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Non-Uniform Integrability and Generalized Young Measures

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Given a bounded sequence (u_n) in $L^1(\Omega,\mu;\mathbb{R}^d)$, we describe the weak limits in the sense of measures of $f(x,u_n)\,\mu$ for a class of continuous integrands with linear growth at infinity. The defect of uniform integrability of the sequence $f(x,u_n)$ is described by a measure m and a family of probability measures on S^{d-1} whereas the classical Young measure is associated with the biting limits in the sense of Chacon's lemma. Some consequences of this new approach are given in Calculus of Variations.

1. Introduction

The oscillatory properties of a weakly convergent sequence (u_n) of functions in $L^1(\Omega,\mu,\mathbb{R}^d)$ can be very well described by the parametrized measure (or Young measure) it generates. This parametrized measure is a family of probabilities $(\nu_x)_{x\in\Omega}$ on $\mathbb{R}^d$ such that:

$$f(x,u_n(x)) \rightharpoonup \int_{\mathbb{R}^d} f(x,z)\nu_x(dz)$$

for every Caratheodory function $f:\Omega\times\mathbb{R}^d\to\mathbb{R}$ such that $f(x,u_n(x))$ is weakly convergent in L^1 (see [1,2,3]). In practice this representation formula is applied in case (u_n) is bounded in some L^p and f satisfies a suitable growth condition: $|f(x,z)|\le C(1+|z|^q)\quad(1\le q<p)$.

In many situations $f(x,u_n(x))$ represents a density of energy and is bounded in $L^1(\Omega)$ but non uniformly integrable (problems in Plasticity and fracture mechanics). A way to overcome this difficulty already used in Calculus of Variations was to replace the weak convergence in L^1 by the convergence in the biting sense introduced by Gaposhkin and Chacon (see for example [4]). In this case, one can construct also a parametrized measure denoted "biting Young measure" by Kinderlehrer and Pedregal [5] (see also [2]). In fact in this approach, we have no information on the singular part of the weak limit of energies $f(x,u_n(x))\,\mu$ in the sense of Radon measures. Note also that the regular part of this weak limit do not coincide in general with the biting limit (see Ball and Murat [3] or Example 3.2 in this paper). An important precursor to the present paper is the work by Di Perna and Majda [6] where an extension of the concept of Young measure was introduced in

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order to study concentrations effects for measure-valued solutions of incompressible fluid equations (see also the notion of defect measure introduced in [7]) .

Here we consider bounded sequences (u_n) in $L^1(\Omega, \mu, \mathbb{R}^d)$ which we see as bounded μ-absolutely continuous vector measures. Using a slicing argument for Radon measures constructed on the product $\Omega \times \bar{B}_d$ where $\bar{B}_d$ denotes the closed unit ball of $\mathbb{R}^d$ (seen as the compactification of $\mathbb{R}^d$), we obtain an integral representation formula for the limit λ of $f(x, u_n(x))\,\mu$ of the form:

$$\lambda = \left(\int_{\mathbb{R}^d} f(x,z)\nu_x(dz)\right)\mu \; + \left(\int_{S^{d-1}} f^\infty(x,z)\nu_x^\infty(dz)\right) m \; ,$$

where:

- ν_x is a probability measure on $\mathbb{R}^d$,
- m is a Radon measure on Ω,
- ν_x^∞ is a probability measure on $S^{d-1}$the unit sphere of $\mathbb{R}^d$,
- $f(x,z)$ belongs to a suitable class $\mathcal{F}(\Omega \times \mathbb{R}^d)$ of continuous functions such that $\dfrac{f(x,z)}{1+|z|}$ is bounded and uniformly continuous with respect to z and

$$f^\infty(x,z) := \lim_{z' \to z, |z'|<1} (1 - |z'|) f(x, \frac{z'}{1-|z'|}) = \lim_{t \to \infty} \frac{f(x,tz)}{t} \; .$$

The paper is organized as follows: in section 2, we fix the notations and state the main theorem (Theorem 2.5) and its corollary (Theorem 2.9) where we evaluate the biting limit of the sequence $f(x, u_n(x))$ in term of the probability measure ν_x and characterize the weak compactness property in L^1_{loc} in term of the measures m and ν_x^∞. Proofs are given in section 4. In section 3, we identify the probability measures ν_x and ν_x^∞ on some relevant examples exhibiting concentrations and oscillations effects. In section 5, we present some applications in Calculus of variations: we give in Theorem 5.1 an improved version of a lower semicontinuity result for convex functional on measures (see [8],[9],[10]) and in Theorem 5.3, we recover and precise a recent result of strong convergence in L^1 under strict convexity obtained by Brezis [11].

2. Notations and main results

In all the sequel Ω denotes either an open subset or a compact subset of $\mathbb{R}^N$, $|x|$ the Euclidian norm of $x \in \mathbb{R}^N$, $\mathcal{B}(\Omega)$ the σ -algebra of Borel subsets of Ω , 1_B the characteristic function of some Borel set B. We will denote by B_d the unit open ball of $\mathbb{R}^d$, by $\bar{B}_d$ its closure. S^{d-1} denotes the unit sphere of $\mathbb{R}^d$. Let us introduce also the following notations:

- $\mathcal{M}(\Omega, \mathbb{R}^d)$ is the space of all vector valued Borel measures $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}^d$ with finite variation on Ω. Recall the variation of λ on B is defined by:

$$|\lambda|(B) := \sup\{\sum_k |\lambda(B_k)| : \bigcup_k B_k \subset B, B_k \text{ pairwise disjoints}\}$$

- The space $\mathcal{M}(\Omega;\mathbb{R}^d)$ is naturally in duality with the space $\mathcal{C}_0(\Omega;\mathbb{R}^d)$ of all continuous functions of Ω to $\mathbb{R}^d$ vanishing at infinity and we will consider the associated weak* topology. The weak* convergence of a sequence (λ_n) in $\mathcal{M}(\Omega;\mathbb{R}^d)$ will be denoted $\lambda_n \stackrel{*}{\rightharpoonup} \lambda$ which means:

$$\int_\Omega \varphi(x)\cdot\frac{d\lambda_n}{d|\lambda_n|}(x)\,|\lambda_n|(dx) \to \int_\Omega \varphi(x)\cdot\frac{d\lambda}{d|\lambda|}(x)\,|\lambda|(dx) \text{ for every } \varphi\in\mathcal{C}_0(\Omega;\mathbb{R}^d) \quad (2.1)$$

When (2.1) holds for every φ in the space $\mathcal{C}_b(\Omega;\mathbb{R}^d)$ of bounded continuous functions of Ω to $\mathbb{R}^d$, we will say that the convergence of (λ_n) is *tight*. Recall that it is the case when the following condition is satisfied (automatic if Ω is compact):

$$\forall\varepsilon>0\ , \qquad \exists K \text{ compact} \subset\Omega : \qquad \sup_n |\lambda_n(\Omega\backslash K)|<\varepsilon \qquad (\textit{tightness})$$

- Given a Radon measure m on Ω and a topological space X, we denote by $\mathcal{P}(\Omega,m,X)$ the set of families $(\nu_x)_{x\in\Omega}$ of probability measures on X (Young family) which depend m-measurably on x. Sometimes the integral $\int_X f(z)\nu_x(dz)$ will be denoted simply $\langle\nu_x,f\rangle$ while in case X is imbedded in a linear space , $[\nu_x]$ will denote the barycenter of ν_x (corresponding to $f(z)=z$ when $\int|z|\nu_x(dz)$ is finite).
- For every function g on $\Omega\times\mathbb{R}^d$, we define the recession function g^∞ by:

$$g^\infty(x,z):=\liminf_{t\to\infty,z'\to z}\frac{g(x,tz')}{t}\ , \qquad (2.2)$$

which is clearly lower semicontinuous and positively 1-homogeneous with respect to z. The definition of the recession cone of a subset F of $\mathbb{R}^d$ is deduced by taking the indicator function of F:

$$F^\infty:=\{z\,\epsilon\,\mathbb{R}^d;\exists z_n\to z\ \exists\ t_n\to\infty\ \ t_nz_n\,\epsilon\,F\}\ . \qquad (2.3)$$

- $\mathcal{F}(\Omega\times\mathbb{R}^d)$ is defined as the class of continuous functions f on $\Omega\times\mathbb{R}^d$ such that the mapping: $(x,z)\in\Omega\times B_d\ \to\ (1-|z|)f(x,\frac{z}{1-|z|})$ can be extended into a bounded continuous function on $\Omega\times\bar{B}_d$. Denoting by $\tilde{f}$ this (unique) extension, we have:

$$\exists\Lambda>0\ :\qquad |f(x,z)|\le\Lambda(1+|z|)\qquad \text{for all } (x,z)\in\Omega\times\mathbb{R}^d\ , \qquad (2.4)$$

$$\tilde{f}(x,z):=\begin{cases}(1-|z|)f(x,\frac{z}{1-|z|}) & \text{if } |z|<1\ ,\\ f^\infty(x,z) & \text{if } |z|=1\ .\end{cases} \qquad (2.5)$$

Note that for such a function f , the liminf in the definition (2.2) is actually a limit (this by the continuity of $\tilde{f}(x,.)$ on $\bar{B}_d$). Moreover $\mathcal{F}(\Omega\times\mathbb{R}^d)$ endowed with the norm $\sup\{\frac{|f(x,z)|}{1+|z|};(x,z)\in\Omega\times\mathbb{R}^d\}$ is a Banach space.

