^m$ admits a regular approximate Jacobian $\partial^* F(x)$ at $x \in S$ if $\partial^* F(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ is closed and for each $v \in \mathbb{R}^m$

$$(vF)^+(x,u) = \sup_{M \in \partial^* F(x)} \langle Mv, u\rangle \ \ \forall u \in \mathbb{R}^n. \tag{2.6}$$

or equivalently

$$(vF)^-(x,u) = \inf_{M \in \partial^* F(x)} \langle Mv, u\rangle \ \ \forall u \in \mathbb{R}^n. \tag{2.7}$$

Note that in the case $m = 1$ this definition collapses to the notion of the regular convexificator [14]. Thus a closed set $\partial^* F(x) \subset \mathbb{R}^n$ is a regular convexificator of the real-valued function F at x if for each $u \in \mathbb{R}^n$,

$$F^-(x,u) = \inf_{\xi \in \partial^* F(x)} \langle \xi, u\rangle \ \text{ and } \ F^+(x,u) = \sup_{\xi \in \partial^* F(x)} \langle \xi, u\rangle.$$

It is evident that these equalities follow from (2.3) by taking $v = -1$ and $v = 1$.

It is immediate from the definition that if F is differentiable at x, then $\{\nabla F(x)\}$ is a regular approximate Jacobian of F at x. It is also clear by Radamacher's Theorem [3] that a locally Lipschitz map $F : S \to \mathbb{R}^m$ is regular on a dense subset of S.

We now establish a form of mean value condition for a vector-valued continuous map $F : S \to \mathbb{R}^m$ extending the result in [15].

Theorem 2.3 (Mean Value Theorem.). *Let $a, b \in S$ and $F : S \to \mathbb{R}^m$. Assume that $F\big|_{[a,b]}$ is continuous and for each $x \in [a, b]$ F admits an approximate Jacobian $\partial^* F(x)$ at x. Then*

$$F(b) - F(a) \in \overline{\mathrm{co}}(\partial^* F([a, b])(b - a)).$$

Proof. Let us first note that the right-hand side above is the closed convex hull of all points of the form $M(b - a)$ where $M \in \partial^* F(\zeta)$ for some $\zeta \in [a, b]$. Let $v \in \mathbb{R}^m$ be arbitrary and fixed. Consider the real-valued function $g : [0, 1] \to \mathbb{R}$

$$g(t) = \langle v, F(a + t(b - a)) - F(a) + t(F(a) - F(b))\rangle.$$

Then g is continuous on $[0, 1]$ with $g(0) = g(1)$. So, g attains a minimum or a maximum at some $t_0 \in (0, 1)$. Suppose that t_0 is a minimum point. Then, for each $\alpha \in \mathbb{R}$, $g^-(t_0, \alpha) \geq 0$. It now follows from direct calculations that

$$g^-(t_0, \alpha) = \langle v, F(a + t_0(b - a))\rangle^-(t_0, \alpha(b - a)) + \alpha\langle v, F(b) - F(a)\rangle.$$

Hence for each $\alpha \in \mathbb{R}$

$$\langle v, F(a + t_0(b - a))\rangle^-(t_0, \alpha(b - a)) \geq \alpha\langle v, F(a) - F(b)\rangle.$$

Now, by taking $\alpha = 1$ and $\alpha = -1$, we obtain that

$$-\langle v, F(a + t_0(b - a))\rangle^-(t_0, a - b) \leq \langle v, F(b) - F(a)\rangle \leq \langle v, F(a + t_0(b - a))\rangle^-(t_0, b - a)\rangle.$$

By (2.1), we get

$$\inf_{M \in \partial^* F(a + t_0(b-a))} \langle Mv, b - a\rangle \leq \langle v, F(b) - F(a)\rangle \leq \sup_{M \in \partial^* F(a + t_0(b-a))} \langle Mv, b - a\rangle.$$

Consequently,

$$\langle v, F(b) - F(a)\rangle \in \overline{\mathrm{co}}(\partial^* F(a + t_0(b - a))v)(b - a))$$

and so,

$$\langle v, F(b) - F(a)\rangle \in \overline{\mathrm{co}}(\partial^* F([a, b])v)(b - a)). \tag{2.8}$$

