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Variational Approximation of Functionals with Curvatures and Related Properties

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We consider the problem of approximating via Γ-convergence a class of functionals depending on curvatures of smooth compact boundaries. We investigate the connections between the approximation problem and the lower semicontinuous envelope of the original functional. We provide some examples of lower semicontinuous functionals and their variational approximation.

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1. Introduction

Functionals depending on curvatures arise in different contexts, such as differential geometry [30], integral geometry [28], and singular perturbations in the theory of phase transitions [12]. The study of minimum problems related to functionals of this type is meaningful in some applications, for instance in the segmentation of images in computer vision [24,25].

In this article we study the $L^1(\mathbb{R}^N)$-lower semicontinuity and the approximation, via Γ-convergence, of a functional F defined on the class $\mathcal{C}_b^\infty(\mathbb{R}^N)$ of all bounded open sets $E \in \mathcal{C}^\infty(\mathbb{R}^N)$, and consisting of two parts: the perimeter of E and an integral over ∂E of a suitable function f of the curvatures of ∂E. If F reduces to the perimeter functional then it is lower semicontinuous and it turns out [13,22,23] that the measures defined by

$$[h^{-1}|\nabla v|^2 + hW(v)]\ dx, \tag{1.1}$$

where v is a smooth function, $W(t) = t^2(1-t)^2$ and $h \in \mathbb{N}$, provide an approximation of the perimeter as $h \to +\infty$. This result has been widely generalized [1,6,26].

The presence here of the curvatures in the energy F is clearly source of new difficulties. We start by expressing the curvature term through the signed distance function d_E from ∂E, and this reflects on the choice of the approximating functionals F_h. To be more precise the following result is a consequence of Theorems 4.2 and 4.3. Let $F, F_h : L^1(\mathbb{R}^N) \to [0, +\infty]$ be the maps defined by

$$F(u) = \int_{\partial E} [1 + f(x, \nabla d_E, \nabla^2 d_E)]\ d\mathcal{H}^{N-1} \qquad \text{if } u = 1_E,\ E \in \mathcal{C}_b^\infty(\mathbb{R}^N), \tag{1.2}$$

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and, if $v \in \mathcal{C}^\infty(\mathbb{R}^N)$ has compact support,

$$F_h(v) = \int_{\mathbb{R}^n \setminus \{|\nabla v|=0\}} \left[1 + f\Big(x, \frac{\nabla v}{|\nabla v|}, \frac{P_{\nabla v} \nabla^2 v P_{\nabla v}}{|\nabla v|}\Big)\right] \, d\mu_h(x). \tag{1.3}$$

Here $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure, f is a non-negative continuous function, 1_E is the characteristic function of E, $\nabla^2 v$ is the Hessian of v, $P_{\nabla v} = \mathrm{Id} - |\nabla v|^{-2} \nabla v \otimes \nabla v$, and μ_h (which depend on v) are as in (1.1). Let us extend F and F_h with value $+\infty$ on the whole $L^1(\mathbb{R}^N)$. Then, if F is $L^1(\mathbb{R}^N)$-lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^N)$, we prove that

$$\Gamma - \lim_{h \to +\infty} F_h = 2c_0 \overline{F} \qquad \text{on } L^1(\mathbb{R}^N), \tag{1.4}$$

where $c_0 = \int_0^1 \sqrt{W(t)}\, dt$, $\overline{F}$ denotes the $L^1(\mathbb{R}^N)$-lower semicontinuous envelope of F, and the Γ-limit is computed with respect to the $L^1(\mathbb{R}^N)$-topology. More in general, we prove that (1.4) holds if the measure $\mathcal{H}^{N-1}$ is replaced by a suitable anisotropic surface measure (see (4.1) and Section (2.1)), provided that the measures μ_h in (1.3) are consequently modified (see (4.2)).

Our results apply, in particular, to the *elastica* functional F (see [14] and [20]) defined, in dimension $N = 2$, by

$$F(1_E) = \int_{\partial E} [1 + |\kappa|^2] \, d\mathcal{H}^1, \tag{1.5}$$

where $\kappa(x)$ is the curvature of ∂E at the point x. Since F is $L^1(\mathbb{R}^2)$-lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^2)$ [2], by (1.4) it can be approximated by the sequence of functionals

$$F_h(v) = \int_{\mathbb{R}^2 \setminus \{|\nabla v|=0\}} \left[1 + \left| |\nabla v|^{-1} \Delta v - |\nabla v|^{-3} \sum_{i,j=1}^{2} \nabla_i v \nabla_j v \nabla^2_{ij} v \right|^2 \right] d\mu_h(x),$$

where $\nabla v = (\nabla_1 v, \nabla_2 v)$ and $(\nabla^2 v)_{ij} = \nabla^2_{ij} v$.

Thanks to (1.4) the approximation problem is reduced to study the lower semicontinuity of F. Conditions on f ensuring that $F = \overline{F}$ on $\mathcal{C}_b^\infty(\mathbb{R}^N)$ are partially investigated in the last part of the paper, in which the cases $N = 2$ and $N \geq 3$ are separately considered. In particular, if $N = 3$, using a slicing argument [9] we prove the lower semicontinuity of F when f is a suitable quadratic form of the principal curvatures (Proposition 6.1). Moreover, if $N \geq 2$, we prove that $F = \overline{F}$ on $\mathcal{C}_b^\infty(\mathbb{R}^N)$ provided that f is a convex function of the length of the second fundamental form (Theorem 6.3).

The outline of the paper is the following. In Section 2 we give some notation. In Section 3 we give some preliminaries concerning the signed distance function and we recall the notion of geometric integrand $\widetilde{f}$ (next used in the definition of the approximating functionals F_h), and we show some examples. In Section 4 we define the functionals F and F_h. In Theorems 4.2 and 4.3 we show the relations between F and the Γ-limit of the sequence $\{F_h\}$. In Section 5 we find, in dimension $N = 2$, some necessary conditions on f that guarantee the lower semicontinuity of F (Theorem 5.2). Finally, in Section 6 we exhibit some examples of lower semicontinuous functionals in dimension $N \geq 3$.

2. Notation

Let E be a subset of $\mathbb{R}^N$, $N \geq 2$; we denote by 1_E the characteristic function of E, i.e., $1_E(x) = 1$ if $x \in E$, and $1_E(x) = 0$ if $x \in \mathbb{R}^N \setminus E$. We write $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ if E is bounded, open, and near each point $x \in \partial E$ the set E can be written as the subgraph of a function of class $\mathcal{C}^\infty$. If $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ its boundary ∂E is oriented by the inner unit normal vector field ν_E; moreover, we say that $u : \mathbb{R}^N \to \mathbb{R}$ is a *defining function* of ∂E if u is of class C^∞ in a neighbourhood of ∂E, $\partial E = \{u = 0\}$, $E = \{u > 0\}$, and $|\nabla u| \neq 0$ on ∂E.

Given $p, q \in \mathbb{R}^N$, by $p \otimes q$ we mean the matrix whose entries are $(p \otimes q)_{ij} = p_i q_j$, for $i, j = 1, \ldots, N$. We set also $p \odot q = (p \otimes q + q \otimes p)/2$.

If $p \in \mathbb{R}^N \setminus \{0\}$, by $P_p : \mathbb{R}^N \to \mathbb{R}^N$ we mean the projection matrix on the orthogonal subspace to p, i.e.,

$$P_p = \mathrm{Id} - |p|^{-2} p \otimes p. \tag{2.1}$$

Note that if $q \in \mathbb{R}^N$ we have

$$P_p \, p \odot q \, P_p = 0. \tag{2.2}$$

We denote by $\mathrm{Sym}(N)$ the space of the real $N \times N$ symmetric matrices. For notational simplicity we set

$$T = \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathrm{Sym}(N). \tag{2.3}$$

2.1. *Definitions of ϕ, ϕ^o, δ, δ_E, d_E, m, and $\mathcal{P}^{N-1}$*

We denote by $|\cdot|$ and d the norm and the distance in $\mathbb{R}^N$ (in the euclidean sense), respectively.