Remark 2.1. In [6], Di Perna and Majda consider also a particular class of admissible integrands f which satisfy a growth condition of order p. They give a representation formula for the weak $*$ limit of $f(x, u_n(x))\,\mu$ when (u_n) is bounded in $L^p(\Omega, \mu; \mathbb{R}^d)$. This formula has a concrete form when $f(x, .)$ is positively homogeneous of order p or when $\frac{f(x,z)}{1+|z|^p}$ vanishes as $|z| \to \infty$. Let us notice that the latter cases are recovered from our results (related to the case p = 1) by setting: $v_n := |u_n|^{p-1} u_n$ and $g(x, z) := f(x, |z|^{\frac{1-p}{p}} z)$. Indeed we can write $f(x, u_n(x)) = g(x, v_n(x))$ where the new sequence (v_n) is bounded in $L^1(\Omega, \mu; \mathbb{R}^d)$ and g belongs to $\mathcal{F}(\Omega \times \mathbb{R}^d)$.

Some properties related to the class $\mathcal{F}(\Omega \times \mathbb{R}^d)$ are collected in the two lemmas below (proved in the appendix).

Lemma 2.2. *Let f be continuous on $\Omega \times \mathbb{R}^d$ and satisfy* (2.4). *Then f belongs to $\mathcal{F}(\Omega \times \mathbb{R}^d)$ in the following cases:*

(i) *f is bounded (in that case $f^\infty \equiv 0$),*

(ii) *$f(x, \lambda z) = \lambda f(x, z)$ for every $\lambda \geq 0$ and $(x, z) \in \Omega \times \mathbb{R}^d$ (in that case $f^\infty = f$),*

(iii) *f is convex (or concave) with respect to z and f^∞ is continuous (for example f independent of x).*

Lemma 2.3. *Let f be lower semicontinuous on $\Omega \times \mathbb{R}^d$ such that for some suitable $\Lambda > 0$: $f(x, z) \geq -\Lambda\,(1 + |z|)$ for every $(x, z) \in \Omega \times \mathbb{R}^d$. Define:*

$$h_f(x, z) \;:=\; \liminf\{\, \frac{f(x', tz')}{t}\;;\; t \to \infty,\; x' \to x,\; z' \to z\} \tag{2.6}$$

Then there exists a non decreasing sequence (f_p) in $\mathcal{F}(\Omega \times \mathbb{R}^d)$ such that:

$f_p(x, z) \geq -\Lambda\,(1 + |z|) \quad , \quad sup f_p = f$

Moreover for any such a sequence (f_p), we have: $sup f_p^\infty(x, z) = h_f(x, z)$.

Remark 2.4. The integrand h_f defined by (2.6) looks as a l.s.c. regularization of f^∞ with respect to x and $h_f(x, .) = f^\infty(x, .)$ for every x when f has a good behaviour (for example $f \in \mathcal{F}(\Omega \times \mathbb{R}^d)$). In particular it is the case when f satisfies the assumptions of the lemma, is convex with respect to z and f(.,0) is locally bounded (see the proof in Appendix).

In all the sequel, μ is a given non negative Radon measure on Ω. We are in the position to state our main result:

Theorem 2.5. *Let (u_n) be a bounded sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$ and F a closed subset of $\mathbb{R}^d$ such that $u_n(x) \in F$ μ a.e.. Then there exists a subsequence (u_{n_k}), a nonnegative Radon measure m and parametrized families of probabilities $(\nu_x) \in \mathcal{P}(\Omega, \mu; \mathbb{R}^d)$, $(\nu_x^\infty) \in \mathcal{P}(\Omega, m; S^{d-1})$ such that:*

$$f(x, u_{n_k}(x))\,\mu \;\stackrel{*}{\rightharpoonup}\; \langle \nu_x, f(x, .)\rangle\,\mu \;+\; \langle \nu_x^\infty, f^\infty(x, .)\rangle\, m$$

for every f in $\mathcal{F}(\Omega \times \mathbb{R}^d)$.

Moreover the following properties hold:

(i) $\int_\Omega \langle \nu_x, |z| \rangle d\mu < +\infty$ *and* $\operatorname{supp} \nu_x \subset F$ *for* μ *a.e.* $x \in \Omega$.

(ii) $\operatorname{supp} m \subset \operatorname{supp} \mu$ *and* $\operatorname{supp} \nu_x^\infty \subset F^\infty \cap S^{d-1}$ *for* m *a.e.* $x \in \Omega$.

(iii) *For every* f *satisfying the conditions of Lemma* 2.3*:*

$$\liminf_{k\to\infty} \int_\Omega f(x, u_{n_k})\ d\mu\ \geq \int_\Omega \langle \nu_x, f(x,.) \rangle\ d\mu\ +\ \int_\Omega \langle \nu_x^\infty, h_f(x,.) \rangle\ dm$$

where h_f *is defined as in* (2.6).

Remark 2.6. a) In fact, it is easy to check that the Theorem 2.5 can be reformulated assuming only that the initial sequence (u_n) is bounded in $L^1_{loc}(\Omega, \mu; \mathbb{R}^d)$. In that case the weak $*$ convergence has to be understood in the sense that (2.1) holds only for those functions φ in $\mathcal{C}_0(\Omega; \mathbb{R}^d)$ which are compactly supported in Ω.
b) Using the argument in Remark 2.10 below, it is possible to prove that the assertion (i) of the theorem still holds if we replace F by any μ- measurable closed valued multifunction F(x).

Now we want to discuss the connections of Theorem 2.5 with the weak convergence in L^1 and also with the Chacon biting's lemma [3]. Let us recall briefly the main features of the latter convergence which has been used in many situations where weak compactness in L^1 fails. Recall that, by the Dunford-Pettis theorem, a bounded sequence (u_n) in $L^1_\mu(\Omega)$ ($\mu(\Omega)$ being finite) is weakly relatively compact if and only if it is uniformly μ-integrable, that is:

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|u_n|\geq R} |u_n| d\mu = 0.$$

Lemma 2.7. **(Chacon)** *Assume that* $\mu(\Omega)$ *is finite and let* (u_n) *be a bounded sequence in* $L^1(\Omega, \mu; \mathbb{R}^d)$*. Then there exist a function* u *, a subsequence* (u_{n_k}) *and a non-increasing sequence of* μ*-measurable subsets* E_p *with* $\mu(E_p) \to 0$ *such that for every* p*,* u_{n_k} *converges weakly to* u *in* $L^1(\Omega \setminus E_p, \mu; \mathbb{R}^d)$*. We will say that* u *is the biting limit of* (u_{n_k})*, which we denote* $u_{n_k} \overset{b}{\to} u$*.*

Remark 2.8. If $\mu(\Omega) = \infty$, the same result holds with weak convergence in $L^1_{loc}(\Omega \setminus E_p, \mu; \mathbb{R}^d)$. We will use the same notation ($u_{n_k} \overset{b}{\to} u$) in that case.

As a corollary of Theorem 2.5, we obtain:

Theorem 2.9. *Let* $(u_{n_k}), \nu_x, \nu_x^\infty$ *and* m *be as in* Theorem 2.5*. Then:*

(i) $f(., u_{n_k}) \overset{b}{\to} \langle \nu_x, f \rangle$ *for every Caratheodory function* f *which satisfies* (2.4).