Since this inclusion holds for each $v \in \mathbb{R}^m$ we claim that

$$F(b) - F(a) \in \overline{\mathrm{co}}(\partial^* F([a, b])(b - a).$$

If this is not so, then it follows from the Separation Theorem

$$\langle p, F(b) - F(a)\rangle - \epsilon > \sup_{u \in \overline{\mathrm{co}}(\partial^* F([a,b])(b-a))} \langle p, u\rangle,$$

for some $p \in \mathbb{R}^m$ since $\overline{\text{co}}(\partial^* F([a,b])(b-a))$ is a closed convex subset of $\mathbb{R}^m$. This implies

$$\langle p, F(b) - F(a) \rangle > \sup\{\alpha : \alpha \in \overline{\text{co}}(\partial^* F([a,b])p)(b-a))\}$$

which contradicts (2.8).

Similarly, if t_0 is a maximum point, then $g^+(t_0, \alpha) \leq 0$, for each $\alpha \in \mathbb{R}$. Using the same line of arguments as above we arrive at the same conclusion, and so the proof is complete. □

If for each $x \in [a,b]$ F admits a bounded approximate Jacobian $\partial^* F(x)$ at x, then the above condition collapses to

$$F(b) - F(a) \in \text{co}(\partial^* F([a,b])(b-a)).$$

Moreover, if $F : S \to \mathbb{R}^m$ is locally Lipschitz on S, then we get

$$F(b) - F(a) \in \text{co}(\partial_C F([a,b])(b-a)).$$

In this case the Clarke generalized Jacobian $\partial_C F(x)$ can be chosen as a convex compact approximate Jacobian of F at x. For corresponding results, see [11].

Suppose $f : S \to \mathbb{R}$ is a C^1- function, that is, a continuously differentiable function. We now introduce the notion of approximate Hessian for a C^1- function. Note that the derivative of f which is denoted by ∇f is a map from S to $\mathbb{R}^n$.

Definition 2.4. The function f admits an **approximate Hessians** $\partial^2_* f(x)$ at x if this set is an approximate Jacobian to ∇f .

Note that $\partial^2_* f(x) = \partial^* \nabla f(x)$ and the matrix $M \in \partial^2_* f(x)$ is an approximate Hessian matrix of F at x. Clearly, if f is twice differentiable at x, then $\nabla^2 f(x)$ is an approximate Hessian matrix of f at x.

If $f : S \to \mathbb{R}$ is $C^{1,1}$, that is, a Gateaux differentiable function with locally Lipschitz derivative, then the generalized Hessian [12] in the sense of Hiriart-Urruty et.al. is given by

$$\partial^2_H f(x) = \text{co}\{M \ : \ M = \lim_{n\to\infty} \nabla^2 f(x_n), \ x_n \in \Delta, \ x_n \to x\},$$

where Δ is the set of points in S where f is twice differentiable. Clearly, $\partial^2_H f(x)$ is a nonempty convex compact set of symmetric matrices and it is an approximate Hessian of f at x.

3. Monotone maps and approximate Jacobians

In this section, we characterize monotonicity of continuous maps in terms of approximate Jacobian matrices. As an application, we also present a second-order characterization of convexity of C^1- functions. Recall that the map $F : S \to \mathbb{R}^n$ is **monotone** if for each $x, \ y \in S$,

$$\langle F(x), \ y - x \rangle + \langle F(y), \ x - y \rangle \leq 0.$$

Theorem 3.1. *Let $F : S \to \mathbb{R}^n$ be a continuous map which admits an approximate Jacobian $\partial^* F(x)$ for each $x \in S$. If for each $x \in S$, the matrices $M \in \partial^* F(x)$ are positive semi-definite, then F is monotone.*

Proof. Let $x,\ y \in S$ be arbitrary; let $u = y - x$. By the Mean Value Theorem,

$$F(x+u) - F(x) \in \overline{\text{co}}(\partial^* F([x,\ x+u])u),$$

and so

$$\langle F(x+u) - F(x),\ u\rangle \in \langle \overline{\text{co}}(\partial^* F([x,\ x+u])u),\ u\rangle.$$

Thus, there exists $z \in [x,\ x+u]$ and $N \in \overline{\text{co}}(\partial^* F(z))$ such that

$$\begin{aligned}\langle F(x+u) - F(x),\ u\rangle &= \langle Nu, u\rangle \\ &\geq \inf_{M \in \overline{\text{co}}(\partial^* F(z))} \langle Mu, u\rangle \\ &= \inf_{M \in \partial^* F(z)} \langle Mu, u\rangle \\ &\geq 0.\end{aligned}$$