In what follows $\phi : \mathbb{R}^N_x \times \mathbb{R}^N_\xi \to [0, +\infty[$ is a symmetric convex Finsler metric, that is a continuous function which is locally equivalent (uniformly with respect to x) to the euclidean norm and such that $\phi(x, \cdot)$ is positively one-homogeneous and convex on $\mathbb{R}^N$.

We denote by $\phi^o : \mathbb{R}^N_x \times \mathbb{R}^N_{\xi^*} \to [0, +\infty[$ the dual of ϕ with respect to ξ [27, p. 128], defined as $\phi^o(x, \xi^*) = \sup\{\xi \cdot \xi^* : \phi(x, \xi) \leq 1\}$. One can show that ϕ^o is a symmetric convex Finsler metric.

We indicate by δ the distance on $\mathbb{R}^N$ induced by the metric ϕ, and by δ_E the δ signed distance function from ∂E positive inside $E \subseteq \mathbb{R}^N$, i.e.,

$$\delta_E(x) = \delta(x, \mathbb{R}^N \setminus E) - \delta(x, E). \tag{2.4}$$

By d_E we mean the euclidean signed distance function from ∂E positive inside E.

In the sequel $m : \mathbb{R}^N \to]0, +\infty[$ is a function of class $\mathcal{C}^1(\mathbb{R}^N)$ such that $0 < \inf m \leq \sup m < +\infty$.

We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, for $k \in \mathbb{N}$, $0 \leq k \leq N$, and by $\mathcal{P}^{N-1}$ the $(N-1)$-dimensional measure associated to ϕ and m, defined, on a smooth compact boundary ∂E, as the restriction to ∂E of $\mathcal{H}^{N-1}$ with density $\phi^o(x, \nu_E)m(x)$, see [5].

2.2. Definitions of Γ*-convergence and lower semicontinuous envelope*

Let $F_h : L^1(\mathbb{R}^N) \to [0, +\infty]$ be a sequence of functionals. For any $v \in L^1(\mathbb{R}^N)$ set

$$(\Gamma - \liminf_{h\to+\infty} F_h)(v) = \inf \left\{ \liminf_{h\to+\infty} F_h(v_h) : \{v_h\} \subseteq L^1(\mathbb{R}^N), v_h \overset{h\to+\infty}{\longrightarrow} v \text{ in } L^1(\mathbb{R}^N) \right\}$$

$$(\Gamma - \limsup_{h\to+\infty} F_h)(v) = \inf \left\{ \limsup_{h\to+\infty} F_h(v_h) : \{v_h\} \subseteq L^1(\mathbb{R}^N), v_h \overset{h\to+\infty}{\longrightarrow} v \text{ in } L^1(\mathbb{R}^N) \right\}.$$

We say that the sequence $\{F_h\}$ is Γ-convergent with respect to the $L^1(\mathbb{R}^N)$-topology if $\Gamma - \liminf_{h\to+\infty} F_h = \Gamma - \limsup_{h\to+\infty} F_h =: \Gamma - \lim_{h\to+\infty} F_h$ [13,11]. If $F_h = F$ for any $h \in \mathbb{N}$, the $\Gamma - \lim_{h\to+\infty} F_h$ is denoted by $\overline{F}$ and is called the $L^1(\mathbb{R}^N)$- lower semicontinuous envelope of F [8].

3. Preliminaries

We recall that if $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ then there exists a neighbourhood U of ∂E such that $d_E \in \mathcal{C}^\infty(U)$ and $|\nabla d_E| = 1$ in U; moreover, on ∂E, $\nabla d_E = \nu_E$ and $-(N-1)^{-1}\Delta d_E$ is the scalar mean curvature of ∂E (see [18, 14.6]). Since $\nabla d_E \in \mathrm{Ker}(\nabla^2 d_E)$ in U, we have

$$P_{\nabla d_E} \nabla^2 d_E P_{\nabla d_E} = \nabla^2 d_E \qquad \text{on } \partial E. \tag{3.1}$$

The following lemma, which is useful for the definition of $\widetilde{f}$ in (3.3), shows the relations between $\nabla^2 d_E$ and $\nabla^2 u$, where u is any defining function of ∂E.

Lemma 3.1. *Let $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ and suppose that $u, v : \mathbb{R}^N \to \mathbb{R}$ are two defining functions of ∂E. Then*

$$|\nabla u|^{-1} P_{\nabla u} \nabla^2 u P_{\nabla u} = |\nabla v|^{-1} P_{\nabla v} \nabla^2 v P_{\nabla v} = \nabla^2 d_E \qquad \textit{on } \partial E. \tag{3.2}$$

Proof. Fix $x \in \partial E$. By [4, Lemma 3.2] we have

$$|\nabla u(x)|^{-1} \nabla^2 u(x) = |\nabla v(x)|^{-1} \nabla^2 v(x) + \nabla u(x) \odot q,$$

for a suitable $q \in \mathbb{R}^N$. Then (3.2) follows from (2.2) and (3.1). □

Let us recall the definition of the geometric integrand $\widetilde{f}$ associated to f [17].

Definition 3.2. Let T be as in (2.3) and let $f : T \to \mathbb{R}$ be a continuous function. We define the function $\widetilde{f} : T \to \mathbb{R}$ as follows: if $(x, p, X) \in T$ we set

$$\widetilde{f}(x, p, X) = f\left(x, \frac{p}{|p|}, \frac{P_p X P_p}{|p|}\right), \tag{3.3}$$

where P_p is defined in (2.1).

The following properties are consequences of (3.3), (2.2) and (3.2).

Lemma 3.3. *The function* $\widetilde{f}$ *is continuous,* $\widetilde{\widetilde{f}} = \widetilde{f}$ *and*

(i) *for any* $(x, p, X) \in T$*, any* $\lambda > 0$*, and any* $q \in \mathbb{R}^N$ *we have*

$$\widetilde{f}(x, \lambda p, \lambda X + p \odot q) = \widetilde{f}(x, p, X); \tag{3.4}$$

(ii) *if* $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ *and* u *is any defining function of* ∂E *then*

$$\widetilde{f}(x, \nabla u, \nabla^2 u) = \widetilde{f}(x, \nabla d_E, \nabla^2 d_E) = f(x, \nabla d_E, \nabla^2 d_E) \qquad \text{on } \partial E. \tag{3.5}$$

Let $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ and let u be any defining function of ∂E.

Example 3.4. Let $\alpha \in [1, +\infty[$ and let $f_1, f_2 : T \to [0, +\infty[$ be defined as

$$f_1(x, p, X) = f_1(X) = ((N-1))^{-\alpha} \left|\mathrm{tr}\,(X)\right|^\alpha,$$
$$f_2(x, p, X) = f_2(X) = ((N-1))^{-1} [\,\mathrm{tr}\left(X^2\right)]^{1/2}.$$

Then $\widetilde{f}_1(\nabla u, \nabla^2 u)$ equals $|H|^\alpha$ on ∂E, where H is the scalar mean curvature of ∂E, while $((N-1)\widetilde{f}_2(\nabla u, \nabla^2 u))^2$ equals, on ∂E, the sum of the squares of the principal curvatures of ∂E, and $(f_2(\nabla^2 d_E))^2 = (N-1)^{-2} \sum_{i,j=1}^N \left|\nabla_{ij}^2 d_E\right|^2$.

Example 3.5. Assume that ϕ^2, $(\phi^o)^2$ are of class $\mathcal{C}^2$ and that $(\phi(x, \cdot))^2$, $(\phi^o(x, \cdot))^2$ are strictly convex. Let $f : T \to [0, +\infty[$ be defined as

$$f(x, p, X) = \left| \sum_{i=1}^N \frac{\partial^2(\phi^o)}{\partial x_i \partial \xi_i}(x, p) + \nabla_\xi^2(\phi^o)(x, p) X + \nabla_\xi(\phi^o)(x, p) \cdot \nabla(\log m) \right|.$$

Then one can show that $f = \widetilde{f}$; $f(x, \nabla u, \nabla^2 u)$ represents the absolute value of the anisotropic scalar mean curvature of ∂E with respect to ϕ and m [5,4].