(ii) *If moreover* f *belongs to* $\mathcal{F}(\Omega \times \mathbb{R}^d)$*, then* $f(., u_{n_k})$ *is weakly convergent in* $L^1_{loc}(\Omega, \mu)$ *if and only if* $\langle \nu_x^\infty, |f^\infty| \rangle = 0$ m *a.e on* Ω*. In particular,* (u_{n_k}) *is weakly convergent in* $L^1_{loc}(\Omega, \mu; \mathbb{R}^d)$ *if and only if the measure* m *vanishes.*

Remark 2.10. a) The representation of the biting limit of $f(.,(u_{n_k}))$ in the form (i) has been already given in [5] (see also [3]) and the associated probability measure ν_x can be seen as a "biting Young measure". By taking $f(x,z) := \dfrac{d(x,z)}{1+d(x,z)}$ where d(x,z) is the distance of z to some closed valued μ-measurable multifunction F(x), we recover that the probability measure ν_x is μ a.e. supported by F(x) whenever $dist(u_{n_k}(x), F(x)) \to 0 \quad \mu$ *a.e.* (see Remark 2.6). The particular case $F(x) = \{u(x)\}$ leads to the well-known characterization of μ-convergence in measure:

$$u_{n_k} \to u \quad \mu\text{-locally in measure} \quad \Longleftrightarrow \quad \nu_x = \delta_{u(x)} \ \mu \text{ a.e.} \tag{2.7}$$

b) The condition $m = 0$ characterizes the weak compactness in L^1_μ of the sequence (u_n) if moreover $(u_n\, \mu)$ is tight.

The next statement gives a geometrical criterium for weak convergence in $L^1(\Omega, \mu; \mathbb{R}^d)$ which has been already given by S. Müller [12] in the special case $d = 1$, μ = Lebesgue measure and $F = [0, \infty]$.

Corollary 2.11. *Let F be a closed line free convex subset of $\mathbb{R}^d$ and $u_n, u \in L^1(\Omega, \mu; \mathbb{R}^d)$ such that:* $u_n \in F \quad \mu$ a.e. , $u_n\, \mu \overset{*}{\rightharpoonup} u\, \mu$, $u_n \overset{b}{\to} u$.
Then u_n converges weakly to u in $L^1_{loc}(\Omega, \mu; \mathbb{R}^d)$.

Proof. By Theorem 2.5 and Theorem 2.9, one has: $u(x) = [\nu_x]\ \mu$ a.e. , $\operatorname{supp} \nu_x^\infty \subset F^\infty \cap S^{d-1}$, $[\nu_x^\infty] = 0 \ m$ a.e. . Since F is closed and convex, we have that, for every $z_0 \in F$, $F^\infty = \{z \in \mathbb{R}^d; z_0 + tz \in F \ , \forall t \geq 0\}$. Hence F^∞ is also line free and 0 is an extreme point F^∞. Then the condition $[\nu_x^\infty] = 0$ implies that the probability measure ν_x^∞ (supported by F^∞) is the Dirac mass at 0. Of course, as ν_x^∞ is also supported by S^{d-1}, this is impossible unless $m = 0$. We conclude by using the last part in assertion (ii) of Theorem 2.9. □

3. Examples

To identify the measures m, ν_x and ν_x^∞ given by Theorem 2.5, it is enough to test the weak $*$ convergence of $f(x, u_n(x))\, \mu$ for integrands of the class $\mathcal{F}(\Omega \times \mathbb{R}^d)$ which do not depend on x. Recall that for such an integrand f, we have (up to a subsequence):

$$f(u_n(x))\ \mu \quad \overset{*}{\rightharpoonup} \quad \langle \nu_x, f \rangle\ \mu \ + \ \langle \nu_x^\infty, f^\infty \rangle\ m \tag{3.1}$$

To identify ν_x, we take f bounded so that $(f(u_n))$ is weakly convergent in L^1_{loc} and (3.1) becomes:

$$f(u_n(x)) \quad \rightharpoonup \quad \langle \nu_x, f \rangle \quad \text{weakly in } L^1_{loc}(\Omega, \mu) \quad \text{for every } f \in \mathcal{C}_b(\mathbb{R}^d) \tag{3.2}$$

In a second step, we may identify m and ν_x^∞ by substituting $f(z) = |z|\varphi(\frac{z}{|z|})$ in (3.1) where φ is any continuous function on S^{d-1}:

$$|u_n|\varphi(\frac{u_n}{|u_n|})\ \mu \quad \overset{*}{\rightharpoonup} \quad \langle \nu_x, |z|\varphi(\frac{z}{|z|}) \rangle\ \mu \ + \langle \nu_x^\infty, \varphi \rangle\ m \tag{3.3}$$

In all examples below, μ is the Lebesgue measure on $\Omega =]-1,1[$, (u_n) is a bounded sequence in $L^1(\Omega)$ which do not satisfy the uniform integrability property.

Example 3.1. Let $u_n(x) := -n\ 1_{]-\frac{1}{n},0[} + n\ 1_{]0,\frac{1}{n}[}$. This sequence is bounded in L^1 and converges $*$-weakly and μ a.e. to 0. However a concentration effect appears at 0 since in that case we have:

$$\nu_x = \delta_0 \quad , \quad m = 2\delta_0 \quad , \quad \nu_x^\infty = \frac{1}{2}\,\delta_1 + \frac{1}{2}\,\delta_{-1}$$

(indeed $u_n^+ \overset{*}{\rightharpoonup} \delta_0$ and $u_n^- \overset{*}{\rightharpoonup} \delta_0$)

Example 3.2. Here we take the sequence already considered in [3]:

$$u_n = \sum_{k=0}^{n-1} n\ 1_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]}$$

(u_n) converges to 0 in measure and in the biting sense in L^1. Nevertheless as pointed out in [3] , $(u_n\mu)$ converges $*$-weakly to $1_{[0,1]}\,\mu$ (where μ is the Lebesgue measure on Ω). In this case the lack of uniform integrability of the sequence is diffused on the whole interval $[0,1]$ since we find:

$$\nu_x = \delta_0 \qquad m = 1_{[0,1]}\ \mu \qquad \nu_x^\infty = \delta_1$$

Proof. The convergence of u_n to 0 in measure implies that for every $f \in \mathcal{C}_b(\Omega)$, one has $f(u_n) \rightharpoonup f(0)$ in L^1_μ. Hence $\nu_x = \delta_0$. Here $S^{d-1} = \{-1,1\}$ and as $u_n \geq 0$, ν_x^∞ is a Dirac mass at $z = 1$ and m is by (3.3) the weak*-limit of $|u_n|\mu$ (take $\varphi(z) := |z|$). An easy computation gives $m = 1_{[0,1]}\,\mu$. $\square$

Example 3.3. Let: $u_n(x) = \sum_{k=0}^{n-1} n\ 1_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]} \quad (cos\ 2\pi n^2 x, sin\ 2\pi n^2 x)$

As in Example 3.2, the singularities of u_n are uniformly distributed on $[0,1]$, but we get that ν_x^∞ is uniform on S^1:

$$\nu_x = \delta_0 \qquad m = 1_{[0,1]}\ \mu \qquad \nu_x^\infty = \frac{1}{2\pi}\mathcal{H}^1 \llcorner S^1$$

Proof. For ν_x and m, the computation is the same as above. To determine ν_x^∞, we parametrize S^1 by $\theta \in [0,2\pi]$ and apply (3.3) with a function $\varphi(\theta)$ where φ is continuous 2π-periodic. We get:

$$\Big[\sum_{k=0}^{n-1} n\ 1_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]} \quad \varphi(2\pi n^2 x)\Big]\ \mu \quad \overset{*}{\rightharpoonup} \quad \langle \nu_x^\infty, \varphi\rangle\ m$$

Applying this convergence with a continuous test function Φ and using the mean value theorem for Riemann integrals, it follows:

$$\begin{aligned}\int_\Omega \langle \nu_x^\infty, \varphi\rangle \Phi(x) m(dx) &= \lim_{n\to\infty} \sum_{k=0}^{n-1} n \int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} \Phi(x)\varphi(2\pi n^2 x)dx \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \Phi(\frac{k}{n}+\frac{\theta_{k,n}}{n^2}) n^2 \int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} \varphi(2\pi n^2 x)dx \quad (\theta_{k,n}\in[0,1]) \\ &= \int_0^1 \Phi(t)dt \int_0^1 \varphi(2\pi u)du\ .\end{aligned}$$

Hence m is the Lebesgue measure on $[0,1]$ and ν_x^∞ is uniform on the torus. □

4. Proofs

To prove Theorem 2.5, we need first:

Proposition 4.1. *Let (u_n) be a bounded sequence in $L^1(\Omega,\mu;\mathbb{R}^d)$ and F a closed subset of $\mathbb{R}^d$ such that $u_n(x)\,\epsilon\, F \quad \mu$ a.e.. Then there exist a subsequence (u_{n_k}), a nonnegative Radon measure $\tilde m$ on Ω and a parametrized family $(\tilde\nu_x)_{x\in\Omega}$ in $\mathcal{P}(\Omega,\tilde m;\bar B_d)$ such that:*

$$(f(x,u_{n_k}(x))\ \mu \quad \overset{*}{\rightharpoonup} \quad (\int_{\bar B_d} \tilde f(x,z)\tilde\nu_x(dz)) \quad \tilde m \tag{4.1}$$

for every f in $\mathcal{F}(\Omega\times\mathbb{R}^d)$.
Moreover, let $\tilde m = \tilde p\mu + \tilde m_s$ with $\mu \perp \tilde m_s$ be the Lebesgue-Nikodym decomposition of $\tilde m$ with respect to μ and let $\tilde F$ be the closed subset of $\bar B_d$ defined by $\tilde F = cl\{\frac{z}{1+|z|}; z\,\epsilon\, F\}$.
Then the following properties hold:

(i) $\operatorname{supp}\tilde m \subset \operatorname{supp}\mu$ *and* $\operatorname{supp}\tilde\nu_x \subset \tilde F$ *for $\tilde m$ a.e. $x\in\Omega$*
(ii) $\tilde p(x) = (\int_{B_d}(1-|z|)\tilde\nu_x(dz))^{-1} \geq 1$ *for μ a.e. $x\in\Omega$*
(iii) $\tilde\nu_x(S^{d-1}) = 1$ *for $\tilde m_s$ a.e. $x\in\Omega$.*

Proof. By (2.4), for every $f\in\mathcal{F}$, the sequence $f(x,u_n)\,\mu$ is bounded hence $*$-weakly relatively compact in $\mathcal{M}(\Omega)$. Therefore we need only to check (4.1) on relatively compact subsets of Ω so that, without any loss of generality, we may assume that $\mu(\Omega)<+\infty$.
Then we define a bounded sequence of Radon measures on $\Omega\times\bar B_d$ by setting for every $\varphi\in\mathcal{C}_0(\Omega\times\bar B_d)$:

$$\langle L_n,\varphi\rangle = \int_{\Omega\times\bar B_d}\varphi dL_n := \int_\Omega \varphi(x,\frac{u_n(x)}{1+|u_n(x)|})(1+|u_n(x)|)d\mu$$

(indeed $|\langle L_n,\varphi\rangle| \leq |\varphi|_\infty(\mu(\Omega)+sup\ \|u_n\|_{L^1})$. Hence there exists a subsequence (L_{n_k}) and a bounded Radon measure L such that for every $\varphi\in\mathcal{C}_0(\Omega\times\bar B_d)$:

$$\lim_{k\to\infty}\langle L_{n_k},\varphi\rangle = \langle L,\varphi\rangle = \int_{\Omega\times\bar B_d}\varphi(x,z)L(dxdz) \tag{4.2}$$

Clearly by taking φ depending only on x in (4.2), we identify $\tilde{m}$ as the canonical projection of L on Ω that is $\tilde{m}(E) = L(E \times \bar{B}_d)$ for every Borel subset E of Ω. Now by a classical slicing argument for measures (often referred as a disintegration theorem, see for example [13]), there exists a family of probability measures in $\mathcal{P}(\Omega, \tilde{m}; \bar{B}_d)$ such that:

$$\langle L, \varphi \rangle = \int_\Omega (\int_{\bar{B}_d} \varphi(x,z)\, \tilde{\nu}_x(dz))\ \tilde{m}(dx)$$

Let f belong to the class $\mathcal{F}(\Omega \times \mathbb{R}^d)$ and take for φ in (4.2) the function defined by $\varphi(x,z) = \Phi(x)\tilde{f}(x,z)$ where Φ runs over $\mathcal{C}_0(\Omega)$. We are led to:

$$\int_\Omega f(x, u_{n_k})\Phi(x)\, d\mu \quad \to \quad \int_\Omega \langle \tilde{\nu}_x, \tilde{f} \rangle \Phi(x)\, d\tilde{m}$$

which is equivalent to (4.1). Moreover by taking $f(z) := 1 + |z|$ in (4.1) we find that $\tilde{m}$ is the weak* limit of the sequence $(1 + |u_{n_k}|)\, \mu$. Hence $\tilde{m} \geq \mu$ and $\operatorname{supp} \tilde{m} \subset \operatorname{supp} \mu$.
To prove (i), we choose a sequence of continuous functions $g_p : \bar{B}_d \to [0,1]$ such that $g_p = 1$ on $\tilde{F}$ and $g_p \to 0$ on $\bar{B}_d \setminus \tilde{F}$. Then applying (4.1) with $f(z) := (1 + |z|) g_p(\frac{z}{1+|z|})$, we obtain that $f(u_{n_k})\mu$ converges $*$-weakly to $\langle \tilde{\nu}_x, g_p \rangle \tilde{m}$. As $f(u_{n_k}) = (1 + |u_{n_k}|)\ \mu$ a.e., this limit coincides with $\tilde{m}$ which yields: $\langle \tilde{\nu}_x, g_p \rangle = 1\ \tilde{m}$ *a.e.*. By letting p tend to ∞, we obtain $\tilde{\nu}_x(\tilde{F}) = 1\ \tilde{m}$ *a.e.* which proves the assertion (i). Let now apply (4.1) with $f(x,z) := 1$ that is $\tilde{f}(x,z) = 1 - |z|$. We get: $\mu = \langle \tilde{\nu}_x, \tilde{f} \rangle \tilde{m}$ which according to the Lebesgue-Nikodym decomposition of $\tilde{m}$ splits into:

$$\langle \tilde{\nu}_x, 1 - |z|) \rangle\ \tilde{p}(x) = 1 \quad \mu \ a.e. \tag{4.3}$$

$$\langle \tilde{\nu}_x, 1 - |z|) \rangle = 0 \qquad \tilde{m}_s \ a.e. \tag{4.4}$$

Since $1 - |z| \leq 1$ on $\bar{B}_d$, (4.3) yields (ii) and (4.4) that for $\tilde{m}_s$ a.e. on Ω, $\tilde{\nu}_x$ is concentrated on $\{|z| = 1\} = S^{d-1}$ that is (iii). The proof of Proposition 4.1 is achieved. □

Proof of Theorem 2.5. Let $\tilde{m}$, $\tilde{p}$, $\tilde{\nu}_x$ defined as in Proposition 4.1. For μ a.e. $x \in \Omega$, we define a probability measure on $\mathbb{R}^d$ by setting for every bounded continuous φ on $\mathbb{R}^d$:

$$\langle \nu_x, \varphi \rangle := \tilde{p}(x) \int_{\{|z|<1\}} (1 - |z|)\varphi(\frac{z}{1 - |z|})\tilde{\nu}_x(dz) = \tilde{p}(x) \int_B \tilde{\varphi}(z)\tilde{\nu}_x(dz) \tag{4.5}$$

(indeed for $\varphi \equiv 1$, we get from Proposition 4.1 (ii): $\langle \nu_x, 1 \rangle = \tilde{p}(x)\langle \tilde{\nu}_x, 1 - |z| \rangle = 1\ \mu$ a.e.). We define also the bounded Radon measure $m := \tilde{\nu}_x(S^{d-1})\, \tilde{m}$. We have $\operatorname{supp} m \subset \operatorname{supp} \tilde{m} \subset \operatorname{supp} \mu$ (see Proposition 4.1 (i)). Moreover the Lebesgue-Nikodym decomposition of m with respect to μ reads as:

$$m = \tilde{p}(x)\tilde{\nu}_x(S^{d-1})\mu + \tilde{m}_s\ . \tag{4.6}$$

Now we define for m a.e. $x \in \Omega$ a probability measure on S^{d-1} by setting:

$$\langle \nu_x^\infty, \varphi \rangle := \frac{1}{\tilde{\nu}_x(S^{d-1})} \int_{S^{d-1}} \varphi(z)\tilde{\nu}_x(dz)\ . \tag{4.7}$$

Using (4.5)(4.6)(4.7) and noticing that $f^\infty(x,.) = \tilde{f}(x,.)$ on S^{d-1} for every f in $\mathcal{F}(\Omega\times\mathbb{R}^d)$ and that $\tilde{\nu}_x(S^{d-1}) = 1$ $\tilde{m}_s$ a.e. (see assertion (iii) of Proposition 4.1), one gets:

$$\begin{aligned}
&\langle \nu_x, f\rangle\mu + \langle \nu_x^\infty, f^\infty\rangle m \\
&\quad = [\langle \nu_x, f\rangle + \tilde{p}(x)\tilde{\nu}_x(S^{d-1})\langle \nu_x^\infty, f^\infty\rangle]\ \mu \ + \ \langle \nu_x^\infty, f^\infty\rangle \tilde{m}_s \\
&\quad = \left(\int_{B_d} \tilde{f}(x,z)\tilde{p}(x)\tilde{\nu}_x(dz) + \int_{S^{d-1}} \tilde{f}(x,z)\tilde{p}(x)\tilde{\nu}_x(dz)\right)\mu \\
&\qquad\qquad + \left(\frac{1}{\tilde{\nu}_x(S^{d-1})}\int_{\bar{B}_d} \tilde{f}(x,z)\tilde{\nu}_x(dz))\right)\tilde{m}_s \\
&\quad = \int_{\bar{B}_d} \tilde{f}(x,z)\tilde{\nu}_x(dz)\ \ [\tilde{p}(x)\mu + \tilde{m}_s] \\
&\quad = \ \langle \tilde{\nu}_x, \tilde{f}\rangle \tilde{m}\ .
\end{aligned}$$

