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Cone-Constrained Linear Equations in Banach Spaces[1]

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This paper presents *necessary and sufficient* conditions for the existence of solutions to cone-constrained linear equations in some function spaces. These conditions yield, in particular, the classical Fredholm alternative for compact operators. We use a formulation that under some conditions permits to apply the Generalized Farkas Theorem of Craven and Koliha. The Poissson Equation for stochastic (Markov) kernels, the Volterra and Fredholm equations for non-compact operators in L_p spaces, are among the particular cases of potential application.

Keywords: Linear equations in Banach spaces, Generalized Farkas Theorem, Fredholm alternative

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1. Introduction

In this paper we are concerned with the existence of solutions to the linear equations

$$Ax = b \tag{1.1}$$

and

$$Ax = b,\ x \in S, \tag{1.2}$$

where $A:\ \mathcal{X} \to \mathcal{X}$ is a linear operator, $\mathcal{X}$ is a Banach space, and $S \subseteq \mathcal{X}$ is a convex cone. Of course, the unconstrained equation (1.1) is a particular case of (1.2) with $S := \mathcal{X}$. Other particular cases are the integral-type equations [7, 10, 11], the Poisson Equation [9] in L_p spaces and the Hilbert-Schmidt-type operators.

The *Generalized Farkas Theorem* of Craven and Koliha [3] permits to characterize existence of solutions to (1.1) or (1.2) *provided* the condition *"$A(S)$ is closed"* is satisfied

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in some appropriate topology. However, this condition is rarely met in practice. For instance, consider the Poisson Equation $(I-P)h=g$ for a Markov chain with associated stochastic kernel P, viewed as an operator on $\mathcal{X} := L_p(\Omega,\mathcal{B},\mu)$ (see [9]). The images $(I-P)(\mathcal{X})$ or $(I-P)(S)$ (where S is the positive cone in $\mathcal{X}$) need not be closed. However, the case of *compact operators* P is a notable exception and we will show that the Fredholm alternative theorem (see e.g. [2, 10]) is an immediate consequence of the Generalized Farkas Theorem of Craven and Koliha [3]. In addition, we will also show that our conditions for existence of a solution to (1.1) also reduce to the standard Fredholm alternative for compact operators when $\mathcal{X}$ is a reflexive Banach space.

To circumvent the non-closedness of $A(S)$, and in the same spirit but differently from [12], we formulate the problem in such a way that, by introducing some appropriate constraint, one may apply the Generalized Farkas Theorem to the modified system, which yields a new Generalized Farkas Theorem without this closure condition (see Theorem 2.4). We then apply this result to the particular case of $L_p(\Omega,\mathcal{B},\mu)$ spaces.

The case $\mathcal{X} := L_1(\Omega,\mathcal{B},\mu)$ needs a special treatment and interestingly enough, although the proof is different, the resulting (necessary and sufficient) conditions of existence can be represented in a single theorem (Theorem 3.2) that covers all the spaces $L_p(\Omega,\mathcal{B},\mu)$, $1\leq p\leq\infty$. Another slightly different Farkas-like theorem is also given for $L_1(\Omega,\mathcal{B},\mu)$ when identified with a subspace of $M(\Omega)$, the space of bounded signed Borel measures on Ω.

The paper is organized as follows. In Section 2, after some preliminary results, we present the new Farkas Lemma without a closure condition. We also consider the special case of compact operators and show that our conditions then reduce to the Fredholm alternative. In Section 3, we consider cone-constrained linear equations in the spaces $L_p(\Omega,\mathcal{B},\mu)$, and particularize the results obtained in Section 2 in a single theorem (Theorem 3.2 below) that covers all the cases. In Section 4, we present the proof of Theorem 3.2 with special attention to the case L_1. Finally, Section 5 is an appendix summarizing some basic results that we extensively use.

2. Notation, Definitions and Main Result

Let $\mathcal{X}$ be a separable Banach space with topological dual $\mathcal{X}^*$. The duality bracket between $\mathcal{X}$ and $\mathcal{X}^*$ is denoted $\langle .,.\rangle$. For a convex cone S in $\mathcal{X}$ we denote by S^* its dual cone, i.e.

$$S^* := \{y\in\mathcal{X}^* |\ \langle x,y\rangle\geq 0\ \ \forall x\in S\}, \tag{2.1}$$

and for a convex cone Ω in $\mathcal{X}^*$ we define

$$\Omega^* := \{x\in\mathcal{X}^{**} |\ \langle x,y\rangle\geq 0\quad \forall y\in\Omega\} \tag{2.2}$$

and

$$\Omega^+ := \{x\in\mathcal{X} |\ \langle x,y\rangle\geq 0\quad \forall y\in\Omega\} \tag{2.3}$$

Remark 2.1. Note that with the natural embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$, $\Omega^+ = \Omega^*\cap\mathcal{X}$. In addition, if S is strongly closed, then $(S^*)^+ = S$ (see e.g. [3]). Moreover, $(\mathcal{X}\times R, \mathcal{X}^*\times R)$ is viewed as a dual pair with duality bracket $\langle (x,r),(y,\rho)\rangle := \langle x,y\rangle + r\rho$.

2.1. A preliminary result

In the sequel we will use the following lemma:

Lemma 2.2. *Let $\Gamma \subset (\mathcal{X}^* \times R)$ be the cone $\{(y,r) \in \mathcal{X}^* \times R| \; ||y|| \leq r\}$. Then Γ is a weak* closed convex cone, and $\Gamma^+ = \Omega$, where*

$$\Omega := \{(x,z) \in \mathcal{X} \times R| \; ||x|| \leq z\}. \tag{2.4}$$

We also have

$$(\Gamma^+)^* = \Gamma, \quad i.e. \quad \Omega^* = \Gamma. \tag{2.5}$$

Proof. The fact that Γ is a convex cone is trivial. Now, to prove that it is weak* closed, from Theorem 5.2 (d), consider a sequence (y_n, r_n) in Γ such that (y_n, r_n) converges in the weak* topology to (a,b) in $\mathcal{X}^* \times R$, i.e.,

$$y_n \stackrel{w*}{\to} a, \quad r_n \to b \quad \text{as} \quad n \to \infty, \tag{2.6}$$

where $\stackrel{w*}{\to}$ denotes the $\sigma(\mathcal{X}^*, \mathcal{X})$ (weak*) topology in $\mathcal{X}^*$. We want to show that $(a,b) \in \Gamma$. From (2.6), $b \geq \liminf_n ||y_n|| \geq ||a||$ (see e.g. [2]) so that $(a,b) \in \Gamma$. Since Γ is weak* closed, then $(\Gamma^+)^* = \Gamma$ (see e.g. Proposition 1 in [1]) which yields (2.5). We shall now prove that $\Gamma^+ = \Omega$.