4. Definitions of the functionals. Γ-convergence result

Given the continuous function $f : T \to [0, +\infty[$ we define the map $F : L^1(\mathbb{R}^N) \to [0, +\infty]$ as

$$F(u) = \int_{\partial E} \left[1 + f(x, \nabla d_E, \nabla^2 d_E)\right] \, d\mathcal{P}^{N-1} \tag{4.1}$$

if u is the characteristic function of a set $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$, and $F = +\infty$ elsewhere. The integral in (4.1) will be also denoted by $F(E)$, thus emphasizing the dependence of F on the set E rather than on its boundary ∂E. Note that by (3.5) we can replace f by $\widetilde{f}$ in (4.1) without affecting the result.

4.1. The approximating functionals F_h

For any $v \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ (the space of functions of class $\mathcal{C}^\infty(\mathbb{R}^N)$ with compact support) and $h \in \mathbb{N}$ we define the non-negative Radon measures μ_h on $\mathbb{R}^N$ by

$$d\mu_h(x) = [h^{-1}(\phi^o(x, \nabla v))^2 + hW(v)]\ m(x)\ dx, \tag{4.2}$$

where $W : \mathbb{R} \to [0, +\infty[$ is defined by $W(t) = t^2(1-t)^2$ and dx stands for the Lebesgue measure. For simplicity of notation we omit the dependence on v in the notation of the measures μ_h. We set $c_0 = \int_0^1 \sqrt{W(t)}\ dt$.

Let $f : T \to [0, +\infty[$ be a continuous function; for any $h \in \mathbb{N}$ we define the map $F_h : L^1(\mathbb{R}^N) \to [0, +\infty]$ as

$$F_h(v) = \int_{\mathbb{R}^N \setminus \{|\nabla v| = 0\}} \left[1 + \widetilde{f}(x, \nabla v, \nabla^2 v)\right]\ d\mu_h(x) \qquad \text{if } v \in \mathcal{C}_c^\infty(\mathbb{R}^N), \tag{4.3}$$

and $F_h = +\infty$ elsewhere. Note the presence of $\widetilde{f}$ in (4.3).

The following remark is a consequence of (3.5) and the definition of $\overline{F}$, and will be useful in the proof of Theorem 4.2.

Remark 4.1. Let $\{E_k\}$ be a sequence in $\mathcal{C}_b^\infty(\mathbb{R}^N)$ converging in $L^1(\mathbb{R}^N)$ to E with $1_E \in L^1(\mathbb{R}^N)$. For any $k \in \mathbb{N}$ let u_k be a defining function of ∂E_k. Then

$$\liminf_{k \to +\infty} \int_{\partial E_k} [1 + \widetilde{f}(x, \nabla u_k, \nabla^2 u_k)]\ d\mathcal{P}^{N-1} \geq \overline{F}(E).$$

The main result of this section is contained in the following two theorems.

Theorem 4.2. *Let T be as in (2.3), let $f : T \to [0, +\infty[$ be a continuous function and let $\widetilde{f}$ be defined as in (3.3). Let F, F_h be the functionals defined in (4.1) and (4.3), respectively. Then*

$$\Gamma - \liminf_{h \to +\infty} F_h \geq 2c_0 \overline{F} \qquad \text{on } L^1(\mathbb{R}^N).$$

Proof. Let $u \in L^1(\mathbb{R}^N)$ and let $\{v_h\}$ be a sequence in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ converging to u in $L^1(\mathbb{R}^N)$ such that $\liminf_{h \to +\infty} F_h(v_h) = (\Gamma - \liminf_{h \to +\infty} F_h)(u) < +\infty$. Passing to a subsequence, we may assume that the $\liminf_{h \to +\infty}$ is a $\lim_{h \to +\infty}$ and that $\{v_h\}$ converges to u almost everywhere. We have $\int_{\mathbb{R}^N} W(v_h)\ dx \leq O(h^{-1})$, so that u is the characteristic function of some measurable set $E \subseteq \mathbb{R}^N$.

As $h^{-1}(\phi^o(x, \nabla v_h))^2 + hW(v_h) \geq 2\phi^o(x, \nabla v_h)\sqrt{W(v_h)}$, using the fact that $\phi^o(x, \cdot)$ is one-homogeneous and the coarea formula [16] we have

$$\begin{aligned} F_h(v_h) &\geq 2 \int_{\{|\nabla v_h| \neq 0\}} |\nabla v_h| \sqrt{W(v_h)} \left[1 + \widetilde{f}(x, \nabla v_h, \nabla^2 v_h)\right] \phi^o\left(x, \frac{\nabla v_h}{|\nabla v_h|}\right)\ m dx \\ &\geq 2 \int_0^1 \sqrt{W(t)} \int_{\{v_h = t\} \cap \{|\nabla v_h| \neq 0\}} \left[1 + \widetilde{f}(x, \nabla v_h, \nabla^2 v_h)\right] d\mathcal{P}^{N-1} dt. \end{aligned}$$

Observe that, given $h \in \mathbb{N}$, $\partial\{v_h > t\}$ is compact for every $t \in]0,1[$. Moreover, Sard's theorem [16, 3.4.3] implies that $|\nabla v_h| \neq 0$ on $\{v_h = t\}$ for almost every $t \in]0,1[$, hence $\{v_h = t\} = \partial\{v_h > t\}$ and $\{v_h > t\} \in \mathcal{C}_b^\infty(\mathbb{R}^N)$. Let $I_h \subseteq]0,1[$ be such that the Lebesgue measure $|I_h|$ of I_h is zero and $\{v_h > t\}$ verifies the properties listed above for any $t \in]0,1[\setminus I_h$. Letting $I := \bigcup_{h\in\mathbb{N}} I_h$ we have $|I| = 0$ and, for any $h \in \mathbb{N}$,

$$F_h(v_h) \geq 2\int_{]0,1[\setminus I} \sqrt{W(t)} \int_{\partial\{v_h>t\}} \left[1 + \widetilde{f}\left(x, \nabla v_h, \nabla^2 v_h\right)\right] \, d\mathcal{P}^{N-1} dt. \tag{4.4}$$

Using the Cavalieri formula there exist a subsequence $\{v_k\}$ of $\{v_h\}$ and a set $J \subseteq]0,1[$ with $|J| = 0$ such that $1_{\{v_k>t\}} \to 1_{\{u>t\}} = E$ in $L^1(\mathbb{R}^N)$ as $k \to +\infty$ for any $t \in]0,1[\setminus(I\cup J)$. Fix $t \in]0,1[\setminus(I \cup J)$ and set $u_k = v_k - t$. Using (4.4), Fatou's lemma, applying Remark 4.1 with E_k replaced by $\{u_k > 0\}$ and observing that $\nabla u_k = \nabla v_k$, $\nabla^2 u_k = \nabla^2 v_k$, we obtain

$$\begin{aligned}
(\Gamma - \liminf_{h\to+\infty} F_h)(u) &= \lim_{h\to+\infty} F_h(v_h) = \lim_{k\to+\infty} F_k(v_k) \\
&\geq 2\int_{]0,1[\setminus I\cup J} \sqrt{W(t)} \, \liminf_{k\to+\infty} \int_{\partial\{u_k>0\}} \left[1 + \widetilde{f}(x, \nabla u_k, \nabla^2 u_k)\right] \, d\mathcal{P}^{N-1} dt \\
&\geq 2\overline{F}(E) \int_{]0,1[\setminus(I\cup J)} \sqrt{W(t)} \, dt = 2c_0 \overline{F}(E).
\end{aligned}$$