Hence (4.1) can be reformulated as:

$$(f(x, u_{n_k}(x))\ \mu \quad \stackrel{*}{\rightharpoonup} \quad \langle \nu_x, f\rangle\mu + \langle \nu_x^\infty, f^\infty\rangle m \tag{4.8}$$

for every f in $\mathcal{F}(\Omega\times\mathbb{R}^d)$. The first statement of (i) is easily obtained by applying (4.8) to $f(x,z) := |z|$. By the boundedness of (u_n) in L^1, we have $\int_\Omega \langle \nu_x, |z|\rangle d\mu < +\infty$. To prove the second statement of (i), we consider a bounded continuous function φ on $\mathbb{R}^d$ such that $\varphi = 0$ on the closed subset F . We have to show that $\langle \nu_x, \varphi\rangle = 0$ μ a.e. on Ω. For such a function φ, the associated $\tilde{\varphi}$ (see (2.5) and Lemma 2.2 (i)) vanishes on $\tilde{F}\cup S^{d-1}$. By assertion (i) of Proposition 4.1, $\tilde{\nu}_x$ is concentrated on $\tilde{F}$ $\tilde{m}$ a.e.(hence μ a.e. since $\tilde{m} \geq \mu$) so that using (4.5), we can write:

$$\langle \nu_x, \varphi\rangle = \tilde{p}(x)\ \int_{\tilde{F}\cap B} \tilde{\varphi}(z)\tilde{\nu}_x(dz) = 0 \qquad \mu \text{ a.e. } x\in\Omega.$$

Similarly let φ be a continuous function on S^{d-1} satisfying $\varphi = 0$ on $F^\infty \cap S^{d-1} (= \tilde{F}\cap S^{d-1})$. From (4.7), we have:

$$\langle \nu_x^\infty, \varphi\rangle := \frac{1}{\tilde{\nu}_x(S^{d-1})}\int_{S^{d-1}\cap\tilde{F}} \varphi(z)\tilde{\nu}_x(dz) = 0 \qquad m \text{ a.e. } x\in\Omega,$$

wich proves that ν_x^∞ is concentrated on F^∞ that is (ii). Let us now prove (iii): we consider the sequence (f_p) in $\mathcal{F}$ given by Lemma 2.3. Apply the first statement of our theorem to f_p yields for every p:

$$\liminf_{k\to\infty}\int_\Omega f(x,u_{n_k})d\mu \geq \int_\Omega \langle \nu_x, f_p(x,.)\rangle\ d\mu \ + \ \int_\Omega \langle \nu_x^\infty, f_p^\infty(x,.)\rangle\ dm \tag{4.9}$$

Set: $g_p(x) := \langle \nu_x, f_p(x,.)\rangle$, $h_p(x) := \langle \nu_x^\infty, f_p^\infty(x,.)\rangle$. As g_p and h_p are non decreasing and have a uniform lower bound in L^1_μ (resp. in L^1_m): $g_p(x) \geq -\Lambda(1 + \langle \nu_x, |z|\rangle$ (resp.

$h_p(x) \geq -\Lambda$), we conclude by letting p tend to ∞ in (4.9) using monotone convergence theorem. The proof of Theorem 2.5 is finished. □

Proof of Theorem 2.9. By a trivial localization argument, we can assume that the sequence $(u_n\mu)$ is tight and that $\mu(\Omega)$ is finite (see the Remark 2.8). As (v_k) is bounded in L^1_μ where $v_k(x) := f(x, u_{n_k}(x))$, by Lemma 2.7 we can assume, possibly passing to a subsequence, that it converges in the biting sense to some limit $l(x)$. If we prove that $l(x) \equiv \langle \nu_x, f\rangle$, then by unicity of the limit, the whole sequence $f(x, u_{n_k}(x))$ will satisfy the assertion (i).

First step: we assume that (v_k) is uniformly integrable so that it converges weakly to $l(x)$ in L^1_μ. Assume first that f is bounded and let us show that $l(x) = \langle \nu_x, f(x,.)\rangle$ μ a.e. on every compact subset K of Ω. By the theorem of Scorzá-Dragoni (see for example [14]), for every $\varepsilon > 0$, there exists a compact $K_\varepsilon \subset K$ and a continuous g_ε on $\Omega \times \mathbb{R}^d$ such that:

$$\mu(K \setminus K_\varepsilon) < \varepsilon \qquad g_\varepsilon \equiv f \text{ on } K_\varepsilon \times \mathbb{R}^d . \tag{4.10}$$

By truncating, we can assume that g_ε is bounded so that g_ε belongs to $\mathcal{F}(\Omega \times \mathbb{R}^d)$. The weak limit of $g_\varepsilon(., u_{n_k})$ given by Theorem 2.5 is μ-absolutely continuous and coincides with l on K_ε, hence by (4.10):

$$l(x) = \langle \nu_x, g_\varepsilon(x,.)\rangle = \langle \nu_x, f(x,.)\rangle \quad \mu \text{ a.e. on } K_\varepsilon.$$

The equality μ a.e. on Ω follows by letting ε tend to 0. To relax the boundedness assumption on f, we approximate f by the sequence (f_p) defined by $f_p(x,z) := sup\{-p, inf\{f(x,z), p\}\}$ and obtain as above the weak convergence of the sequence $v_k^{(p)} := f_p(., u_{n_k})$. More precisely:

$$v_k^{(p)} \rightharpoonup l_p \text{ weakly in } L^1_\mu \text{ as } k \to \infty \quad , \qquad l_p(x) = \langle \nu_x, f_p(x,.)\rangle. \tag{4.11}$$

By the uniform integrability of (v_k), by (4.11) and the weak lower semicontinuity of the L^1-norm, we are led to:

$$\limsup_{p\to\infty} ||l_p - l||_{L^1} \leq \limsup_{p\to\infty} \liminf_{k\to\infty} ||v_k^{(p)} - v_k||_{L^1} = 0$$

which proves that l is the strong limit of l_p in L^1_μ . By the growth condition (2.5) and the assertion (i) of Theorem 2.5, we can apply dominated convergence theorem to the integral $\int f_p(x,z)\nu_x(dz)$ to see that l_p converges also μ a.e. to $l = \int f(x,z)\nu_x(dz)$.

Second step: Now we consider the general case. By Lemma 2.7, there exists a nonincreasing sequence (E_p) such that $\mu(E_p) \to 0$ and v_k converges weakly to l in $L^1(\Omega \backslash E_p, \mu; \mathbb{R}^d)$. For every p, we set $A_p = \Omega \backslash E_p$ and consider the Caratheodory function $g_p(x,z) := 1_{A_p}(x) f(x,z)$ which satisfies (2.4). Noticing that $v_k 1_{A_p} := g_p(., u_{n_k})$, we can apply step 1 so that: $v_k 1_{A_p} \rightharpoonup \langle \nu_x, g_p(x,.)\rangle$ weakly in L^1_μ. Then by the unicity of the weak limit of (v_k) on every set A_p, we get $l(x) = \langle \nu_x, f(x,.)\rangle$ μ a.e. on A_p. The conclusion follows by letting p tend to ∞.