1. $\Omega \subseteq \Gamma^+$. It is obvious that

$$(x,z) \in \Omega \Rightarrow \langle (y,r),(x,z) \rangle = \langle x,y \rangle + rz \geq 0 \quad \forall (y,r) \in \Gamma,$$

since (note that both r and z are nonnegative)

$$|\langle x,y \rangle| \leq ||x||.||y|| \leq zr,$$

yields $\langle x,y \rangle + rz \geq 0$.

2. $\Gamma^+ \subseteq \Omega$.
$(x,z) \in \Gamma^+ \Rightarrow \langle x,y \rangle + rz \geq 0$ for all $(y,r) \in \Gamma$. For any $x \in \mathcal{X}$, $\exists y(x) \in \mathcal{X}^*$ such that $||x|| = \langle y(x), x \rangle$ and $||y(x)|| = 1$ (see e.g. [2]). Thus, for any $(x,z) \in \Gamma^+$, $(\pm y(x), 1) \in \Gamma$ so that

$$\langle x, \pm y(x) \rangle + z.1 \geq 0 \Rightarrow |\langle y(x), x \rangle| \; (:= ||x||) \leq z \Rightarrow (x,z) \in \Omega,$$

which is the desired result. $\square$

Remark 2.3. Note that if $\mathcal{X}$ is a reflexive Banach space, then in Lemma 2.2, $(\Gamma^*)^* = \Gamma$. Let H be a locally convex space with topological dual H' equipped with the weak* topology $\sigma(H', H)$. If C and D are two convex cones in H with closures $\overline{C}$ and $\overline{D}$, then $(\overline{C} \cap \overline{D})^* = \overline{C^* + D^*}$ (see e.g. [1]).

2.2. A Farkas Lemma without a closedness condition

In this section, we consider the general linear system

$$Ax = b, \quad x \in S^* \tag{2.7}$$

where $A : \mathcal{X}^* \to Z$ is a linear mapping, $\mathcal{X}$ and Z are Banach spaces and S is a convex cone in $\mathcal{X}$. W is another Banach space such that (Z, W) is a dual pair. Existence of solutions to (2.7) can be characterized via e.g. the *Generalized Farkas Theorem* of Craven and Koliha [3] (see Theorem 5.1 below). However, in addition to the usual continuity assumptions on the mapping A, a (restrictive) *closure* assumption is required on $A(S)$. The purpose of this section is to present a new Farkas-like theorem without such a restrictive assumption.

Theorem 2.4. *Let $\mathcal{X}$ be a separable Banach space with dual $\mathcal{X}^*$, equipped with the weak topology $\sigma(\mathcal{X}, \mathcal{X}^*)$ and the weak* topology $\sigma(\mathcal{X}^*, \mathcal{X})$ respectively. Let (Z, W) be a dual pair of Banach spaces, Z (resp. W) being equipped with the weak topology $\sigma(Z, W)$ (resp. $\sigma(W, Z)$). Let $A : \mathcal{X}^* \to Z$ be a weakly continuous linear mapping and $S \subset \mathcal{X}$ a strongly closed convex cone in $\mathcal{X}$. Let $A^* : W \to \mathcal{X}$ be the adjoint mapping of A, and let Γ be the convex cone $\{(x, r) \in \mathcal{X}^* \times R^+ |\ ||x|| \leq r\}$. Then,*

(a) *the following two propositions are equivalent for $b \in Z$:*
   - (i) *the system $Ax = b$ has a solution $x \in S^* \subset \mathcal{X}^*$.*
   - (ii) *$[w \in W,\ z \in R$ and $(A^*w, z) \in (\Gamma \cap (S^* \times R))^+] \Rightarrow \langle b, w\rangle + Mz \geq 0$, for some $M > 0$.*

(b) *If, in addition, $\mathcal{X}$ is reflexive, or if the cone $\Omega + (S \times \{0\})$ is strongly closed (where $\Omega := \{(x, r) \in \mathcal{X} \times R^+ |\ ||x|| \leq r\}$), then the following two propositions are equivalent for $b \in Z$:*
   - (i) *the system $Ax = b$ has a solution $x \in S^* \subset \mathcal{X}^*$.*
   - (ii) *$w \in W,\ s \in S \Rightarrow \langle b, w\rangle + M||A^*w - s|| \geq 0$, for some $M > 0$.*

Proof. (a) The system $\{Ax = b,\ x \in S^*\}$ has a solution if and only if the system

$$Ax = b;\ r = M; ||x|| \leq r;\ x \in S^* \tag{2.8}$$

has a solution for some $M > 0$, or equivalently, if and only if

$$T(x, r) = (b, M), \quad (x, r) \in \Gamma \cap (S^* \times R) \quad (\text{with } T(x, r) := (Ax, r)) \tag{2.9}$$

has a solution for some $M > 0$.

We first prove that $T(\Gamma \cap (S^* \times R))$ is weakly-closed. Let (x_α, r_α) be a net ($\alpha \in D$ for some directed set $(D, \geq)$) in $\Gamma \cap (S^* \times R)$ such that $T(x_\alpha, r_\alpha)$ converges weakly to some (a_1, a_2) in $Z \times R$, i.e.,

$$Ax_\alpha \text{ converges weakly to } a_1, \quad \text{and} \quad r_\alpha \text{ converges to } a_2. \tag{2.10}$$

If $a_2 = 0$, then obviously, since $(x_\alpha, r_\alpha) \in \Gamma$, $||x_\alpha||$ converges to zero so that Ax_α converges weakly to $a_1 = 0$, and $T(0, 0) = (0, 0)$.