This shows that F is monotone. □

The map $F : S \to \mathbb{R}^n$ is **strictly monotone** if for each $x, y \in S, x \neq y$,

$$\langle F(x),\ y - x\rangle + \langle F(y),\ x - y\rangle < 0.$$

Theorem 3.2. *Let $F : S \to \mathbb{R}^n$ be a continuous map which admits a bounded approximate Jacobian $\partial^* F(x)$ for each $x \in S$. If the matrices $M \in \partial^* F(x)$ are positive definite for each $x \in S$, then F is strictly monotone.*

Proof. Let $x,\ y \in S$, $x \neq y$, be arbitrary and $u = y - x$. By the Mean Value Theorem, there exist $z \in [x,\ x+u]$ and $N \in \overline{\text{co}}(\partial^* F(z))$ such that

$$\begin{aligned}\langle F(x+u) - F(x),\ u\rangle &= \langle Nu, u\rangle \\ &\geq \inf_{M \in \overline{\text{co}}(\partial^* F(z))} \langle Mu, u\rangle \\ &= \inf_{M \in \partial^* F(z)} \langle Mu, u\rangle \\ &= \min_{M \in \partial^* F(z)} \langle Mu, u\rangle \\ &> 0.\end{aligned}$$

This shows that F is strictly monotone. □

Theorem 3.3. *Let $F : S \to \mathbb{R}^n$ be a continuous map which admits an approximate Jacobian $\partial^* F(x)$ for each $x \in S$. Suppose there exists a dense subset K of S such that for each $x \in K$ $\partial^* F(x)$ is regular and for each $x \notin K$*

$$\partial^* F(x) \subset \left\{ \lim_{n \to \infty} M_n : \{x_n\} \subset K,\ x_n \to x,\ M_n \in \partial^* F(x_n) \right\}.$$

If F is monotone, then for each $x \in S$ the matrices $M \in \partial^* F(x)$ are positive semi-definite.

Proof. Suppose, to the contrary,

$$\langle M_0 u_0, u_0\rangle < 0,$$

for some $x_0, u_0 \in S$ and $M_0 \in \partial^* F(x_0)$. If $x_0 \in K$, then by regularity,

$$(u_0 F)^-(x_0, u_0) = \inf_{M \in \partial^* F(x_0)} \langle M u_0, u_0\rangle < 0.$$

So, there exists t sufficiently small and positive such that

$$\langle u_0, F(x_0 + t u_0)\rangle - \langle u_0, F(x_0)\rangle < 0.$$

This contradicts monotonicity of F.

If on the other hand $x_0 \notin K$, then by hypothesis we can find a sequence $\{x_n\} \subset K,\ x_n \to x_0$ and $M_n \in \partial^* F(x_n)$ such that

$$\lim_{n\to\infty} M_n = M_0.$$

So for n_0 sufficiently large $M_{n_0} \in \partial^* F(x_{n_0})$ and $\langle M_{n_0} u_0, u_0\rangle < 0$. Hence,

$$(u_0 F)^-(x_{n_0}, u_0) = \inf_{M \in \partial^* F(x_{n_0})} \langle M_{u_0}, u_0\rangle < 0.$$

Then for sufficiently small $t > 0$

$$\langle u_0, F(x_{n_0} + t u_0)\rangle - \langle u_0, F(x_{n_0})\rangle < 0.$$

This again contradicts monotonicity of F, and so the proof is completed. □

It is worth noting that the above theorem is no longer true without the regularity condition. This can be seen by choosing $L(\mathbb{R}^n, \mathbb{R}^n)$ as an approximate Jacobian at each point.

As a special case of the above theorem we see that if F is locally Lipschitz, then monotonicity of F is characterized by positive semi-definiteness of the Jacobian matrices. This was shown in [22] and [17].