Let us now prove (ii): by the Dunford-Pettis theorem, the weak compactness in L^1_μ of $v_k := f(x, u_{n_k})$ holds if and only if $\eta(R) := \limsup_{k\to\infty} \int_{|v_k|\geq R} |v_k| d\mu$ tends to 0 as $R \to \infty$. Let us define:

$$f_R(x,z) := \; R\, h\left(\frac{|f(x,z)|}{R}\right)$$

where

$$h(t) = 0 \text{ if } 0 \leq t < \frac{1}{2}\ , \ h(t) = 2t-1 \text{ if } \frac{1}{2} \leq t \leq 1\ , \ h(t) = t \text{ if } t > 1\ .$$

It is easy to check that f_R belongs to the class $\mathcal{F}(\Omega \times \mathbb{R}^d)$ and that:

$$f_{2R}(.,u_{nk}) \leq |v_k| 1_{\{|v_k|>R\}} \leq f_R(.,u_{n_k}) \quad \text{and} \quad f_R^\infty = f_{2R}^\infty = |f|^\infty\ .$$

Passing to the limit as $k \to \infty$ and using Theorem 2.5, we get for every R:

$$\begin{aligned} &\int_\Omega \langle \nu_x, f_{2R}\rangle d\mu + \int_\Omega \langle \nu_x^\infty, |f^\infty|\rangle dm \\ &\qquad \leq\ \eta(R)\ \leq \int_\Omega \langle \nu_x, f_R\rangle d\mu + \int_\Omega \langle \nu_x^\infty, |f^\infty|\rangle dm \end{aligned} \tag{4.12}$$

Then letting R tend to ∞ in (4.12) and using dominated convergence theorem:

$$\eta_\infty \ :=\ \lim_{R\to\infty}\ \eta(R)\ =\ \int_\Omega \langle \nu_x^\infty, |f^\infty|\rangle dm \tag{4.13}$$

The first equivalence in the assertion (ii) of Theorem 2.9 follows. In the particular case $f(x,z) = |z|$, (4.13) reduces to $\eta_\infty = m(\Omega)$. Hence the weak compactness of (u_{n_k}) in L^1_μ is equivalent to $m = 0$. □

Remark 4.2. The real η_∞ in (4.13) is the modulus of uniform integrability associated with the sequence (v_k) intoduced by H.P.Rosenthal. Note also that in the proof of the assertion (i), the measurable dependence in x is overcomed as in [14] by a Lusin's type argument which is very far from the technique used in [1,2,3].

5. Applications in Calculus of Variations

In connection with lower semicontinuity results of integral functionals with respect to the weak∗- convergence of measures, we are looking for an optimal lower bound for $\liminf_{n\to\infty} \int f(x,u_n) d\mu$ when (u_n) is a sequence in $L^1(\Omega,\mu;\mathbb{R}^d)$ converging ∗-weakly to some measure $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^d)$. The lower bound we propose hereafter takes into account the possible gap between the biting limit of u_n and the μ-absolutely continuous part of the limit λ.

Theorem 5.1. *Let $f(x,z) : \Omega \times \mathbb{R}^d \to (-\infty, \infty]$ be a lower semicontinuous integrand, convex with respect to z such that $f(x,0)$ is locally bounded on Ω and verifying:*

$$\exists C > 0\ , \quad \exists a \in L^1_\mu(\Omega)\ :\qquad f(x,z) \geq -C|z| + a(x) \tag{5.1}$$

Let $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^d)$ with Lebesgue-Nikodym decomposition $\lambda = u^a\mu + \lambda_s$. Then for every sequence (u_n) such that:

$$u_n\ \mu \xrightharpoonup{*} \lambda \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^d) \quad , \quad u_n \xrightarrow{b} u^b$$

one has:

$$\liminf_{n\to\infty} \int_\Omega f(x,u_n)d\mu \geq \int_\Omega f(x,u^b)d\mu + \int_\Omega f^\infty(x,u^a-u^b)d\mu + \int_\Omega f^\infty(x,\frac{d\lambda_s}{d|\lambda_s|})d|\lambda_s|$$

Notation: When h(x,z) is a Borel function on $\Omega \times \mathbb{R}^d$ positively 1-homogeneous with respect to z and satisfy the lower bound $h(x,z) > -C(1+|z|)$ the integral $\int_B h(x,\frac{d\lambda}{d|\lambda|})d|\lambda|$ is well defined for any measure in $\mathcal{M}(\Omega;\mathbb{R}^d)$ and any Borel subset B. As the expression $\int h(x,\frac{d\lambda}{d\theta})d\theta$ does not depend on the positive measure θ provided $\lambda \ll \theta$, this integral will be denoted simply $\int_B h(x,\lambda)$ and the corresponding Borel measure by $h(x,\lambda)$ (see [8]).

Remark 5.2. a) From Theorem 5.1 and assertion (i) of Lemma 5.4 below, we recover the lower bound inequality obtained by Goffmann&Serrin [8]:

$$\liminf_{n\to\infty} \int_\Omega f(x,u_n)d\mu \geq \int_\Omega f(x,u^a)d\mu + \int_\Omega f^\infty(x,\lambda_s) \tag{5.2}$$

Let us notice that this inequality is not optimal: let us consider the sequence (u_n) of Example 3.2 and the integrand $f(x,z) := 1+|z|$. We find:

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega f(x,u_n(x))d\mu \quad &= \int_\Omega f(x,u^b)d\mu + \int_\Omega f^\infty(x,u^a-u^b)d\mu + \int_\Omega f^\infty(x,\lambda_s) = 3 \\ &> \int_\Omega f(x,u^a)d\mu + \int_\Omega f^\infty(x,\lambda_s) = 2\end{aligned}$$

b) It is possible to extend Theorem 5.1 by relaxing the lower semicontinuity assumption of f with respect to x. In that case, the integrand h_f has to be replaced by a "μ-essentiel regularization already used in [10] (see Proposition 5.5 below).

In the following theorem, we obtain a strong convergence result under strict convexity assumption:

Theorem 5.3. *Let $f(x,z) : \Omega\times\mathbb{R}^d \to (-\infty,\infty]$ be a lower semicontinuous integrand, strictly convex with respect to z, such that (5.1) holds and $f(.,0)$ is locally bounded on Ω. Let (u_n) be a sequence in $L^1(\Omega,\mu;\mathbb{R}^d)$ such that:*

$$u_n\ \mu \xrightharpoonup{*} \lambda \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^d) \quad , \quad \lambda = u\ \mu + \lambda_s \quad \text{with} \quad \lambda_s \perp \mu\ ,$$

$$\limsup_{n\to\infty} \int_\Omega f(x,u_n)d\mu \ \leq \ \int_\Omega f(x,u)d\mu + \int_\Omega f^\infty(x,\lambda_s) \ < \ +\infty\ .$$

Then the following properties hold:

(i) *u_n converges to u μ-locally in measure (and $u_n \xrightarrow{b} u$).*
(ii) *$u_n \to u$ strongly in $L^1_{loc}(\Omega, \mu; \mathbb{R}^d)$ if moreover $\lambda_s = 0$.*
(iii) *Assume that $f^\infty(x,.)$ is strictly convex for all $x \in \Omega$, then*

$$g(x, u_n)\mu \overset{*}{\rightharpoonup} g(x, u)\ \mu + g^\infty(x, \lambda_s) \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^d),$$

for every g in the class $\mathcal{F}(\Omega \times \mathbb{R}^d)$.

Moreover in (ii) *the convergence is strong in $L^1(\Omega, \mu; \mathbb{R}^d)$ and in* (iii) *the convergence is tight in $\mathcal{M}(\Omega; \mathbb{R}^d)$ if f satisfies: $\exists C > 0\ \exists b \in L^1_\mu \quad f(x, z) \geq C|z| - b(x)$.*

Remark 5.4. a) Recall that for a convex 1-homogeneous function $h : \mathbb{R}^d \to (-\infty, +\infty]$ strict convexity means that $h(y + z) = h(y) + h(z)$ with $|y| = |z| = 1$ implies $y = z$. Note that the strict convexity of a function g on $\mathbb{R}^d$ does not imply the strict convexity of the 1-homogeneous function g^∞ (Ex. Let $g : (z_1, z_2) \in \mathbb{R}^2 \to (1 + |z_1|^2)^{1/2} + (1 + |z_2|^2)^{1/2}$, then $g^\infty(z_1, z_2) = |z_1| + |z_2|$).
b) The property (ii) has been proved in [11] in case f is independent of x. The property (iii) appears already in [9] in case of 1-homogeneous integrands.