Now, if $a_2 \neq 0$ then, from (2.10) and the fact that $(x_\alpha, r_\alpha) \in \Gamma$ there exists some $\alpha_0 \in D$ such that for all $\alpha \geq \alpha_0$, $||x_\alpha|| \leq 2a_2$. Since $\mathcal{X}^*$ is the dual of a separable Banach space, the set $\{x|\ ||x|| \leq 2a_2\}$ is weak* sequentially compact (see Theorem 5.2 (b),(c)). Thus,

from the net $\{x_\alpha\}$, one can extract a sequence $\{x_{\alpha_i}\}$ that converges to some x in the weak* topology $\sigma(\mathcal{X}^*, \mathcal{X})$. Moreover, since A is weakly continuous, Ax_{α_i} converges weakly to $Ax = a_1$. In addition, since r_{α_i} converges to a_2 and $\liminf_i ||x_{\alpha_i}|| \geq ||x||$ (see e.g. [2]), we get $||x|| \leq a_2$ i.e. $(x, a_2) \in \Gamma$. Finally, noting that S^* is weak* closed, then $x \in S^*$ so that $(x, a_2) \in (\Gamma \cap (S^* \times R^+))$. This combined with $T(x, a_2) = (a_1, a_2)$ implies that $T(\Gamma \cap (S^* \times R))$ is weakly closed.
Since $T((\Gamma \cap (S^* \times R))$ is weakly-closed, we can appply the Generalized Farkas Theorem (Theorem 5.1 below) which states that the system $\{T(x,r) = (b, M), (x,r) \in \Gamma \cap (S^* \times R)\}$ has a solution if and only if

$$[(w,z) \in W \times R, \quad T^*(w,z) \in (\Gamma \cap (S^* \times R))^+] \Rightarrow \langle b, w\rangle + Mz \geq 0, \tag{2.11}$$

where $T^*(w,z) = (A^*w, z)$, which yields part (a).

(b) To prove part (b), note that from Lemma 2.2, Γ is the dual cone of the strongly closed convex cone $\Omega := \{(x,z) \in \mathcal{X} \times R |\ ||x|| \leq z\}$. Note also that $S^* \times R$ is the dual cone of the strongly closed convex cone $S \times \{0\}$. Thus, as $(0,0) \in \Omega \cap (S \times \{0\})$,

$$\Gamma \cap (S^* \times R) = (\Omega + (S \times \{0\}))^*. \tag{2.12}$$

As $\mathcal{X}$ is reflexive, the cone $\Omega + (S \times \{0\})$ is strongly closed in $\mathcal{X} \times R$. Indeed, consider any sequence (x_n, s_n, r_n) with $(x_n, r_n) \in \Omega$ and $s_n \in S$, such that

$$x_n + s_n \to a \quad \text{and } r_n \to r \tag{2.13}$$

for some $(a, r) \in \mathcal{X} \times R$ and where the first convergence is in the strong topology of $\mathcal{X}$. From (2.13) and the fact that $||x_n|| \leq r_n$ we conclude that both x_n and s_n are uniformly bounded. From the weak* sequential compactness of the unit ball in $\mathcal{X}$ (since $\mathcal{X}$ is reflexive), there is a subsequence $(x_{n_i}, s_{n_i}, r_{n_i})$ such that $x_{n_i} \stackrel{w*}{\to} x$ and $s_{n_i} \stackrel{w*}{\to} s$. Both Ω and S are weak* closed (as dual cones of Γ and S^* respectively) so that $(x, r) \in \Omega$ and $s \in S$. Combining this and (2.13) yields $(a, r) = (x, r) + (s, 0)$, i.e. $(a, r) \in \Omega + (S \times \{0\})$, which proves that $\Omega + (S \times \{0\})$ is strongly closed.
Thus, by (2.12) and Remark 2.3,

$$(\Gamma \cap (S^* \times R))^+ = (\Omega + (S \times \{0\}))^{**} = \Omega + (S \times \{0\}). \tag{2.14}$$

Hence, (2.11) reads

$$\begin{aligned} &(w,z) \in W \times R, \ (A^*w, z) = (u + s, z), \ ||u|| \leq z, \ s \in S \\ &\Rightarrow \langle b, w\rangle + Mz \geq 0 \end{aligned} \tag{2.15}$$

for some $M > 0$, or, equivalently,

$$w \in W, \ s \in S \Rightarrow \langle b, w\rangle + M||A^*w - s|| \geq 0,$$

since it suffices to check (2.15) for $z := ||A^*w - s||$. $\square$

2.3. Compact operators

Let $A := (I - P)$ where $P : \mathcal{X}^* \to \mathcal{X}^*$ is a *compact* operator, i.e. P maps the unit ball of $\mathcal{X}^*$ into a relatively compact set in $\mathcal{X}^*$. We show that the Fredholm alternative (see e.g. [2]) is a particular case of the (Generalized Farkas) Theorem 5 of Craven and Koliha in [3]. We then show that our condition in Theorem 2.4 (b)(ii) also reduces to the Fredholm alternative when $\mathcal{X}$ is reflexive.
Note that if P is a compact operator then range(A) is closed (see e.g. [2] Th. VI.6). In addition, A is also strongly continuous. Thus, one may apply Theorem 5.1 below, with $\mathcal{X} := \mathcal{X}^*$, $Y = \mathcal{X}^*$, $S := \mathcal{X}^*$, $A := (I - P)$ so that, since $S^* = \{0\}$, we obtain

Corollary 2.5. *Assume that $P : \mathcal{X}^* \to \mathcal{X}^*$ is compact. Let $A := (I - P)$. Then*

$$Ax = b \;\text{ has a solution } x \text{ in } \mathcal{X}^* \;\text{ iff }\; [A^*w = 0,\; w \in \mathcal{X}] \Rightarrow \langle b, w\rangle = 0, \tag{2.16}$$

which is *the Fredholm alternative*.