This concludes the proof. □

Theorem 4.3. *Assume that ϕ does not depend on x and that $\phi^2, (\phi^o)^2$ are of class $\mathcal{C}^\infty$ and strictly convex. Let F and F_h be as in the statement of Theorem 4.2. Suppose that F is $L^1(\mathbb{R}^N)$-lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^N)$. Then*

$$\Gamma - \limsup_{h\to+\infty} F_h \leq 2c_0\overline{F} \qquad \text{on } L^1(\mathbb{R}^N). \tag{4.5}$$

Proof. Let $u \in L^1(\mathbb{R}^N)$; we can assume that $\overline{F}(u) < +\infty$, so that $u = 1_E$ for some measurable set $E \subseteq \mathbb{R}^N$. Let us first suppose that $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$, so that $\overline{F}(E) = F(E)$. Denote by γ the minimizer of the one-dimensional functional $\int_{\mathbb{R}}(|\zeta'|^2 + W(\zeta)) \, dt$ under the constraints $\zeta \in H^1_{\rm loc}(\mathbb{R})$, $\lim_{t\to+\infty} \zeta(t) = 1$, $\lim_{t\to-\infty} \zeta(t) = 0$, $\zeta(0) = 1/2$. With our choice of W it turns out that $\gamma(t) = (\mathrm{tgh}(t/2) + 1)/2$. For any $h \in \mathbb{N}$ let $t_h = \log h$. Denote by $\chi_h : [0,+\infty[\to [0,1]$ a function of class $\mathcal{C}^\infty$ with the following properties: $\chi_h = 1$ on $[0,t_h]$, $\chi_h = 0$ on $[2t_h,+\infty[$, $\chi'_h < 0$ in $]t_h, 2t_h[$ and $\|\chi'_h\|_{L^\infty([t_h,2t_h])} = O(1/\log h)$. Let

$$\hat{\gamma}_h(t) = \begin{cases} \gamma(t)\chi_h(t) + 1 - \chi_h(t) & \text{if } t \geq 0, \\ 1 - \hat{\gamma}_h(-t) & \text{if } t < 0, \end{cases} \qquad \gamma_h(t) = \hat{\gamma}_h(ht),$$

and

$$u_h(x) = \gamma_h(\delta_E(x)) \qquad \forall x \in \mathbb{R}^N, \tag{4.6}$$

where δ_E is defined in (2.4). One can show that $u_h \to 1_E$ in $L^1(\mathbb{R}^N)$ as $h \to +\infty$ and that, under our assumptions on ϕ and ϕ^o, the function δ_E is of class $\mathcal{C}^\infty$ in a suitable neighbourhood of ∂E. Hence $u_h \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ for any h large enough. Set

$$A_h = \{x \in \mathbb{R}^N : |\delta_E(x)| < h^{-1}t_h\}, \quad B_h = \{x \in \mathbb{R}^N : h^{-1}t_h < |\delta_E(x)| < 2h^{-1}t_h\}.$$

Using the definition of F_h we have

$$\begin{aligned} F_h(u_h) &= \int_{A_h} d\mu_h(x) + \int_{A_h} \widetilde{f}(x, \nabla u_h, \nabla^2 u_h)\ d\mu_h(x) \\ &+ \int_{B_h} d\mu_h(x) + \int_{B_h} \widetilde{f}(x, \nabla u_h, \nabla^2 u_h)\ d\mu_h(x) \\ &=: F_h^{(1)}(u_h) + F_h^{(2)}(u_h) + F_h^{(3)}(u_h) + F_h^{(4)}(u_h). \end{aligned}$$

We shall show that

$$\lim_{h\to+\infty} \left(F_h^{(1)}(u_h) + F_h^{(2)}(u_h)\right) = 2c_0 F(E), \tag{4.7}$$

$$\lim_{h\to+\infty} \left(F_h^{(3)}(u_h) + F_h^{(4)}(u_h)\right) = 0, \tag{4.8}$$

and this will conclude the proof of (4.5) when $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$. We recall a basic property of δ_E [5, Theorem 3.2]: if $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ then

$$\phi^o(\nabla \delta_E) = 1 \tag{4.9}$$

in a suitable neighbourhood of ∂E, which implies, in particular, that the infimum of $|\nabla \delta_E|$ in this neighbourhood is strictly positive.
Let us prove (4.7). First it is well known [5,6,26] that

$$\lim_{h\to+\infty} F_h^{(1)}(u_h) = 2c_0 \int_{\partial E} d\mathcal{P}^{N-1}. \tag{4.10}$$

By (4.6), for h large enough we have

$$\nabla u_h = \gamma_h'(\delta_E)\nabla\delta_E, \qquad \nabla^2 u_h = \gamma_h''(\delta_E)\nabla\delta_E \otimes \nabla\delta_E + \gamma_h'(\delta_E)\nabla^2\delta_E. \tag{4.11}$$

Hence by (4.9) and since $\gamma' = \sqrt{W(\gamma)}$, on A_h we have

$$\begin{aligned} &h^{-1}(\phi^o(\nabla u_h))^2 + hW(u_h) = h^{-1}\left(\gamma_h'(\delta_E)\right)^2 + hW(\gamma_h(\delta_E)) \\ &= 2\gamma_h'(\delta_E)\sqrt{W(\gamma_h(\delta_E))}. \end{aligned} \tag{4.12}$$

In addition, using (4.11) and (3.4) (for $x \in A_h$) applied with $\lambda = \gamma_h'(\delta_E(x)) > 0$, $p = \nabla\delta_E(x) \in \mathbb{R}^N \setminus \{0\}$, and $q = \gamma_h''(\delta_E(x))\nabla\delta_E(x)/2$, we obtain that

$$\widetilde{f}(x, \nabla u_h, \nabla^2 u_h) = \widetilde{f}(x, \nabla\delta_E, \nabla^2\delta_E) \qquad \text{on } A_h. \tag{4.13}$$

By (4.12) and (4.13), the fact that $|\nabla\delta_E|^{-1} = \phi^o\Big(\frac{\nabla\delta_E}{|\nabla\delta_E|}\Big)$ in a neighbourhood of ∂E (see (4.9)) and the coarea formula we get, for h large enough,

$$\begin{aligned} F_h^{(2)}(u_h) &= 2\int_{A_h} |\nabla\delta_E|\gamma_h'(\delta_E)\sqrt{W(\gamma_h(\delta_E))}\ \widetilde{f}(x,\nabla\delta_E,\nabla^2\delta_E)\phi^o\Big(\frac{\nabla\delta_E}{|\nabla\delta_E|}\Big)m\ dx \\ &= 2\int_{-h^{-1}t_h}^{h^{-1}t_h} \gamma_h'(t)\sqrt{W(\gamma_h(t))}\int_{\{\delta_E=t\}} \widetilde{f}(x,\nabla\delta_E,\nabla^2\delta_E)\ \ d\mathcal{P}^{N-1}dt. \end{aligned}$$