To prove Theorems 5.1 and 5.3, we will use some facts of convex analysis collected in Lemma 5.5 below (proved in Appendix):

Lemma 5.5. *Let $g : \mathbb{R}^d \to (-\infty, +\infty]$ be a convex proper l.s.c. function and ν a probability on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |z|\nu(dz) < +\infty$. Then:*

(i) *$g(y + z) \leq g(y) + g^\infty(z)$ for all $y, z \in \mathbb{R}^d$. $g(y + z) = g(y) + g^\infty(z) < +\infty \Rightarrow z = 0$ whenever g is strictly convex.*
(ii) *$g([\nu]) \leq \langle \nu, g\rangle$ (Jensen inequality) $g([\nu]) = \langle \nu, g\rangle < +\infty \Rightarrow$ ν is a Dirac mass whenever g is strictly convex.*
(iii) *$\langle \nu, g^\infty\rangle > 0$ whenever g is strictly convex, $[\nu] = 0$ and ν is not the Dirac mass at 0 (observe that g^∞ is not assumed to be strictly convex).*

Proof of Theorem 5.1. Applying Theorem 2.5 to the sequence (u_n), we can associate a subsequence (u_{n_k}), a measure m and parametrized probabilities ν_x and ν_x^∞ such that all conclusions of this theorem are fulfilled and such that: $\liminf_{n\to\infty} \int_\Omega f(x, u_n)d\mu = \lim_{k\to\infty} \int_\Omega f(x, u_{n_k})d\mu$. As $u_n\ \mu \overset{*}{\rightharpoonup} \lambda$, we find that: $\lambda = [\nu_x]\mu + [\nu_x^\infty]m$. By Theorem 2.9, we have also: $u^b = [\nu_x]$. Hence:

$$[\nu_x^\infty]m = \lambda - u^b\ \mu = (u^a - u^b)\mu + \lambda_s. \tag{5.3}$$

By the assumptions on f (see the Remark 2.4), we have $h_f = f^\infty$ so that by the assertion (iii) of Theorem 2.5, Jensen inequality and (5.3), we conclude:

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega f(x, u_n)d\mu &= \lim_{k\to\infty} \int_\Omega f(x, u_{n_k})d\mu \\ &\geq \int_\Omega \langle \nu_x, f(x,.)\rangle d\mu \ + \ \int_\Omega \langle \nu_x^\infty, f^\infty(x,.)\rangle dm\end{aligned}$$

$$\geq \int_\Omega f(x,[\nu_x])\ d\mu\ +\ \int_\Omega f^\infty(x,[\nu_x^\infty])dm \tag{5.4}$$
$$= \int_\Omega f(x,u^b)\ d\mu\ +\ \int_\Omega f^\infty(x,(u^a-u^b)\mu+\lambda_s)$$
$$= \int_\Omega f(x,u^b)\ d\mu\ +\ \int_\Omega f^\infty(x,(u^a-u^b))d\mu + \int_\Omega f^\infty(x,\lambda_s)\ .$$

□

Proof of Theorem 5.3. Take (u_{n_k}), m, ν_x and ν_x^∞ as in the proof of Theorem 5.1 and set:

$$I_1 := \int_\Omega f(x,u)d\mu + \int_\Omega f^\infty(x,\lambda_s)$$
$$I_2 := \int_\Omega f(x,u^b)\ d\mu\ +\ \int_\Omega f^\infty(x,(u-u^b))d\mu + \int_\Omega f^\infty(x,\lambda_s) \qquad (u^b=[\nu_x])$$
$$I_3 := \int_\Omega f(x,[\nu_x])\ d\mu\ +\ \int_\Omega f^\infty(x,[\nu_x^\infty])\ dm$$

By (5.4), we have: $\liminf_{n\to\infty}\int_\Omega f(x,u_n)d\mu \geq I_3 \geq I_2$.
By assumption and by Lemma 5.5 (i): $I_2 \geq I_1 \geq \limsup_{n\to\infty}\int_\Omega f(x,u_n)d\mu$.
It follows: $I_1 = I_2 = I_3$. Since $I_2 = I_1 < +\infty$, we deduce from (i) of Lemma 5.5 and (5.3) that:

$$u = u^b \quad \mu\ a.e. \quad , \quad \lambda_s = [\nu_x^\infty]\ m \tag{5.5}$$

Now since $I_2 = I_3$, by the assertion (ii) of Lemma 5.5 we get:

$$\langle \nu_x, f(x,.)\rangle = f(x,[\nu_x]) \qquad \mu \text{ a.e.} \tag{5.6}$$
$$\langle \nu_x^\infty, f^\infty(x,.)\rangle = f^\infty(x,[\nu_x^\infty]) \quad m \text{ a.e.} \tag{5.7}$$

The strict convexity of $f(x,.)$, (5.6) and (5.5) imply that ν_x is μ a.e. concentrated at $u(x)$ which proves the assertion (i) of our theorem (see (2.7)).
Let us prove (ii): by Vitali's theorem and the μ-convergence in measure of u_n already proved in (i), it is enough to show that (u_n) is locally uniformly integrable which from Theorem 2.9 (ii) is equivalent to say that $m = 0$. Assume that m does not vanish. Then by (5.5) and (5.7), the assumption $\lambda_s = 0$ yields: $[\nu_x^\infty] = 0$ and $< \nu_x^\infty, f^\infty(x,.) >= 0 \quad m$ a.e.. This is incompatible with the assertion (iii) of Lemma 5.5.
Assume now that for every x, the function $f^\infty(x,.)$ is strictly convex. Then by (5.7) and by Lemma 5.5 (ii)), ν_x^∞ is for a.e. x a Dirac mass concentrated at $[\nu_x^\infty]$ which is by (5.5) nothing else but the Radon Nikodym derivative of λ_s with respect to m. Let g a function in the class $\mathcal{F}(\Omega\times\mathbb{R}^d)$. The sequence $g(x,u_n(x))\,\mu$ is bounded hence weak $*$-relatively compact in $\mathcal{M}(\Omega;\mathbb{R}^d)$. Consider a weakly convergent subsequence. Noticing that all previous arguments appply to the corresponding subsequence (u_{n_k}) of the original sequence (u_n), we can identify the limit through Theorem 2.5:

$$g(x,u_{n_k})\ \mu \quad \overset{*}{\rightharpoonup} \quad \langle \nu_x, g(x,.)\rangle\mu\ + \langle \nu_x^\infty, g^\infty(x,.)\rangle m$$
$$= \quad g(x,u(x))\ \mu\ +\ g^\infty(x,\frac{d\lambda_s}{dm})\ m\ .$$

By the usual unicity argument, the whole sequence $g(x, u_n)\mu$ converges $*$-weakly. The assertion (iii) is proved.
To complete the proof we have only to show that the sequence $\alpha_n := f(x, u_n)\ \mu$ is tight. Indeed under the coercivity condition on f, this will imply the tightness of $|u_n|\mu$ and then that of the sequence $g(x, u_n)\mu$ for all g in $\mathcal{F}(\Omega \times \mathbb{R}^d)$ (use the growth condition (2.4)). Let $\alpha := f(x, u)\ \mu\ +\ f^\infty(x, \lambda_s)$. By localizing the semicontinuity result of Theorem 5.1 and the inequality (5.2), we have for every open subset ω of Ω: $\liminf_{n\to\infty} \alpha_n(\omega) \geq \alpha(\omega)$. By assumption, we also have: $\limsup_{n\to\infty} \alpha_n(\Omega) \leq \alpha(\Omega) < +\infty$. Since by (5.1) the negative part of α_n is uniformly bounded in variation, the conclusion follows. $\square$

Finally we indicate breefly how to relax in Theorem 5.1 the lower semicontinuity assumption of f with respect to x: let $f(x, z)$ be a convex normal integrand (that is measurable in (x, z), convex and l.s.c. in z). Assume that:

$$\exists \varphi_0 \in \mathcal{C}_0(\Omega; \mathbb{R}^d)\ ,\ \exists a \in L^1_\mu\ :\quad f(x, z)\ \geq\ \varphi_0(x).z - a(x)\ . \tag{5.8}$$

Let us define for every (x, z) in $\Omega \times \mathbb{R}^d$:

$$h^\mu_f(x, z) := sup\ \left\{ \varphi(x).z\ \mid\quad \varphi \in \mathcal{C}_0(\Omega; \mathbb{R}^d)\ ,\quad \int_\Omega f^\star(x, \varphi(x))\mu(dx)\quad < \infty \right\} \tag{5.9}$$

where $f^\star(x, z^\star) := \sup\limits_z \{z^\star.z - f(x, z)\}$. The function $h^\mu_f(x, z)$ is l.s.c.,convex and positively 1-homogeneous in z and we have (see [10], Prop. 7):

$$\begin{cases} h^\mu_f(x, .) \leq f^\infty(x, .) & \text{for } \mu \text{ a.e. } \ x \in \Omega \\ h^\mu_f \geq f^\infty & \text{if the multifunction } x \to epif^\star(x, .) \text{ is l.s.c.} \end{cases} \tag{5.10}$$

In particular $h^\mu_f \geq f^\infty$ holds if f satisfies the conditions of Theorem 5.1 and $h^\mu_f(x, .) \equiv h_f(x, .) \equiv f^\infty(x, .)$ for every $x \in \operatorname{supp}\mu$ if moreover f belongs to $\mathcal{F}(\Omega \times \mathbb{R}^d)$ (see [10], Theorem 8).