We now prove that our condition in Theorem 2.4 (b)(ii) also reduces to (2.16), assuming that $\mathcal{X}$ is reflexive.

Corollary 2.6. *Assume that $\mathcal{X}$ is reflexive and let P and A be as in* Corollary 2.5. *Then the condition in* Theorem 2.4 (b)(ii) *reduces to* (2.16).

Proof. As $P : \mathcal{X}^* \to \mathcal{X}^*$, in Theorem 2.4 let (Z, W) be the dual pair $(\mathcal{X}^*, \mathcal{X})$.
With $S := \{0\}$, the condition (b)(ii) in Theorem 2.4 now reads

$$\langle b, w\rangle + M||A^*w|| \geq 0 \quad \forall w \in \mathcal{X}, \tag{2.17}$$

for some $M > 0$, with $A^* = I - P^*$.
Let $V := \mathrm{N}(A^*) = \{w \in \mathcal{X} |\; A^*w = 0\}$. As P is compact then so is P^* (see Schauder Theorem in e.g. [2]) and thus V has finite dimension (see [2] p. 90). Therefore, it admits a topological supplement V^c such that V^c is closed, $V \cap V^c = \{0\}$ and $\mathcal{X} = V + V^c$.
Note that if $w \in V$, then $-w \in V$ so that from (2.17) we must have $\langle b, w\rangle = 0$. Hence it remains to show that

$$\langle b, w\rangle + M||A^*w|| \geq 0 \quad \forall w \in V^c \tag{2.18}$$

is always satisfied for some $M > 0$, so that (2.17) reduces to the Fredholm alternative.
Without loss of generality we may and will assume that $||w|| = 1$ in (2.18). Let $\delta := \inf\{||A^*w|| \mid ||w|| = 1,\; w \in V^c\}$ and consider a minimizing sequence $\{w_n\}$ in V^c such that $||w_n|| = 1$ and $||A^*w_n|| \downarrow \delta$. We prove that $\delta > 0$.
By the weak* sequential compactness of the unit ball in $\mathcal{X}$ (recall that $\mathcal{X}$ is separable and reflexive, and see Theorem 5.2 (c)), $\exists w$ and a subsequence $\{n_i\}$ such that $w_{n_i} \stackrel{w*}{\to} w$ and also $A^*w_{n_i} \stackrel{w*}{\to} A^*w$, where $\stackrel{w*}{\to}$ denotes the (weak* or weak) $\sigma(\mathcal{X}, \mathcal{X}^*)$ convergence. As V^c is closed, it is also weakly-closed (i.e. $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed). Thus $w \in V^c$. Let us now consider the two cases, $w \neq 0$ and $w = 0$.

- If $w \neq 0$ then $A^*w \neq 0$ and as $A^*w_{n_i} \stackrel{w}{\to} A^*w$ we have $\delta = \liminf_i ||A^*w_{n_i}|| \geq ||A^*w|| > 0$.
- Consider now the case where $w = 0$. Since P^* is compact and $||w_{n_i}|| = 1$ for all i, $\{P^*w_{n_i}\}$ is in a relatively compact set for the strong topology in $\mathcal{X}$. Thus, for a

subsequence again denoted $\{w_{n_i}\}$, $P^* w_{n_i} \stackrel{s}{\to} q$ in $\mathcal{X}$. Moreover, $P^* w_{n_i} \stackrel{w*}{\to} P^* w = 0$ and thus $q = 0$, which implies $||P^* w_{n_i}|| \downarrow 0$. Now, from

$$||A^* w_{n_i}|| \geq ||w_{n_i}|| - ||P^* w_{n_i}||$$

we conclude that $\exists \epsilon > 0$ such that for i large enough

$$||A^* w_{n_i}|| \geq ||w_{n_i}|| - \epsilon = 1 - \epsilon$$

so that $\delta > 0$.

Moreover, $|\langle b, w_{n_i} \rangle| \leq ||b||.||w_{n_i}|| = ||b||$ so that for M large enough, and $w \in V^c$

$$\langle b, w \rangle + M||A^* w|| \geq 0.$$

Hence, in Theorem 2.4, the condition (b)(ii)

$$\langle b, w \rangle + M||A^* w|| \geq 0 \quad \forall w \in \mathcal{X}$$

for some $M > 0$ reduces to

$$w \in \mathcal{X}, \quad A^* w = 0 \Rightarrow \langle b, w \rangle = 0,$$

and the proof is complete. □

3. Linear systems in L_p spaces

General assumption. *$(X, \mathcal{B}, \mu)$ is a σ-finite complete measure space, with X a topological space, and $\mathcal{B}$ the completion (with respect to μ) of the σ-algebra of Borel subsets of X. In addition, for the particular case of L_1, we assume that X is a locally compact separable metric space.*

For $1 \leq p \leq \infty$, let q be the exponent conjugate to p, i.e. $(1/p) + (1/q) = 1$. We write L_p for $L_p(X, \mathcal{B}, \mu)$, and L_p^+ denotes the convex cone of nonnegative functions in L_p. Recall that L_p is a Banach space for every $1 \leq p \leq \infty$, with topological dual L_q when $1 \leq p < \infty$, the corresponding "inner product" being

$$\langle u, v \rangle := \int_X uv d\mu, \quad u \in L_p, \; v \in L_q.$$

In this section, we are concerned with the existence of solutions $h \in L_p$ to the equation

$$(I - P)h = b, \tag{3.1}$$

and

$$(I - P)h = b, \quad h \in S, \tag{3.2}$$

where $P: \; L_p \to L_p$ is a linear operator, $b \in L_p$ a given function, and S a convex cone in L_p.

For instance, in solving equation (3.2) with $S := L_p^+$, one looks for *nonnegative* solutions $h \in L_p$ to (3.1). The following examples show that (3.1), (3.2) include well known equations in analysis and probability.

3.1. Examples

$P:\ L_p \to L_p$ is a linear operator and there exists a measurable function $K(x,y)$ on $X \times X$ such that

$$Pu(x) = \int_X K(x,y)u(y)\mu(dy) \quad x \in X, \quad \forall u \in L_p.$$

Among particular cases of the above type of linear operators, let us mention:

Fredholm-type kernel. In this case, take for instance $X := [a,b]$ a closed interval on the real line, and μ the Lebesgue measure. Then, define

$$Pu(x) := \lambda \int_a^b K(x,y)u(y)dy, \qquad x \in X \tag{3.3}$$

where λ is some fixed scalar.