Corollary 3.4. *Let $F : S \to \mathbb{R}^n$ be locally Lipschitz. Then F is monotone if and only if for each $x \in S$ the matrices $M \in \partial_C F(x)$ are positive semi-definite.*

Proof. Let $x \in S$ be arbitrary. Since F is locally Lipschitz by Radamacher's Theorem, there exists a dense subset K of S on which ∇F exists. Define

$$\partial^* F(x) = \begin{cases} \{\nabla F(x)\}, & x \in K \\ \{\lim_{n\to\infty} \nabla F(x_n) : x_n \to x,\ \{x_n\} \subset K\}, & x \notin K. \end{cases}$$

Then $\partial^* F(x)$ is an approximate Jacobian of F at x. If F is monotone, then the hypotheses of the preceding theorem are satisfied, and so the matrices $M \in \partial^* F(x)$ are positive semi-definite. Hence, the matrices $M \in \mathrm{co}(\partial^* F(x)) = \partial_C F(x)$ are also positive semi-definite.

Conversely, if for each $x \in S$ the matrices $M \in \partial_C F(x)$ are positive semi-definite, then the monotonicity of F follows from Theorem 3.1 since $\partial_C F(x)$ is an approximate Jacobian of F at x. □

Corollary 3.5. *Let $f : S \to \mathbb{R}$ be a C^1-function which admits an approximate Hessian $\partial_*^2 f(x)$ at each $x \in \mathbb{R}^n$. Suppose there exists a dense subset K of S such that for each $x \in K$ $\partial_*^2 f(x)$ is regular and for each $x \notin K$*

$$\partial_*^2 f(x) \subset \left\{ \lim_{n\to\infty} \nabla^2 f(x_n) : \{x_n\} \subset K,\ x_n \to x \right\}.$$

If f is convex, then for each $x \in S$ the matrices $M \in \partial_*^2 f(x)$ are positive semi-definite.

Proof. The conclusion follows from Theorem 3.3 by replacing F by ∇f. □

Corollary 3.6. *Let $f : S \to \mathbb{R}$ be $C^{1,1}$. Then f is convex if and only if for each $x \in S$ the matrices $M \in \partial_H^2 f(x)$ are positive semi-definite.*

Proof. The conclusion follows by applying Corollary 3.4 to ∇f. □

4. Quasimonotone continuous maps

In this section, we will see that quasi-monotonicity can be characterized using approximate Jacobians. The map $F : S \to \mathbb{R}^n$ is said to be **quasimonotone** [10, 19] if for each $x, y \in S$

$$\min\{\langle F(x), y - x\rangle,\ \langle F(y),\ x - y\rangle\} \le 0.$$

Theorem 4.1. *Assume that $F : S \to \mathbb{R}^n$ is continuous and admits an approximate Jacobian $\partial^* F(x)$ at each $x \in S$. If F is quasimonotone, then*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \sup_{M\in\partial^* F(x)}\langle Mu, u\rangle \ge 0$,
(ii) $\langle F(x), u\rangle = 0$ *and* $\langle F(x + t'u), u\rangle > 0$ *for some* $t' < 0$ *imply the existence of* $t_0 > 0$ *such that* $\langle F(x + tu), u\rangle \ge 0$ *for all* $t \in [0, t_0]$.

Proof. Suppose (i) does not hold. Then, there exist $x,\ u \in S$, such that

$$\langle F(x), u\rangle = 0 \quad \text{and} \quad \sup_{M\in\partial^* F(x)} \langle Mu, u\rangle < 0.$$

Then from the definition of the approximate Jacobian we get

$$(uF)^-(x, u) \le \sup_{M\in\partial^* F(x)} \langle Mu, u\rangle < 0$$

and

$$(-uF)^-(x, -u) \le \sup_{M\in\partial^* F(x)} \langle Mu, u\rangle < 0.$$

Hence, for sufficiently small $t > 0$

$$\langle u, F(x + tu) - F(x)\rangle < 0$$

and

$$\langle -u, F(x + t(-u)) - F(x)\rangle < 0.$$

These give us that

$$\langle u, F(x+tu)\rangle < 0 \quad \text{and} \quad \langle u, F(x-tu)\rangle > 0.$$

Thus

$$\langle F(x+tu),\ (x-tu)-(x+tu)\rangle > 0$$

and

$$\langle F(x-tu),\ (x+tu)-(x-tu)\rangle > 0.$$

This contradicts quasi-monotonicity of F, and so (i) holds.