Proposition 5.6. *Let $f(x, z) : \Omega \times \mathbb{R}^d \to (-\infty, \infty]$ be a convex normal integrand satisfying* (5.8). *Let $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^d)$ with Lebesgue-Nikodym decomposition $\lambda = u^a\mu + \lambda_s$. Then for every sequence (u_n) such that:*

$$u_n\mu \overset{*}{\rightharpoonup} \lambda \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^d) \qquad , \qquad u_n \overset{b}{\to} u^b$$

one has:

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n)d\mu \geq \int_\Omega f(x, u^b)d\mu + \int_\Omega h^\mu_f(x, u^a - u^b)d\mu + \int_\Omega h^\mu_f(x, \lambda_s)$$

Remark 5.7. By eliminating u^b in the inequality above, we derive an inequality similar to (5.2):

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n)d\mu \geq \int_\Omega g(x, u^a)d\mu + \int_\Omega h^\mu_f(x, \lambda_s)$$

where the integrand g is defined for every $(x,z) \in \Omega \times \mathbb{R}^d$ by: $g(x,z) := inf\{f(x,z') + h_f^{\mu}(x, z-z'); z' \in \mathbb{R}^d\}$. This lower bound has been already obtained in [10].

Proof of Proposition 5.5. As done before we can reduce easily to the case $\mu(\Omega) < +\infty$. Let (E_p) a nonincreasing sequence such that: $\mu(E_p) \to 0$ and $u_n \rightharpoonup u^b$ in $L^1_\mu(\Omega \backslash E_p)$ (see Lemma 2.7). By a classical lower semicontinuity result, we have:

$$\liminf_{n\to\infty} \int_{\Omega\backslash E_p} f(x, u_n(x))d\mu \quad \geq \quad \int_{\Omega\backslash E_p} f(x, u^b(x))d\mu \ . \tag{5.11}$$

On the other hand by Fenchel's inequality we have for every φ in $\mathcal{C}_0(\Omega \times \mathbb{R}^d)$:

$$\int_{E_p} f(x, u_n(x))d\mu \quad \geq \quad \int_\Omega \varphi(x).u_n(x)1_{E_p}d\mu \quad - \quad \int_{E_p} f^\star(x,\varphi(x))d\mu$$

Since $u_n 1_{\Omega\backslash E_p}\,\mu \overset{*}{\rightharpoonup} u^b\, 1_{\Omega\backslash E_p}\,\mu$ we can write: $u_n 1_{E_p}\mu \overset{*}{\rightharpoonup} (u^a - u^b)\mu + u^b 1_{E_p}\mu + \lambda_s$ so that:

$$\liminf_{n\to\infty} \int_{E_p} f(x, u_n(x))d\mu \ \geq$$
$$\int_\Omega \varphi(x).(u^a - u^b)d\mu \ + \int_\Omega \varphi(x).d\lambda_s \ + \int_{E_p} (\varphi(x).u^b(x) - f^\star(x,\varphi(x)))d\mu \ . \tag{5.12}$$

Collecting (5.11)(5.12) and passing to the limit as $p \to \infty$, we obtain using dominated convergence theorem:

$$\liminf_{n\to\infty} \int_\Omega f(x,u_n(x))d\mu \ \geq \ \int_\Omega f(x,u^b)d\mu \ + \int_\Omega \varphi(x).(u^a - u^b)d\mu \ + \int_\Omega \varphi(x).d\lambda_s \tag{5.13}$$

for every $\varphi \in \mathcal{C}_0(\Omega; \mathbb{R}^d)$ such that $\int_\Omega f^\star(x,\varphi(x))d\mu < \infty$. Then we take the supremum in (5.13) with respect to all functions φ satisfying the condition above and conclude by applying a result of commutativity of $\int$ and sup (see [10], Theorem 1). □

Appendix

Proof of Lemma 2.2. Under (i), $f^\infty := 0$ and the continuity of $\tilde{f}$ is trivial. Under the homogeneity assumption (ii), $\tilde{f}$ is the restriction of f to $\Omega \times \bar{B}_d$. Assume now (iii) and let (x_n, z_n) a sequence converging to (x,z) in $\Omega \times \bar{B}_d$. We have to show that $\tilde{f}(x_n,z_n) \to \tilde{f}(x,z)$.
If $|z| < 1$, it comes down to the continuity of f. In case $|z| = 1$,we use the convexity of $f(x,.)$(which guarantees the existence of f^∞). Moreover for every $t > 0$, we have eventually $t > 1 - |z_n|$, hence:

$$tf(x_n, \frac{z_n}{t}) + (1 - |z_n| - t)f(x_n, 0) \leq \tilde{f}(x_n, z_n) \leq f^\infty(x_n, z_n) + (1 - |z_n|)f(x_n, 0) \ .$$

We conclude by passing to the limit as $n \to \infty$ (using the continuity of f and f^∞) and then as $t \to 0$. □

Proof of Lemma 2.3. Let g be defined on $\Omega \times B_d$ by $g(x,z) := (1-|z|) f(x, \dfrac{z}{1-|z|})$ and denote by $\overline{g}$ its lower semicontinuous extension on $\Omega \times \overline{B}_d$. As f is l.s.c. and satisfies (5.1), we have $\overline{g} = g$ on $\Omega \times B_d$ and $\overline{g}$ is lower bounded by $-\Lambda$. Moreover we have: $\overline{g} = h_f$ on $\Omega \times S^{d-1}$. Indeed let $(x,z) \in \Omega \times S^{d-1}$ and (x_k, z_k, t_k) a sequence such that $x_k \to x,\ z_k \to z,\ t_k \to +\infty$. Then:

$$\liminf_k \frac{f(x_k, t_k z_k)}{t_k} \;=\; \liminf_k (\frac{1}{t_k} + |z_k|) g(x_k, \frac{z_k}{\frac{1}{t_k} + |z_k|}) \;\geq\; \overline{g}(x,z)$$

Hence $h_f(x,z) \geq \overline{g}(x,z)$. Conversely choosing the particular sequence $t_k = \frac{1}{1-|z_k|}$, we get: $\liminf_k g(x_k, z_k) \;=\; \liminf_k \frac{1}{t_k} f(x_k, t_k z_k) \;\geq\; h_f(x,z)$ (hence $\overline{g}(x,z) \geq h_f(x,z)$).
We can construct a non decreasing sequence (g_p) in $\mathcal{C}_b(\Omega \times \bar{B}_d)$ such that $g_p \geq -\Lambda$ and $\sup_p g_p = \overline{g}$. Then the sequence (f_p) defined by $f_p(x,z) := (1+|z|) g_p(x, \dfrac{z}{1+|z|})$ satisfies all conditions of the lemma (note that $f_p^\infty(x,z) = g_p(x,z)$ on $\Omega \times S^{d-1}$) .
Let us prove finally that $h_f \equiv f^\infty$ when $f(x,.)$ is convex and $f(x,0)$ locally upper bounded (in order to justify the remark after the lemma). Let $(x_k, z_k) \to (x,z)$ and $t_k \to \infty$. Then for every $t > 0$, we have using the convexity assumption, the lower semicontinuity of f and the upper bound (2.4):

$$\liminf_k \frac{f(x_k, t_k z_k)}{t_k} \geq \liminf_k \left[\frac{f(x_k, t z_k)}{t} - (\frac{1}{t} - \frac{1}{t_k}) f(x_k, 0)\right] \geq \frac{f(x,tz)}{t} - \frac{\Lambda}{t} \ .$$

The conclusion follows by letting $t \to \infty$. □

Proof of Lemma 5.4. Let $y, z \in \mathbb{R}^d$ such that $g(y) < +\infty$ and $z \neq 0$. By the convexity of g, the function $h(t) := \frac{g(y+tz) - g(y)}{t}$ is nondecreasing on $(0, +\infty)$ and converges to $g^\infty(z)$ as $t \to \infty$. Hence $g^\infty(z) \geq h(1)$. Assuming moreover g strictly convex and $g^\infty(z)$ finite yields $g^\infty(z) > h(1)$. The assertion (i) is proved. The assertion (ii) is classical (see for example [13, p.273]). To prove (iii), let us assume that $\langle \nu, g^\infty \rangle = 0$. Using (i) and applying (ii) with $g_y(z) := g(y+z) \quad (g(y) < +\infty)$, we get:

$$g(y + [\nu]) \leq \int g(y+z) \nu(dz) \leq g(y) + \langle \nu, g^\infty \rangle = g(y) \ .$$

If $[\nu] = 0$, we deduce: $g(y + [\nu]) = \langle \nu, g(y + .) \rangle$, hence by the strict convexity of g (see (ii)) ν would be the Dirac mass at 0. □

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