We observe now that for any h large enough

$$\begin{aligned} &\int_{\{\delta_E=t\}} \widetilde{f}(x,\nabla\delta_E,\nabla^2\delta_E)\ d\mathcal{P}^{N-1} \\ &= \int_{\partial E} \widetilde{f}(x,\nabla\delta_E,\nabla^2\delta_E)\ d\mathcal{P}^{N-1} + o(h^{-1}t_h) \end{aligned} \tag{4.14}$$

for any $t\in[-h^{-1}t_h,h^{-1}t_h]$. Indeed, if $t\in[-h^{-1}t_h,h^{-1}t_h]$, $x\in\{\delta_E=t\}$ and if $\pi(x)$ is the (unique) projection of x onto ∂E with respect to the metric ϕ, we have $x=\pi(x)+t(\nabla\phi^o)(\nu_E(\pi(x)))$ [4], $\nabla_i\delta_E(x) = \nabla_i\delta_E(\pi(x)) + O(h^{-1}t_h)$, $\nabla^2_{ij}\delta_E(x) = \nabla^2_{ij}\delta_E(\pi(x)) + O(h^{-1}t_h)$, and similar relations hold for ϕ^o and m. Hence, as all $\overline{A_h}$ are contained (for h large enough) in a fixed neighbourhood of ∂E where δ_E is of class $\mathcal{C}^\infty$ and $\inf|\nabla\delta_E|>0$, and since $\widetilde{f}$, ϕ^o and m are continuous, we have

$$\begin{aligned} &\widetilde{f}(x,\nabla\delta_E(x),\nabla^2\delta_E(x))\phi^o\Big(\frac{\nabla\delta_E(x)}{|\nabla\delta_E(x)|}\Big)m(x) \\ &= \widetilde{f}(\pi(x),\nabla\delta_E(\pi(x)),\nabla^2\delta_E(\pi(x)))\phi^o\Big(\frac{\nabla\delta_E(\pi(x))}{|\nabla\delta_E(\pi(x))|}\Big)m(\pi(x)) + o(h^{-1}t_h) \end{aligned}$$

for any $x\in A_h$, and this, together with a change of variables, implies (4.14). We then get

$$\begin{aligned} F_h^{(2)}(u_h) = 2\int_{\partial E} &\widetilde{f}(x,\nabla\delta_E,\nabla^2\delta_E)\ d\mathcal{P}^{N-1}\int_{-h^{-1}t_h}^{h^{-1}t_h}\gamma_h'(t)\sqrt{W(\gamma_h(t))}\ dt \\ &+ o(h^{-1}t_h)\int_{-h^{-1}t_h}^{h^{-1}t_h}\gamma_h'(t)\sqrt{W(\gamma_h(t))}\ dt =: \mathrm{I}_h + \mathrm{II}_h. \end{aligned}$$

Now

$$\int_{-h^{-1}t_h}^{h^{-1}t_h}\gamma_h'(t)\sqrt{W(\gamma_h(t))}\ dt = \int_{\gamma(-t_h)}^{\gamma(t_h)}\sqrt{W(\tau)}\ d\tau \to c_0 \quad \text{as } h\to+\infty.$$

Therefore $\lim_{h\to+\infty}\mathrm{II}_h = 0$, and using (3.5)

$$\lim_{h\to+\infty}\ \mathrm{I}_h = 2c_0\int_{\partial E}\widetilde{f}(x,\nabla\delta_E,\nabla^2\delta_E)\ d\mathcal{P}^{N-1} = 2c_0\int_{\partial E} f(x,\nabla d_E,\nabla^2 d_E)\ d\mathcal{P}^{N-1}.$$

Then (4.7) follows recalling (4.10).

Let us prove (4.8). One can check that

$$\|\hat{\gamma}_h - 1\|_{L^\infty([t_h,2t_h])} = O(h^{-1}), \qquad \|\hat{\gamma}_h'\|_{L^\infty([t_h,2t_h])} = o(h^{-1}). \tag{4.15}$$

Applying (4.9), the coarea formula and (4.15) we have, for h large enough,

$$\begin{aligned} F_h^{(3)}(u_h) &= \int_{B_h} |\nabla \delta_E| \phi^o\Big(\frac{\nabla \delta_E}{|\nabla \delta_E|}\Big)\, d\mu_h(x) \\ &\le O(1) \int_{h^{-1}t_h}^{2h^{-1}t_h} h^{-1}|\gamma_h'(t)|^2 + hW(\gamma_h(t))\, dt = o(1). \end{aligned}$$

It remains to show that $\lim_{h\to+\infty} F_h^{(4)}(u_h) = 0$.
Reasoning as in (4.13) we have $\widetilde{f}(x, \nabla u_h, \nabla^2 u_h) = \widetilde{f}(x, \nabla\delta_E, \nabla^2\delta_E)$ on B_h. As $\delta_E \in \mathcal{C}^\infty(B_h)$ and $\inf_{B_h} |\nabla\delta_E| > 0$ for h large enough and since $\widetilde{f}$ is continuous, we then have

$$F_h^{(4)}(u_h) \le \int_{B_h} \widetilde{f}(x, \nabla\delta_E, \nabla^2\delta_E)\, d\mu_h(x) \le O(1) \int_{B_h} d\mu_h(x) = o(1).$$

The proof of (4.8) is concluded.

We have showed that (4.5) holds for $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$. If $u = 1_E \in L^1(\mathbb{R}^N)$ we can find a sequence $\{E_h\}$ in $\mathcal{C}_b^\infty(\mathbb{R}^N)$ converging to u in $L^1(\mathbb{R}^N)$ such that $F(E_h) \to \overline{F}(E) < +\infty$ as $h \to +\infty$. Then (4.5) follows from the previous case and a diagonal procedure. □

Using Theorems 4.2 and 4.3 we have the following result, which shows that the Γ-approximability of the functional F is reduced to study its lower semicontinuous envelope $\overline{F}$.

Corollary 4.4. *Under the assumptions of* Theorem 4.3 *we have*

$$\Gamma - \lim_{h\to+\infty} F_h = \begin{cases} 2c_0 F & \text{on } \mathcal{C}_b^\infty(\mathbb{R}^N), \\ 2c_0\overline{F} & \text{elsewhere on } L^1(\mathbb{R}^N). \end{cases}$$

Remark 4.5. Let $c \in [0, +\infty[$ and assume that $\{v_h\}$ is a sequence in $L^1(\mathbb{R}^N)$ such that $F_h(v_h) \le c$ for any $h \in \mathbb{N}$. Then $\{v_h\}$ admits a subsequence converging in $L^1(\mathbb{R}^N)$ to a characteristic function of a set E of finite perimeter such that $\overline{F}(E) < +\infty$.

Indeed, as $c \ge F_h(v_h) \ge O(1) \int_{\mathbb{R}^N} h^{-1}|\nabla v_h|^2 + hW(v_h)\, dx$, we have [22] that $\{v_h\}$ has a subsequence converging in $L^1(\mathbb{R}^N)$ to a characteristic function of a set E of finite perimeter. Reasoning as in the proof of Theorem 4.2, one then obtains $\overline{F}(E) < +\infty$.

5. Lower semicontinuity of F in two dimensions

In this section we find some necessary conditions on f which ensure the L^1-lower semicontinuity of F in two dimensions. In view of Corollary 4.4, F can be approximated via Γ-convergence. A particular case has been illustrated in Section 1, see (1.5).

Let $N = 2$; if $p = (p_1, p_2) \in \mathbb{R}^2$ we set $p^\perp = (-p_2, p_1)$.

Remark 5.1. If $N = 2$ we have

$$\nabla^2 d_E = \Delta d_E (\nabla d_E)^\perp \otimes (\nabla d_E)^\perp \qquad \text{on } \partial E. \tag{5.1}$$

Proof. It is enough to observe that if $p \in \mathbb{R}^2$, $|p| = 1$, and $X \in \mathrm{Sym}(2)$ is such that $P_p X P_p = X$, then $X = \mathrm{tr}(X) p^\perp \otimes p^\perp$. Letting $x \in \partial E$, $p = \nabla d_E(x)$, $X = \nabla^2 d_E(x)$ and using (3.1), the assertion follows. □

Let $E \in \mathcal{C}_b^\infty(\mathbb{R}^2)$ and let $\gamma : [0, l(\gamma)] \to \mathbb{R}^2$ be an oriented parametrization by arc length of a connected component of ∂E (here $l(\gamma)$ denotes the length of γ). Then at $x = \gamma(s) \in \partial E$, we have

$$\nabla d_E = (\dot\gamma)^\perp, \quad \Delta d_E = -\kappa, \quad \nabla^2 d_E = -\kappa \dot\gamma \otimes \dot\gamma, \quad \ddot\gamma = \kappa \nabla d_E,$$

where κ is the curvature (with sign) of ∂E.