Volterra-type kernel. Again, take for instance $X := [a,b]$ a closed interval on the real line, and μ the Lebesgue measure. Then, define

$$Pu(x) := \lambda \int_a^x K(x,y)u(y)dy, \qquad x \in X \tag{3.4}$$

The Poisson Equation. Let P be a stochastic kernel on $(X, \mathcal{B})$, i.e. $P(x,.)$ is a probability measure on X for every $x \in X$, and $P(.,B)$ is a measurable function on X for every $B \in \mathcal{B}$. Let

$$Pu(x) := \int P(x,dy)u(y), \qquad x \in X, \tag{3.5}$$

and suppose that $P(x,.)$ is absolutely continuous with respect to μ, with density $K(x,.)$, i.e.

$$Pu(x) := \int_X K(x,y)u(y)\mu(dy), \qquad x \in X. \tag{3.6}$$

3.2. Existence of solutions in L_p

With P as in (3.1), (3.2), we suppose that for some given $p \in [1,\infty]$:

Assumption 3.1.

(a) *P maps L_p into itself.*

(b) *The adjoint P^* of P maps L_q into itself.*

(c) *In addition, if $p = 1$, P^* maps $C_0(X)$ into itself, where $C_0(X)$ is the separable Banach space of real-valued continuous functions on X that vanish at infinity, endowed with the sup-norm (see e.g.* [4] *or* [13]*).*

We now state the following main result:

Theorem 3.2. *Suppose that* Assumption 3.1 *holds for a given* $p \in [1, \infty]$. *Then:*

(a) *the equation* (3.1) *has a solution in* L_p *if and only if*

$$\langle b, w \rangle + M||(I - P^*)w||_q \geq 0 \quad \forall w \in L_q, \tag{3.7}$$

for some $M > 0$.

(b) *The equation* (3.1) *has a solution in* L_p^+ *if and only if*

$$\langle b, w \rangle + M|| \min[0, (I - P^*)w]||_q \geq 0 \quad \forall w \in L_q, \tag{3.8}$$

for some $M > 0$.

The proof of Theorem 3.2 requires different arguments depending on whether $p = 1$ or $1 < p \leq \infty$. The proof is given in the next section.

4. Proof of Theorem 3.2

4.1. The case $1 < p \leq \infty$

Suppose that $p \in (1, \infty]$ is fixed and b is a given function in L_p, and we wish to find a solution h in L_p (case (a)) or a *nonnegative* solution h in L_p to (3.1) (case (b)).
Then, Theorem 2.4 with the identification

$$\mathcal{X}^* := L_p; \; \mathcal{X} := L_q; \; Z := L_p; \; W := L_q; \; A := (I - P)$$

and $S^* := \mathcal{X}^*$ (case (a)) yields Theorem 3.2 (a) in the case $1 < p \leq \infty$. Similarly, to obtain part (b) for $1 < p < \infty$, let S^* be the positive cone in L_p with dual cone $S =$ the positive cone in L_q, and recall that the spaces L_p are reflexive when $1 < p < \infty$.
For the case $p = \infty$, although L_∞ is not reflexive, the cone $\Omega + (S \times \{0\})$ in Theorem 2.4, is strongly closed when $\mathcal{X} := L_1$.
Indeed, let (f_n, g_n, r_n) be a sequence in L_1 such that

$$||f_n||_1 \leq r_n, \; r_n \to r, g_n \geq 0 \;\text{ and }\; \lim_n ||f_n + g_n - u||_1 = 0.$$

Then, using the standard notation $u^+ := \max[u, 0]$ and $u^- := \max[-u, 0]$, we wish to prove that $u = u^+ - u^-$ is in the cone $\Omega + (S \times \{0\})$, for which (as $u^+ \geq 0$) it is sufficient to show that $||u^-||_1 \leq r$. To prove this, let $\{m\}$ be a subsequence of $\{n\}$ such that $f_m + g_m$ converges to u μ-a.e., so that, in particular,

$$(f_m + g_m)^- \to u^- \quad \mu\text{-a.e..}$$

This, in turn (as $g_m \geq 0$ implies $(f_m + g_m)^- \leq f_m^-$), yields

$$u^- \leq \liminf f_m^-,$$

and we get $||u^-||_1 \leq r$ since, by Fatou's Lemma,

$$||u^-||_1 \leq \liminf ||f_m^-||_1 \leq \liminf ||f_m||_1 \leq r.$$

This proves that $\Omega + (S \times \{0\})$ is strongly closed and, therefore, Theorem 2.4 (b) is valid. To see that (3.8) in Theorem 3.2 is equivalent to Theorem 2.4 (b)(ii), note that if S is the positive cone in L_q, then 2.4 (b)(ii) with $b \in L_p$ is true if and only if

$$\langle b, w \rangle + M || \min[0, A^* w] ||_q \geq 0 \quad \forall w \in L_q,$$

since for any $s \in S$, $||A^* w - s||_q \geq || \min[0, A^* w]||_q$ and thus it suffices to check the condition for $s := \max[0, A^* w]$.