Furthermore, if (ii) does not hold, then there exists $t_0 > 0$ such that $\langle F(x), u\rangle = 0, \langle F(x+t'u), u\rangle > 0$ for some $t' < 0$ and $\langle F(x+t_0u), u\rangle < 0$. Let $x_0 = x + t'u$ and let $y_0 = x + t_0 u$. Then, we have

$$\langle F(y_0), x_0 - y_0\rangle = \langle F(x+t_0u), (t'-t_0)u\rangle > 0,$$

$$\langle F(x_0), y_0 - x_0\rangle = \langle F(x+t'u), (t_0-t')u\rangle > 0.$$

These inequalities contradict the quasimonotonicity of F. □

In general, it is not true that quasimonotonicity of F implies

$$\inf_{M\in\partial^* F(x)} \langle Mu, u\rangle \geq 0,$$

for each $x, u \in S$ as in the differentiable case. Moreover, the conditions (i) and (ii) may not be sufficient without certain restrictions on the approximate Jacobian. This can be seen by taking $\partial^* F(x) = L(\mathbb{R}^n, \mathbb{R}^n)$ for each $x \in S$. We now obtain sufficient conditions under the additional hypotheses that approximate Jacobians are bounded and regular.

Theorem 4.2. *Let $F : S \to \mathbb{R}^n$ be a continuous map which admits a bounded approximate Jacobian, $\partial^* F(x)$, at each $x \in S$. Suppose there exits a dense subset K of S such that for each $x \in K$ $\partial^* F(x)$ is regular and for each $x \notin K$*

$$\partial^* F(x) \subset \left\{ \lim_{n\to\infty} M_n : \{x_n\} \subset K,\ x_n \to x,\ M_n \in \partial^* F(x_n)\right\}.$$

Furthermore, assume that the following conditions hold for each $x, u \in \mathbb{R}^n$:

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \max_{M\in\partial^* F(x)} \langle Mu, u\rangle \geq 0,$

(ii) $\langle F(x), u\rangle = 0, 0 \in \{\langle u, Mu\rangle : M \in \partial^* F(x)\}$ *and* $\langle F(x+t'u), u\rangle > 0$ *for some* $t' < 0$ *imply the existence of* $t_0 > 0$ *such that* $\langle F(x+tu), u\rangle \geq 0$ *for all* $t \in [0, t_0]$.

Then F is quasimonotone.

Proof. Suppose there exist $x, y \in S$ such that

$$\langle F(x), y - x\rangle > 0 \text{ and } \langle F(y), x - y\rangle > 0.$$

Let $u = y - x$ and let $g(t) = \langle F(x + tu), u\rangle$. Then g is continuous, $g(0) > 0$ and $g(1) < 0$. So, there exists $t_1 \in (0, 1)$ such that

$$g(t_1) = 0 \text{ and } g(t) < 0 \text{ for all } \ t \in (t_1, 1).$$

Define $x_1 = x + t_1 u$. Then, $g(t_1) = \langle F(x_1), u\rangle = 0$ and $(uF)^-(x_1, u) \leq 0$. Now we claim that

$$0 \in \{\langle u, Mu\rangle : M \in \partial^* F(x_1)\}.$$

To see this, consider first the case where $x_1 \in K$. If $\langle u, Mu\rangle > 0$ for each $M \in \partial^* F(x_1)$ then by regularity of $\partial^* F(x_1)$ we get a contradiction since

$$0 < \min_{M\in\partial^* F(x_1)} \langle Mu, u\rangle = \inf_{M\in\partial^* F(x_1)} \langle Mu, u\rangle = (uF)^-(x_1, u) \leq 0.$$

If $\langle u, Mu\rangle < 0$ for each $M \in \partial^* F(x_1)$, then by (i) we get a contradiction since

$$0 > \max_{M\in\partial^* F(x_1)} \langle Mu, u\rangle \geq 0.$$

Now consider the case where $x_1 \notin K$. Then for each $M \in \partial^* F(x_1)$ we can find a sequence $\{y_n\} \subset K, y_n \to x_1,\ M_n \in \partial^* F(y_n)$ such that $\lim_{n\to\infty} M_n = M$. The claim holds as in the above case by applying the arguments in the two subcases to $M_{n_0} \in \partial^* F(y_{n_0}), y_{n_0} \in K$, for sufficiently large n_0. By continuity of g there exits $t' < 0$ such that

$$g(t_1 + t') = \langle F(x_1 + t'u), u\rangle > 0.$$

Condition (ii) gives us that there exists $t_0 > 0$ such that

$$g(t_1 + t) = \langle F(x_1 + tu), u\rangle \geq 0 \ \text{ for all } \ t \in [0, t_0].$$

This contradicts the condition that $g(t) < 0$ for all $t \in (t_1, 1)$. Hence F is quasimonotone. □

As a special case, we obtain a characterization of quasimonotone locally Lipschitz maps [22].