If $f : T = \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \times \mathrm{Sym}(2) \to [0, +\infty[$ is a continuous function, using also (5.1) it follows that at $x = \gamma(s) \in \partial E$ we have

$$f\left(x, \nabla d_E, \nabla^2 d_E\right) = g(\gamma, \dot\gamma, \ddot\gamma), \tag{5.2}$$

where $g : \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \to [0, +\infty[$ is the continuous function defined by

$$g(x, \xi, \eta) = f(x, \xi^\perp, \xi \otimes \eta^\perp). \tag{5.3}$$

Theorem 5.2. *Let $f : T \to [0, +\infty[$ be a continuous function such that $f = \widetilde{f}$ (see (3.3)) and satisfying the following properties:*

(i) *there exist constants $c > 0$ and $\alpha > 1$ such that $f(x, p, X) \geq c \left| \frac{\mathrm{tr}(P_p X P_p)}{|p|} \right|^\alpha$ for any $(x, p, X) \in T$;*

(ii) *for any $x \in \mathbb{R}^2$ and any $p \in \mathbb{R}^2 \setminus \{0\}$ the function $q \to f(x, p^\perp, p \otimes q^\perp)$ is convex on $\mathbb{R}^2$.*

Then the map F defined in (1.2) (with $N = 2$) is $L^1(\mathbb{R}^2)$-lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^2)$.

Proof. Let $E \in \mathcal{C}_b^\infty(\mathbb{R}^2)$ and let $\{E_h\}$ be a sequence in $C_b^\infty(\mathbb{R}^2)$ converging to E in $L^1(\mathbb{R}^2)$ as $h \to +\infty$. We must prove that

$$F(E) \leq \liminf_{h \to +\infty} F(E_h). \tag{5.4}$$

We can suppose, possibly passing to a subsequence, that the right hand side of (5.4) is a finite limit. For simplicity, this subsequence (and any further subsequence) will be still denoted by $\{E_h\}$. Using assumption (i) we then have

$$\sup_h \int_{\partial E_h} |\Delta d_{E_h}|^\alpha \, d\mathcal{H}^1 \leq c^{-1} \sup_h \int_{\partial E_h} f(x, \nabla d_{E_h}, \nabla^2 d_{E_h}) \, d\mathcal{H}^1 < +\infty,$$

so that the $L^\alpha_{\mathcal{H}^1}$-norm of the curvature of ∂E_h is uniformly bounded with respect to h. Since also $\{\mathcal{H}^1(\partial E_h)\}$ is bounded, possibly passing to a subsequence, for any h the set ∂E_h has a finite number $\bar m$ of connected components [2], and $\bar m$ is independent of h. Let Δ_h be an oriented parametrization of ∂E_h [2, Definition 3.2]; then $\Delta_h = \{\gamma_h^1, \ldots, \gamma_h^{\bar m}\}$, where γ_h^i are closed regular simple smooth disjoint curves parametrized with constant velocity on $[0,1]$. Let $\{\gamma_h^{i_1}, \ldots, \gamma_h^{i_n}\}$ be the $H^{2,\alpha}$-equibounded sequence and let $\{\gamma^{i_1}, \ldots, \gamma^{i_n}\}$ be the limit system of curves of class $H^{2,\alpha}$ constructed (starting from Δ_h) in the proof of Lemma 3.3 of [2]. Then $\bar m \geq n$,

$$\bigcup_{j=1}^{n} \gamma^{i_j}([0,1]) \supseteq \partial E, \tag{5.5}$$

and for any $j = 1, \ldots, n$ we have that $\gamma_h^{i_j} \rightharpoonup \gamma^{i_j}$ weakly in $H^{2,\alpha}([0,1])$ as $h \to +\infty$ (in particular $\lim_{h\to+\infty} l(\gamma_h^{i_j}) = l(\gamma^{i_j})$). Recalling $\bar m \geq n$, (5.5) and (5.2), we obtain

$$\begin{aligned} \liminf_{h\to+\infty} F(E_h) &\geq \liminf_{h\to+\infty} \Big[\sum_{j=1}^{n} l(\gamma_h^{i_j}) + \sum_{j=1}^{n} \int_0^{l(\gamma_h^{i_j})} g(\gamma_h^{i_j}, \dot\gamma_h^{i_j}, \ddot\gamma_h^{i_j})\, ds\Big] \\ &\geq \mathcal{H}^1(\partial E) + \sum_{j=1}^{n} \liminf_{h\to+\infty} \int_0^{l(\gamma_h^{i_j})} g(\gamma_h^{i_j}, \dot\gamma_h^{i_j}, \ddot\gamma_h^{i_j})\, ds. \end{aligned} \tag{5.6}$$

By assumption (ii), for any $x \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$, the function $\eta \to g(x,\xi,\eta)$ is convex on $\mathbb{R}^2$. Consequently the functional $\gamma \to \int_I g(\gamma,\dot\gamma,\ddot\gamma)\, ds$ is weakly $H^{2,\alpha}(I)$-lower semicontinuous (see, for instance, [8, Section 2.3]), where $I \subseteq [0, l(\gamma)]$ is an open interval. For any $j = 1, \ldots, n$ let $0 < \lambda_j < l(\gamma^{i_j})$. We have $l(\gamma_h^{i_j}) > \lambda_j$ for any $j = 1, \ldots, n$ provided that h is large enough. Therefore by (5.6)

$$\liminf_{h\to+\infty} F(E_h) \geq \mathcal{H}^1(\partial E) + \sum_{j=1}^{n} \int_0^{\lambda_j} g\left(\gamma^{i_j}, \dot\gamma^{i_j}, \ddot\gamma^{i_j}\right)\, ds.$$

Since this inequality holds for any $\lambda_j < l(\gamma^{i_j})$ we have

$$\liminf_{h\to+\infty} F(E_h) \geq \mathcal{H}^1(\partial E) + \sum_{j=1}^{n} \int_0^{l(\gamma^{i_j})} g\left(\gamma^{i_j}, \dot\gamma^{i_j}, \ddot\gamma^{i_j}\right)\, ds. \tag{5.7}$$

Using (5.5) and reasoning as in [2, Lemma 3.4] we also have

$$\sum_{j=1}^{n} \int_0^{l(\gamma^{i_j})} g\left(\gamma^{i_j}, \dot\gamma^{i_j}, \ddot\gamma^{i_j}\right)\, ds \geq \int_{\partial E} f(x, \nabla d_E, \nabla^2 d_E)\, d\mathcal{H}^1. \tag{5.8}$$

Then (5.4) follows from (5.7) and (5.8). $\square$

Example 5.3. The function $f(p,X) = |p|^{-\alpha}|\operatorname{tr}(P_p X P_p)|^\alpha$ for $\alpha > 1$ (which corresponds to $\widetilde{f}_1$ of Example 3.4) verifies the assumptions of Theorem 5.2. Indeed given $p = (p_1, p_2) \in \mathbb{R}^2 \setminus \{0\}$, we have, for $q = (q_1, q_2) \in \mathbb{R}^2$,

$$f(p^\perp, p \otimes q^\perp) = |p|^{-\alpha}|\operatorname{tr}(p \otimes q^\perp P_{p^\perp})|^\alpha = |p|^{-\alpha}|p_2 q_1 - p_1 q_2|^\alpha,$$

so that the map $q \to f(p^\perp, p \otimes q^\perp)$ is convex on $\mathbb{R}^2$.

The case in which the integrand f has linear growth at infinity is useful for some applications, see [25]. In particular the functional $F(E) = \int_{\partial E}[1 + \zeta(\kappa)]\, d\mathcal{H}^1$, where $\zeta(t) = c_1|t|^2 1_{\{|t| \le \tau\}} + [c_2|t| - c_3]1_{\{|t| > \tau\}}$, for $\tau > 0$, $c_1 > 0$, $c_1\tau^2 = c_2\tau - c_3$, $c_2 = 2c_1\tau$, is lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^2)$, see [3, Theorem 6.1]. Hence by Corollary 4.4 F is the Γ-limit of the sequence in (4.3) where $m \equiv 1$, $\phi(x,\xi) = |\xi|$, and $f(X) = c_1\,|\operatorname{tr}(X)|^2\, 1_{\{X \in \mathrm{Sym}(2): |\operatorname{tr}(X)| \le \tau\}} + (c_2|\operatorname{tr}(X)| - c_3)\, 1_{\{X \in \mathrm{Sym}(2): |\operatorname{tr}(X)| > \tau\}}$.