4.2. The case $p = 1$

We now consider the special case of L_1 where $(X, \mathcal{B}, \mu)$ is a σ-finite complete measure space, X is a locally compact separable metric space, and $\mathcal{B}$ is the completion (with respect to μ) of the σ-algebra of Borel subsets of X.
As L_1 is not the dual of L_∞, we cannot use the weak* topology as we extensively did in the proof of Theorem 2.4.
Suppose that b is a given function in L_1, and we wish to find a nonnegative solution h in L_1 to (3.1). Then, (3.1) has a solution in L_1^+ if and only if the following system

$$(I - P)h = b, \quad \langle h, 1 \rangle \leq M, \quad h \in L_1^+ \tag{4.1}$$

has a solution for some $M > 0$, or equivalently, if and only if the system

$$(I - P)h = b, \quad \langle h, 1 \rangle + r = M, \tag{4.2}$$

has a solution (h, r) in $L_1^+ \times R^+$ for some $M > 0$.
The dual pair $(L_1 \times R, L_\infty \times R)$ is endowed with the inner product

$$\langle (h, r), (u, \rho) \rangle := \langle h, u \rangle + r\rho$$

where $\langle h, u \rangle := \int h u d\mu$ for $h \in L_1$ and $u \in L_\infty$.
Thus we now consider the linear operator $A_1 : \ L_1 \times R \to L_1 \times R$ and its adjoint $A_1^* : \ L_\infty \times R \to L_\infty \times R$ given by

$$A_1(h, r) := ((I - P)h, \langle h, 1 \rangle + r), \tag{4.3}$$

$$A_1^*(u, \rho) := ((I - P^*)u + \rho, \rho). \tag{4.4}$$

Note that, by Assumption 3.1 (b), A_1 is weakly continuous and, on the other hand, (4.1) is equivalent to

$$A_1(h, r) = (b, M) \quad \text{has a solution } \ (h, r) \ \text{ in } \ L_1^+ \times R^+ \tag{4.5}$$

for some $M \geq 0$. Similarly, if we wish to find solutions $h = h^+ - h^-$ in L_1, we consider the operators

$$A_1 : \ (L_1)^2 \times R \to L_1 \times R, \quad \text{and} \quad A_1^* : \ L_\infty \times R \to (L_\infty)^2 \times R$$

given by

$$A_1(h_1, h_2, r) := ((I - P)(h_1 - h_2), \langle h_1 + h_2, 1 \rangle + r), \tag{4.6}$$

$$A_1^*(u,\rho) := ((I-P^*)u+\rho, \rho-(I-P^*)u, \rho). \tag{4.7}$$

Again, A_1 is weakly continuous, and (3.1) has a solution in L_1 if and only if

$$A_1(h_1,h_2,r) = (b,M) \quad \text{has a solution} \quad (h_1,h_2,r) \in (L_1^+)^2 \times R^+ \tag{4.8}$$

for some $M \geq 0$. Thus Lemma 4.2 below and Theorem 5.1 yield the following proposition $\equiv$ Theorem 3.2 for $p=1$.

Proposition 4.1. *Suppose that* $b \in L_1$ *and* Assumption 3.1 *holds. Then:*

(a) *The equation* (3.1) *has a solution in* L_1 *if and only if*

$$[u \in L_\infty, \rho \in R^+, \text{ and } -\rho \leq (I-P^*)u \leq \rho] \Rightarrow \langle b,u\rangle + M\rho \geq 0$$

for some $M \geq 0$, *or, equivalently, if and only if*

$$\langle b,u\rangle + M||(I-P^*)u||_\infty \geq 0 \quad \forall u \in L_\infty$$

for some $M \geq 0$.

(b) *The equation* (3.1) *has a solution in* L_1^+ *if and only if*

$$[u \in L_\infty, \rho \in R^+, \text{ and } (I-P^*)u \geq -\rho] \Rightarrow \langle b,u\rangle + M\rho \geq 0$$

for some $M \geq 0$, *or, equivalently, if and only if*

$$\langle b,u\rangle + M||\min[0,(I-P^*)u||_\infty \geq 0 \quad \forall u \in L_\infty$$

for some $M \geq 0$.

Lemma 4.2.

(a) *With* A_1 *as in* (4.3), $A_1(L_1^+ \times R^+)$ *is weakly closed.*

(b) *With* A_1 *as in* (4.6), $A_1((L_1^+)^2 \times R^+)$ *is weakly closed.*

Proof. The proof of Lemma 4.2 requires in particular Lemma 5.3 (a) in the appendix, which is an extension of the Vitali-Hahn-Saks theorem.

Remark 4.3. We use below the following *notation*: $M(X)$ denotes the Banach space of finite signed measures on $(X,\mathcal{B})$, endowed with the total variation norm. By the Riesz theorem (see e.g. [13] p. 130) $M(X)$ is the dual of the *separable* Banach space $C_0(X)$ in Assumption 3.1 (c).

Proof of Lemma 4.2 (b). We first give the proof of part (b), and then show that it also contains the proof of (a). Let us write the convex cone $(L_1^+)^2 \times R^+$ as S_1, and for some directed set $(D,\leq)$, let $\{(v_\alpha,w_\alpha,r_\alpha),\ \alpha \in D\}$ be a net in S_1 such that $A_1(v_\alpha,w_\alpha,r_\alpha)$, with A_1 as in (4.6), converges to $(a,b) \in L_1 \times R$ in the weak topology $\sigma(L_1 \times R, L_\infty \times R)$; that is, for all (u,ρ) in $L_\infty \times R$:

$$\langle (I-P)(v_\alpha - w_\alpha), u\rangle + (\langle v_\alpha + w_\alpha, 1\rangle + r_\alpha)\rho \to \langle a,u\rangle + b\rho. \tag{4.9}$$

We wish to show that (a,b) is in $A_1(S_1)$, i.e. there is (h_1,h_2,r) in S_1 with

$$(I-P)(h_1-h_2) = a, \quad \text{and} \quad \langle h_1+h_2, 1\rangle + r = b. \tag{4.10}$$