Corollary 4.3. *Assume $F : S \to \mathbb{R}^n$ is locally Lipschitz on S. Then F is quasimonotone if and only if the following conditions hold for each $x, u \in S$:*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \max_{M\in\partial_C F(x)}\langle Mu, u\rangle \geq 0$,

(ii) $\langle F(x), u\rangle = 0, 0 \in \{\langle u, Au\rangle : A \in \partial_C F(x)\}$ *and* $\langle F(x + t'u), u\rangle > 0$ *for some* $t' < 0$ *imply the existence of* $t_0 > 0$ *such that* $\langle F(x + tu), u\rangle \geq 0$ *for all* $t \in [0, t_0]$.

Proof. The conclusion follows from the preceding theorems by noting that

$$\partial^* F(x) = \begin{cases} \{\nabla F(x)\}, & x \in K \\ \{\lim_{n\to\infty} \nabla F(x_n) : x_n \to x,\ \{x_n\} \subset K\}, & x \notin K. \end{cases}$$

is an approximate Jacobian of F at x which satisfies the hypotheses of the previous theorem and observing that $\partial_C F(x) = \mathrm{co}(\partial^* F(x))$. □

Corollary 4.4. *Assume $F : S \to \mathbb{R}^n$ is differentiable on S. Then F is quasimonotone if and only if the following conditions hold for each $x, u \in \mathbb{R}^n$:*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \langle u, \nabla F(x)u\rangle \geq 0$,

(ii) $\langle F(x), u\rangle = \langle u, \nabla F(x)u\rangle = 0$ *and* $\langle F(x + t'u), u\rangle > 0$ *for some* $t' < 0$ *imply the existence of* $t_0 > 0$ *such that* $\langle F(x + tu), u\rangle \geq 0$ *for all* $t \in [0, t_0]$.

Proof. Since F is differentiable, $\{\nabla F(x)\}$ is a regular and bounded approximate Jacobian for each $x \in S$. So, the conclusion follows from Theorems 4.1 and 4.2. □

Corollary 4.5. *Let $f : S \to \mathbb{R}$ be a C^1-function which admits an approximate Hessian $\partial^2_* f(x)$ at each $x \in \mathbb{R}^n$. If f is quasiconvex, then for each $x, u \in S$ with $\langle \nabla f(x),\ u\rangle = 0$,*

$$\sup_{M \in \partial^2_* f(x)} \langle Mu, u\rangle \geq 0.$$

Proof. The conclusion follows from Theorem 4.1 by replacing F by ∇f and noting that f is quasiconvex if and only if ∇f is quasimonotone [21]. □

5. Pseudomonotone maps

In this section, we will see that pseudomonotonicity of a continuous map can be characterized using approximate Jacobians.The map $F : S \to \mathbb{R}^n$ is said to be **pseudomonotone** [18, 19] if for each $x, y \in S$

$$\langle F(x), y - x\rangle > 0 \Longrightarrow \langle F(y), y - x\rangle > 0.$$

Theorem 5.1. *Assume $F : S \to \mathbb{R}^n$ is continuous and admits an approximate Jacobian $\partial^* F(x)$ at each $x \in S$. If F is pseudomonotone, then*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \sup_{M\in\partial^* F(x)} \langle Mu, u\rangle \geq 0$,

(ii) $\langle F(x), u\rangle = 0 \Longrightarrow \exists t_0 > 0,\ \langle F(x + tu), u\rangle \geq 0,\ \forall t \in [0, t_0]$.