6. Lower semicontinuity of F in dimension $N \ge 3$

In this section we show some examples of functionals F in dimension $N \ge 3$ depending on curvatures and which are $L^1(\mathbb{R}^N)$-lower semicontinuous on smooth compact boundaries. In the proof of the following proposition we use a slicing argument [9].

Proposition 6.1. *Let $N = 3$, let f_1, f_2 be as in Example 3.4 with $\alpha = 2$, let $a, b \in]0, +\infty[$, and set*

$$f(p,X) = a\widetilde{f}_1(p,X) + b(\widetilde{f}_2(p,X))^2 \qquad \forall p \in \mathbb{R}^3 \setminus \{0\},\ \forall X \in \mathrm{Sym}(3). \tag{6.1}$$

Then if $b \ge 2a$ the map F defined in (1.2) (with $N = 3$) is $L^1(\mathbb{R}^3)$-lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^3)$.

Note that, if κ_1, κ_2 are the principal curvatures of ∂E, letting $H = (\kappa_1 + \kappa_2)/2$ and $K = \kappa_1\kappa_2$, then the functional F in (1.2) corresponding to f in (6.1) can be written as

$$\begin{aligned} F(E) &= \int_{\partial E} \Big[1 + \frac{a}{4}|\Delta d_E|^2 + \frac{b}{4}\sum_{i,j=1}^{3} |\nabla^2_{ij} d_E|^2\Big] d\mathcal{H}^2 \\ &= \int_{\partial E} [1 + \frac{a}{4}|\kappa_1 + \kappa_2|^2 + \frac{b}{4}(\kappa_1^2 + \kappa_2^2)] d\mathcal{H}^2 = \int_{\partial E} [1 + (a+b)|H|^2 - \frac{b}{2}K]\, d\mathcal{H}^2. \end{aligned}$$

Proof of Proposition 6.1.
Let $E \in \mathcal{C}_b^\infty(\mathbb{R}^3)$ and let $\{E_h\}$ be a sequence in $\mathcal{C}_b^\infty(\mathbb{R}^3)$ converging to E in $L^1(\mathbb{R}^3)$ as $h \to +\infty$. We must prove inequality (5.4). We can suppose, possibly passing to a subsequence, that the right hand side of (5.4) is a finite limit. Hence, in view of [7, Remark 29.4.9 with $N - 1 = m = 2$], possibly passing to a further subsequence, we can assume that each ∂E_h has a finite number of connected components independent of h.

We recall now a particular case of a kinematic formula due to C.-S. Chen [9, Section 5] : if π denotes a plane in $\mathbb{R}^3$, one has

$$\int_{\partial E} [6|H|^2 - 2K] \, d\mathcal{H}^2 = c_2 \int_\pi \int_{\pi\cap\partial E} \kappa^2 \, d\mathcal{H}^1 \, d\pi, \tag{6.2}$$

where $d\pi$ denotes the density for the planes in $\mathbb{R}^3$ [28, II.12.5], $c_2 > 0$ depends only on the dimension, and κ denotes the curvature of the curve $\pi \cap \partial E$ (note that for $d\pi$-almost every π the curve $\pi \cap \partial E$ is of class $\mathcal{C}^\infty$ by Sard's theorem, and $\pi \cap \partial E = \partial(\pi \cap E)$). Moreover

$$\mathcal{H}^2(\partial E) = c_1 \int_\pi \mathcal{H}^1(\pi \cap \partial E) \, d\pi, \tag{6.3}$$

where $c_1 > 0$ depends only on the dimension (see [10, formula (81) with $e = 0$]).
Observe now that if $E_h \to E$ in $L^1(\mathbb{R}^3)$, then by Fubini's theorem for $d\pi$-almost every π we have $\pi \cap E_h \to \pi \cap E$ in $L^1(\pi)$.
Let us first suppose that $b = 2a$. For any h denote by J_h the complement of all planes π so that $\pi \cap \partial E_h = \partial(\pi \cap E_h)$ and $\pi \cap E_h$ is of class $\mathcal{C}^\infty$. Using the $L^1(\pi)$-lower semicontinuity on $\mathcal{C}^\infty_b(\pi)$ of the map

$$\pi \cap E \to \int_{\partial(\pi\cap E)} [c_1 + c_2\kappa^2] \, d\mathcal{H}^1$$

proved in [2, Theorem 3.2], using Fatou's lemma, letting $J = \cup_h J_h$ (which has zero $d\pi$-measure), and denoting by κ_h the curvature of $\partial(\pi \cap E_h)$, by (6.2) and (6.3) we have

$$\begin{aligned}
\liminf_{h\to+\infty} F(E_h) &\geq \frac{a}{2} \int_{\pi\setminus J} \liminf_{h\to+\infty} \int_{\partial(\pi\cap E_h)} [c_1 + c_2\kappa_h^2] \, d\mathcal{H}^1 \, d\pi \\
&\geq \frac{a}{2} \int_{\pi\setminus J} \int_{\partial(\pi\cap E)} [c_1 + c_2\kappa^2] \, d\mathcal{H}^1 \, d\pi = F(E).
\end{aligned}$$

If $b \geq 2a$ split F as

$$\begin{aligned}
F(E) = &\int_{\partial E} \Big[\frac{1}{2} + a\{f_1(\nabla^2 d_E) + 2(f_2(\nabla^2 d_E))^2\}\Big] \, d\mathcal{H}^2 \\
&+ \int_{\partial E} [\frac{1}{2} + (b - 2a)(f_2(\nabla^2 d_E))^2] \, d\mathcal{H}^2 =: F_1(E) + F_2(E).
\end{aligned}$$

Then F_1 is lower semicontinuous by the previuos case and F_2 is lower semicontinuous by Theorem 6.3 below. This concludes the proof. □

The next remark shows the connection between the second fundamental form of ∂E and $\nabla^2 d_E$, and is useful in the proof of Theorem 6.3.

Remark 6.2. If B^k_{ij}, $i, j, k \in \{1, \dots, N\}$, denotes the second fundamental form of ∂E with respect to the canonical basis of $\mathbb{R}^N$, then $B^k_{ij} = -\nabla^2_{ij} d_E \nabla_k d_E$ on ∂E. In particular $\sum\limits_{i,j,k=1}^{N} |B^k_{ij}|^2 = \sum\limits_{i,j=1}^{N} |\nabla^2_{ij} d_E|^2$ on ∂E.

Indeed, following [19, Proposition 5.1.1 (i)], if δ_{ij} is the Kronecker delta, summing over repeated indices from 1 to N and using the fact that $\nabla d_E \in \mathrm{Ker}(\nabla^2 d_E)$ on ∂E, we get, at $x \in \partial E$,

$$\begin{aligned} B_{ij}^k &= -(\delta_{lj} - \nabla_l d_E \nabla_j d_E)(\delta_{ih} - \nabla_i d_E \nabla_h d_E)(\nabla^2_{kh} d_E \nabla_l d_E + \nabla^2_{lh} d_E \nabla_k d_E) \\ &= -(\delta_{lj} - \nabla_l d_E \nabla_j d_E)(\nabla^2_{ki} d_E \nabla_l d_E + \nabla^2_{li} d_E \nabla_k d_E) = -\nabla^2_{ij} d_E \nabla_k d_E. \end{aligned}$$

Theorem 6.3. *Let $\alpha > 1$ and let $\zeta : [0, +\infty[\to [0, +\infty[$ be a strictly convex function such that $ct^\alpha \le \zeta(t)$ for any $t \in [0, +\infty[$, for some constant $c > 0$. Let $F : \mathcal{C}_b^\infty(\mathbb{R}^N) \to [0, +\infty]$ be the map defined by*

$$F(E) = \int_{\partial E} \left[1 + \zeta\Big(\Big[\sum_{i=1}^{N-1} \kappa_i^2\Big]^{1/2}\Big)\right] d\mathcal{H}^{N-1}, \tag{6.4}$$

where $\kappa_1, \dots, \kappa_{N-1}$ are the principal curvatures of ∂E. Then F is $L^1(\mathbb{R}^N)$-lower semicontinuous on $\mathcal{C}_b^\infty(\mathbb{R}^N)$.