Now, in (4.9) take $\rho = 0$, and then $\rho = 1, u = 0$ to get

$$\langle (I-P)(v_\alpha - w_\alpha), u\rangle \to \langle a, u\rangle \quad \forall u \in L_\infty, \tag{4.11}$$

and

$$\langle v_\alpha + w_\alpha, 1\rangle + r_\alpha \to b \tag{4.12}$$

respectively. If $b = 0$, then we are done because in such a case r_α, $\langle v_\alpha, 1\rangle$ and $\langle w_\alpha, 1\rangle$ $\to 0$ and we may take $h_1 = h_2 = 0$ and $r = 0$ in (4.10) since a has to be 0.
Let us now consider the case $b > 0$. By (4.10), there is $\alpha_0 \in D$ such that

$$0 \le ||v_\alpha||_1 + ||w_\alpha||_1 + r_\alpha \le 2b \quad \forall \alpha \ge \alpha_0, \tag{4.13}$$

where we have used that $\langle v_\alpha, 1\rangle := \int v_\alpha d\mu = ||v_\alpha||_1$ and similarly for w_α. For every $\alpha \ge \alpha_0$ consider the (nonnegative) measures $\varphi_\alpha, \psi_\alpha$ defined as

$$\varphi_\alpha(B) := \int_B v_\alpha d\mu, \quad \text{and} \quad \psi_\alpha(B) := \int_B w_\alpha d\mu, \quad B \in \mathcal{B}, \tag{4.14}$$

which, by (4.13), are *uniformly bounded* by $2b$. Therefore (see Remark 4.3), by Theorem 5.2 (b),(c), there is a sequence $\{\alpha_i\}$ in D, such that $\{\varphi_{\alpha_i}\}$ and $\{\psi_{\alpha_i}\}$ converge in the weak* topology $\sigma(M(X), C_0(X))$ to measures φ and ψ respectively, i.e., $\forall u \in C_0(X)$:

$$\langle \varphi_{\alpha_i}, u\rangle \to \langle \varphi, u\rangle \quad \text{and} \quad \langle \psi_{\alpha_i}, u\rangle \to \langle \psi, u\rangle. \tag{4.15}$$

From (4.14)–(4.15) and Lemma 5.3 (together with the Radon-Nikodym Theorem and the fact that φ_α and ψ_α are uniformly bounded, *finite* measures) there exist functions h_1 and h_2 in L_1^+ such that

$$\varphi(B) = \int_B h_1 d\mu \quad \text{and} \quad \psi(B) = \int_B h_2 d\mu \quad \forall B \in \mathcal{B}. \tag{4.16}$$

Moreover, (4.15)–(4.16) yield $\forall u \in C_0(X)$:

$$\langle v_{\alpha_i}, u\rangle \to \langle h_1, u\rangle \tag{4.17}$$

since

$$\langle v_{\alpha_i}, u\rangle = \int u v_{\alpha_i} d\mu = \langle \varphi_{\alpha_i}, u\rangle \to \langle \varphi, u\rangle = \langle h_1, u\rangle.$$

Similarly,

$$\langle w_{\alpha_i}, u\rangle \to \langle h_2, u\rangle \quad \forall u \in C_0(X). \tag{4.18}$$

In addition (as $p = 1$), Assumption 3.1 (c) yields, $\forall u \in C_0(X)$:

$$\langle P v_{\alpha_i}, u\rangle \to \langle P h_1, u\rangle \tag{4.19}$$

since

$$\langle P v_{\alpha_i}, u\rangle = \langle v_{\alpha_i}, P^* u\rangle = \langle \varphi_{\alpha_i}, P^* u\rangle \to \langle \varphi, P^* u\rangle = \langle h_1, P^* u\rangle = \langle P h_1, u\rangle.$$

Similarly,

$$\langle Pw_{\alpha_i}, u\rangle \to \langle Ph_2, u\rangle \quad \forall u \in C_0(X). \tag{4.20}$$

Thus combining (4.18)–(4.20) and (4.11)–(4.12) we see that h_1, h_2 and the nonnegative number $r := b - \langle h_1 + h_2, 1\rangle$ satisfy (4.10). As h_1, h_2 are in L_1^+ this completes the proof of part (b).

In fact, the latter also yields part (a), taking $w_\alpha = h_2 = 0$ in (4.9)–(4.20) - - i.e. "deleting" w_α and h_2 (in which case note that (4.6) reduces to (4.3)). □

4.3. The case $p = 1$: another Farkas-like lemma

In this section we provide another Farkas-like theorem for linear systems in L_1. We now identify L_1 with the linear subspace N of finite signed measures in $M(X)$ which are absolutely continuous with respect to μ, and we shall use again Remark 4.3.

Note that by Theorem 5.2 (d) and Lemma 5.3, N is weak* closed in $M(X)$. Moreover, with $p = 1$, consider the case where P has a kernel $K(x, y)$ on $X \times X$. Let $P(B|x) := \int_B K(x, y)\mu(dy)$, $B \in \mathcal{B}$, and assume that $P\nu(B) := \int P(B|x)\nu(dx)$ is finite for all $B \in \mathcal{B}$, $\nu \in M(X)$.

Then, P may be viewed as a linear operator on $M(X)$ and (3.1) (with $p = 1$) is equivalent to

$$(I - P)\varphi = \nu_b, \ \varphi \in N, \tag{4.21}$$

with $\nu_b \in M(X)$ and $\nu_b(B) := \int_B bd\mu \ \forall B \in \mathcal{B}$; moreover, if we look for a nonnegative solution, (3.2) is equivalent to

$$(I - P)\varphi = \nu_b, \ \varphi \in \Delta \cap N, \tag{4.22}$$

where now Δ is the positive cone in $M(X)$.

The orthogonal complement of N, i.e. $N^\perp := \{f \in C_0(X) | \ \langle f, \varphi\rangle = 0 \ \forall \varphi \in N\}$, is (weakly) $\sigma(C_0(X), M(X))$-closed and thus strongly closed. In addition, $(N^\perp)^\perp := \{\varphi \in M(X) | \ \langle f, \varphi\rangle = 0 \ \forall f \in N^\perp\}$ coincides with the (weak*) $\sigma(M(X), C_0(X))$-closure of N (see [2] p. 24) and therefore $(N^\perp)^\perp = N$ since N is weak* closed. Then, we can apply Theorem 2.4 with

$$\mathcal{X} := C_0(X); \ \mathcal{X}^* := M(X); \ Z := M(X); \ W := C_0(X); \ A := (I - P)$$

($\mathcal{X}$ being equipped with the sup norm $||.||$) and $S^* := N = (N^\perp)^\perp = (N^\perp)^*$ in the case of equation (3.1) or $S^* := \Delta \cap N = \Delta \cap (N^\perp)^*$ in the case of (3.2), which yields

Theorem 4.4. *Suppose that $b \in L_1$ and* Assumption 3.1 *holds. Then:*

(a) *The equation* (3.1) *has a solution in L_1 if and only if*

$$u, w \in C_0(X), \ w \in N^\perp \ \Rightarrow \langle b, u\rangle + M||(I - P)^*u - w|| \geq 0$$

for some $M > 0$.