Proof. Since pseudomonotonicity implies quasimonotonicity (i) follows from Theorem 4.1. If (ii) does not hold, then there exist $x, u \in S$, and $t' > 0$ such that $\langle F(x), u\rangle = 0$ and $\langle F(x + t'u), u\rangle < 0$. Define $y = x + t'u$. Then

$$\langle F(x), y - x\rangle = \langle F(x), t'u\rangle = 0. \tag{5.1}$$

On the other hand,

$$\langle F(y), x - y\rangle = \langle F(x + t'u), -t'u\rangle > 0.$$

Now it follows from pseudomonotonicity that $\langle F(x), y - x\rangle > 0$ which is a contradiction to (5.1). □

Theorem 5.2. *Let $F : S \to \mathbb{R}^n$ be a continuous map which admits a bounded approximate Jacobian, $\partial^* F(x)$, at each $x \in S$. Assume there exits a dense subset K of S such that for each $x \in K$ $\partial^* F(x)$ is regular and for each $x \notin K$*

$$\partial^* F(x) \subset \left\{ \lim_{n\to\infty} M_n : \{x_n\} \subset K,\ x_n \to x,\ M_n \in \partial^* F(x_n) \right\}.$$

Furthermore, assume that the following conditions hold for each $x, u \in \mathbb{R}^n$:

(i) $\langle F(x), u\rangle = 0 \Rightarrow \max_{M\in\partial^* F(x)}\langle Mu, u\rangle \geq 0,$

(ii) $\langle F(x), u\rangle = 0$ *and* $0 \in \{\langle u, Mu\rangle : M \in \partial^* F(x)\} \Rightarrow \exists\ t_0 > 0,\ \langle F(x + tu), u\rangle \geq 0$ $\forall t \in [0, t_0]$

Then F is pseudomonotone.

Proof. Suppose F is not pseudomonotone. Then there exist $x, y \in S$ such that

$$\langle F(x), y - x\rangle \geq 0 \ \text{ and } \ \langle F(y), x - y\rangle > 0.$$

Let $u = y - x$ and $g(t) = \langle F(x + tu), u\rangle$. Then g is continuous, $g(0) \geq 0$ and $g(1) < 0$. So there exists $t_1 \in [0, 1]$ such that

$$g(t_1) = 0 \ \text{ and } \ g(t) < 0 \ \text{ for all } \ t \in (t_1, 1]. \tag{5.2}$$

Define $x_1 = x + t_1 u$. As in the proof of Theorem 4.2, $\langle F(x_1), u\rangle = 0$, $(uF)^-(x_1, u) \leq 0$ and

$$0 \in \{\langle u, Mu\rangle : M \in \partial^* F(x_1)\}.$$

Now it follows from (ii) that there exists $t_0 > 0$ such that

$$\langle F(x_1 + tu), u\rangle \geq 0,\ \forall t \in [0, t_0].$$

Thus $g(t_1 + t) = \langle F(x_1 + tu), u\rangle \geq 0\ \forall\ t \in [0, t_0]$ for sufficiently small t close to t_0. This is a contradiction to (5.2), and hence, F is pseudomonotone. □

Corollary 5.3. *Assume $F : S \to \mathbb{R}^n$ is locally Lipschitz on S. Then F is pseudomonotone if and only if the following conditions hold for each $x, u \in S$:*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \max_{M\in\partial_C F(x)}\langle Mu, u\rangle \geq 0$,

(ii) $\langle F(x), u\rangle = 0$ *and* $0 \in \{\langle u, Mu\rangle : M \in \partial_C F(x)\} \Rightarrow \exists\ t_0 > 0, \langle F(x+tu), u\rangle \geq 0, \forall t \in [0, t_0]$.

Proof. The proof follows along the same line of arguments as in Corollary 4.3, and so the details are left to the reader. □

Corollary 5.4. *Assume $F : S \to \mathbb{R}^n$ is differentiable on S. Then F is pseudomonotone if and only if the following conditions hold for each $x, u \in \mathbb{R}^n$:*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow \langle u, \nabla F(x)u\rangle \geq 0$,

(ii) $\langle F(x), u\rangle = \langle u, \nabla F(x)u\rangle = 0 \Rightarrow \exists\ t_0 > 0,\ \langle F(x+tu), u\rangle \geq 0,\ \forall t \in [0, t_0]$.

Proof. Since F is differentiable, $\{\nabla F(x)\}$ is a bounded regular approximate Jacobian for each $x \in S$. So the conclusion follows from the preceding theorems. □

References

[1] M. Bianchi, N. Hadjisavvas, S. Schaible: Vector equilibrium problems with generalized monotone bifunctions, J. Optim. Theor. Appl. 92(3) (1997) 527–542.

[2] M. Bianchi, S. Schaible: Generalized monotone bifunctions and equilibrium problems, J. Optim. Theor. Appl. 90 (1) (1996) 31–43.

[3] F. H. Clarke: Optimization and Nonsmooth Analysis, Wiley-Interscience, New York, 1983.