Proof. Let $E \in \mathcal{C}_b^\infty(\mathbb{R}^N)$ and let $\{E_h\}$ be a sequence in $\mathcal{C}_b^\infty(\mathbb{R}^N)$ converging to E in $L^1(\mathbb{R}^N)$ as $h \to +\infty$. We must prove inequality (5.4). We can suppose, possibly passing to a subsequence, that the right hand side of (5.4) is a finite limit. Set

$$L(\partial E_h) := \int_{\partial E_h} \zeta\Big(\Big[\sum_{i=1}^{N-1} (\kappa_i^h)^2\Big]^{1/2}\Big)\, d\mathcal{H}^{N-1},$$

where $\kappa_1^h, \dots, \kappa_{N-1}^h$ are the principal curvatures of ∂E_h. We have

$$\sup_h \mathcal{H}^{N-1}(\partial E_h) < +\infty, \qquad \sup_h L(\partial E_h) < +\infty, \tag{6.5}$$

hence, if H_h denotes the scalar mean curvature of ∂E_h, by the Hölder inequality and the properties of ζ we deduce that

$$\int_{\partial E_h} |H_h|\, d\mathcal{H}^{N-1} \le O(1) \Big(\mathcal{H}^{N-1}(\partial E_h)\Big)^{1/\alpha'} [L(\partial E_h)]^{1/\alpha} = O(1), \tag{6.6}$$

where $\alpha^{-1} + (\alpha')^{-1} = 1$. We need now some tools of geometric measure theory and we refer to the book of Simon [28]. We associate to each ∂E_h the integer $(N-1)$-dimensional rectifiable varifold $(\partial E_h, 1_{\partial E_h})$, which can be seen as a measure V_h on $\mathbb{R}^N$ times the Grassmannian of the $(N-1)$-dimensional linear subspaces of $\mathbb{R}^N$. The weight λ_h of V_h is the measure defined as the restriction of $\mathcal{H}^{N-1}$ to ∂E_h. Using (6.6) and [28, Theorem 42.7 and Remark 42.8] there exists an integer $(N-1)$-dimensional rectifiable varifold $V = (M, \theta)$ with weight λ_V and a subsequence (still denoted by $\{V_h\}$) such that $V_h \rightharpoonup V$ in the sense of varifolds. We claim that

$$\partial E \subseteq M. \tag{6.7}$$

Indeed, let $x \in \partial E$ and let $B_r(x)$ be the open ball centered at x with radius $r > 0$. Since $V_h \rightharpoonup V$ we have $\lambda_h \rightharpoonup \lambda_V$ weakly as measures. Using [15, Section 1.9], for any $r > 0$ such that $\lambda_V(\partial B_r(x)) = 0$ (hence for all $r > 0$ up to a countable set) we have

$$\lambda_V(B_r(x)) = \lim_{h\to+\infty} \lambda_h(B_r(x)) = \lim_{h\to+\infty} \mathcal{H}^{N-1}(B_r(x) \cap \partial E_h) \geq \mathcal{H}^{N-1}(B_r(x) \cap \partial E),$$

where the last inequality follows by using the L^1-lower semicontinuity of the perimeter of a set relatively to $B_r(x)$ [15, Section 5.2.1]. Since $M = \mathrm{spt}(\lambda_V)$, we have $x \in M$, and this proves the claim.

Observe also that by (6.5) and [19, Theorem 5.3.2] V is a varifold with generalised second fundamental form in L^α in the sense of Hutchinson, and

$$\liminf_{h\to+\infty} L(\partial E_h) \geq L(V), \tag{6.8}$$

where

$$L(V) := \int_M \zeta([\sum_{i,j,k=1}^{N} |B_{ij}^k|^2]^{1/2})\theta \; d\mathcal{H}^{N-1},$$

and B_{ij}^k are defined in [19, Definition 5.2.5].

Recall now that the $L^1(\mathbb{R}^N)$-lower semicontinuity of the perimeter implies that

$$\liminf_{h\to+\infty} \mathcal{H}^{N-1}(\partial E_h) \geq \mathcal{H}^{N-1}(\partial E);$$

hence, in view of (6.8), to prove (5.4) it is enough to show that

$$L(V) \geq L(\partial E) := \int_{\partial E} \zeta([\sum_{i,j,k=1}^{N} |B_{ij}^k|^2]^{1/2}) \; d\mathcal{H}^{N-1}. \tag{6.9}$$

We firstly remark that M is a measurable countably $(\mathcal{H}^{N-1}, N-1)$-rectifiable set. Hence by (6.7) the tangent space $T_x^{\partial E}$ to ∂E at $x \in \partial E$ coincides with the approximate tangent space T_x^M to M at $x \in \partial E$ for $\mathcal{H}^{N-1}$-almost every $x \in \partial E$. Therefore the orthogonal projection $P_{\partial E}$ on $T_x^{\partial E}$ coincides with the orthogonal projection P_M on T_x^M for $\mathcal{H}^{N-1}$-almost every $x \in \partial E$. We recall now that each $B_{ij}^k(x)$, $x \in M$ (resp. $x \in \partial E$) can be obtained by tangentially differentiating in the λ_V-approximate sense (resp. in the classical sense) the function P_M (resp. $P_{\partial E}$), see [21, Theorem 5.4] and [19, Proposition 5.1.1 (i)]. Since $P_{\partial E}$ and P_M coincide $\mathcal{H}^{N-1}$-almost everywhere on $\partial E \cap \{\theta = n\}$, $n \in \mathbb{N}$, $n \geq 1$, they must have on $\partial E \cap \{\theta = n\}$ the same approximate tangential differential

$\mathcal{H}^{N-1}$-almost everywhere. Hence by (6.7) and Remark 6.2

$$
\begin{aligned}
L(V) &\geq \int_{\partial E} \zeta([\sum_{i,j,k=1}^{N} |B_{ij}^k|^2]^{1/2})\ \theta\ d\mathcal{H}^{N-1} \\
&\geq \sum_{n=1}^{\infty} \int_{\partial E \cap \{\theta = n\}} \zeta([\sum_{i,j,k=1}^{N} |B_{ij}^k|^2]^{1/2})\ d\mathcal{H}^{N-1} \\
&= \sum_{i=1}^{\infty} \int_{\partial E \cap \{\theta = n\}} \zeta([\sum_{i,j=1}^{N} |\nabla_{ij}^2 d_E|^2]^{1/2}) d\mathcal{H}^{N-1} = \int_{\partial E} \zeta([\sum_{i,j=1}^{N} |\nabla_{ij}^2 d_E|^2]^{1/2}) d\mathcal{H}^{N-1},
\end{aligned}
$$

which yields (6.9) and concludes the proof. □

Remark 6.4. In view of Corollary 4.4 we have the following approximation results. Assume that F is as in Theorem 5.2, (6.1) and (6.4) respectively. Then

$$
(\Gamma - \lim_{h \to +\infty} F_h)(u) = \begin{cases} 2c_0 F(u) & \text{if } u = 1_E,\ E \in \mathcal{C}_b^\infty(\mathbb{R}^N), \\ 2c_0 \overline{F}(u) & \text{elsewhere in } L^1(\mathbb{R}^N), \end{cases}
$$

where $N = 2$, $N = 3$, $N \geq 2$ respectively, F_h is defined in (4.3) with $m \equiv 1$, $\phi(x, \xi) = |\xi|$ and, concerning (6.4), $\widetilde{f} = \zeta((N-1)\widetilde{f}_2)$, where f_2 is defined in Example 3.4.

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