[4] J. P. Crouzeix, J. A. Ferland: Criteria for differentiable generalized monotone maps, Math. Progr. 75(3) (1996) 399–406.

[5] J. P. Crouzeix, S. Schaible: Generalized monotone affine maps, SIAM J. Matrix Anal. Appl. 17(4) (1996) 992–997.

[6] V. F. Demyanov, V. Jeyakumar: Hunting for a smaller convex subdifferential, J. Global Optim. 10 (1997) 305–326.

[7] V. F. Demyanov, A. M. Rubinov: Constructive Nonsmooth Analysis, Verlag Peter Lang, Frankfurt am Main, 1995.

[8] M. S. Gowda: Affine pseudomonotone maps and the linear complementarity problem, SIAM J. Matrix Anal. Appl. 11 (1990) 373–380.

[9] N. Hadjisavvas, S. Schaible: Quasimonotone variational inequalities in Banach spaces, J. Optim. Theor. Appl. 90(1) (1996) 95–111.

[10] A. Hassouni: Opérateurs Quasimonotone - Applications a Certain Problèmes Variationnels, Thesis, Mathématiques Appliquées, Université Paul Sabatier, Toulouse, France, 1993.

[11] J. B. Hiriart-Urruty: Mean value theorems for vector valued mappings in nonsmooth optimization, Numer. Funct. Anal. Optim. 2 (1980) 1–30.

[12] J. B. Hiriart-Urruty, J. J. Strodiot, V. Hien Nguyen: Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data, Appl. Math. Optim. 11 (1984) 43–56.

[13] V. Jeyakumar, V. F. Demyanov: A mean value theorem and a characterization of convexity using convexificators, Applied Mathematics Research Report AMR96/13, University of New South Wales, Australia, 1996.

[14] V. Jeyakumar, D. T. Luc: Nonsmooth calculus, minimality and monotonicity of convexificators, Applied Mathematics Research Report AMR96/29, University of New South Wales, Australia, 1996 (submitted).

[15] V. Jeyakumar, D. T. Luc: Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization, Applied Mathematics Report AMR97/1, University of New South Wales, Australia, SIAM J. Control Optim., 1997 (to appear).

[16] V. Jeyakumar, X. Q. Yang: Approximate generalized Hessians and Taylor's expansions for continuously Gâteaux differentiable functions, Applied Mathematics Research Report AMR96/20, University of New South Wales, Australia, 1996 (submitted).

[17] H. Jiang, L. Qi: Local uniqueness and convergence of iterative methods for nonsmooth variational inequalities, J. Math. Anal. Appl. 196 (1995) 314–331.

[18] S. Karamardian: Complementarity over cones with monotone and pseudomonotone maps, J. Optim. Theor. Appl. 18 (1976) 445–454.

[19] S. Karamardian, S. Schaible: Seven kinds of monotone maps, J. Optim. Theor. Appl. 66 (1990) 37–46.

[20] S. Karamardian, S. Schaible, J. P. Crouzeix: Characterizations of generalized monotone maps, J. Optim. Theor. Appl. 76 (1993) 399–413.

[21] D. T. Luc: Characterizations of quasi-convex functions, Bull. Austr. Math. Soc. 49 (1993) 393–405.

[22] D. T. Luc, S. Schaible: On generalized monotone nonsmooth maps, J. Convex Analysis 3(2) (1996) 195–205.

[23] S. Schaible: Generalized monotonicity-concepts and uses, in: Variational Inequalities and Network Equilibrium Problems, F. Giannessi and A. Maugeri (eds.), Proceedings of the 19^{th} Course of the International School of Mathematics, G. Stampacchia, Erice/Italy, June 19-25, Plenum Publishing Corporation, New York (1995) 289–299.

[24] S. Schaible: Criteria for generalized monotonicity, in F. Giannessi, S. Komlosi, and T. Rapcsak, (eds.), Proceedings of the 13^{th} International Conference on Mathematical Programming in Matrahaza (Hungary), March 24-27, 1996, Kluwer Academic Publishers, Dordrecht et al., 1997.

[25] S. Schaible, J. C. Yao: On the equivalence of nonlinear complementarity problems and least element problems, Math. Progr. 70 (1995) 191–200.

[26] J. C. Yao: Variational inequalities with generalized monotone operators, Math. Oper. Res. 19 (1994) 691–705.