(b) *The equation* (3.2) *has a nonnegative solution in L_1 if and only if*

$$u, w, h \in C_0(X), \ w \geq 0, \ h \in N^\perp \Rightarrow \langle b, u\rangle + M||(I - P)^*u - w - h|| \geq 0$$

for some $M > 0$, or equivalently, if and only if

$$u, h \in C_0(X),\ \ h \in N^{\perp} \Rightarrow \langle b, u\rangle + M||\min[0, (I-P)^* u] - h|| \geq 0$$

for some $M > 0$.

Proof. Because of Assumption 3.1, the hypotheses of Theorem 2.4 (a) are satisfied. In the case of (3.2),

$$S^* = (\Delta \cap N) = (G + N^{\perp})^*$$

with $G := \{f \in C_0(X),\ f \geq 0\}$. As $\mathcal{X}$ is not reflexive, it remains to show that $\Omega+(S\times\{0\})$ is strongly closed in $C_0(X)$.
We first consider the case $S = N^{\perp}$.
Let (f_n, g_n, r_n) be a sequence in $C_0(X) \times N^{\perp} \times R^+$ such that

$$||f_n|| \leq r_n;\ g_n \in N^{\perp};\ r_n \to r \ \text{ and } \ \lim_n ||f_n + g_n - f|| = 0,$$

where $||.||$ denotes the sup norm in $C_0(X)$.
$C_0(X)$ with the sup norm is complete so that $f \in C_0(X)$. Let $B_1 := \{x \in X |\ f(x) > r\}$ and $B_2 := \{x \in X|\ f(x) < -r\}$. Assume that $\mu(B_1) > 0$. Then as strong convergence implies weak convergence, and $g_n \in N^{\perp}$, we have

$$\int (f_n + g_n) d\varphi = \int f_n d\varphi \to \int f d\varphi,\ \ \forall \varphi \in N.$$

In particular, take a nonnegative measure φ in N with $\varphi(B_1) = 1$ and $\varphi(B_1^c) = 0$. We would have $\int f_n d\varphi \to \int f d\varphi = r+\delta$ for some $\delta > 0$. On the other hand, as $||f_n|| \leq r_n\ \forall n$, for n sufficiently large, $||f_n|| \leq r + \delta/2$ so that $|\int f_n d\varphi| \leq r+\delta/2 < r+\delta$ a contradiction. Therefore, we must have $\mu(B_1) = 0$ and similarly $\mu(B_2) = 0$.
In addition, $\{x \in X|\ |f(x)| \geq r\}$ is compact as $f \in C_0(X)$. Consider the functions $f_1(x) := f(x)$ if $|f(x)| \leq r$ and $\text{sign}(f(x))r$ otherwise, $f_2(x) := f(x) - r$ if $f(x) \geq r$, $f(x) + r$ if $f(x) \leq -r$ and 0 otherwise. Both are in $C_0(X)$. In addition, f_2 is in $N^{\perp}$, and $f = f_1 + f_2$. It then suffices to note that $||f_1|| \leq r$.
For the case where $S = G+N^{\perp}$ consider a sequence (f_n, h_n, g_n, r_n) in $C_0(X)\times G\times N^{\perp}\times R^+$ such that

$$||f_n|| \leq r_n;\ g_n \in N^{\perp};\ r_n \to r \ \text{ and } \ \lim_n ||f_n + h_n + g_n - f|| = 0,$$

$f_n = f_n^+ + f_n^-$ with $||f_n^-|| \leq r_n$. Rewrite $f_n + h_n$ as $w_n^+ + w_n^-$ so that as $h_n \geq 0$, $||w_n^-|| \leq r_n$. Again consider the set B_2 as above. $\mu(B_2) = 0$ for the same reasons. Indeed, with φ such that $\varphi(B_2) = 1$ and $\varphi(B_2^c) = 0$

$$\int (f_n + h_n + g_n) d\varphi = \int (w_n^+ + w_n^-) d\varphi \to \int f d\varphi = -r - \delta$$

for some $\delta > 0$. But $\int (w_n^+ + w_n^-) d\varphi \geq \int w_n^- d\varphi \geq -r-\delta/2$ for n sufficiently large. Consider the functions $f_1(x) := f(x)$ if $-r \leq f \leq 0$, $-r$ if $f \leq -r$ and 0 if $f \geq 0$; $f_2(x) := f(x) + r$

if $f \leq -r$ and 0 otherwise; $f_3(x) := \max[0, f(x)]$. Again $f_i \in C_0(X)$, $\forall i$; $f = f_1 + f_2 + f_3$, $f_2 \in N^{\perp}$, $f_3 \geq 0$, and $||f_1|| \leq r$, which proves that $\Omega + (S \times \{0\})$ is closed in $C_0(X)$. □

Note that with this Farkas-like theorem, one uses functions in $C_0(X)$ and the sup-norm rather than functions in L_∞ with $||.||_\infty$ as in Theorem 3.2.

5. Appendix

For ease of reference we collect in this appendix some results used in the paper, including Theorem 5.1 below that is a special case of the *Generalized Farkas Theorem* of Craven and Koliha ([3], Theor. 2).

If $\mathcal{X}$ is Banach space with topological dual $\mathcal{X}^*$, the *weak topology* on $\mathcal{X}$ is denoted $\sigma(\mathcal{X}, \mathcal{X}^*)$ and the *weak* topology* on $\mathcal{X}^*$ is denoted $\sigma(\mathcal{X}^*, \mathcal{X})$. U denotes the closed *unit sphere* in $\mathcal{X}^*$, i.e. $U := \{f \in \mathcal{X}^* |\ ||f|| \leq 1\}$. If S is a convex cone in $\mathcal{X}$, its *dual cone* is

$$S^* := \{f \in \mathcal{X}^* |\ \langle f, x \rangle \geq 0 \quad \forall x \in S\}.$$

Theorem 5.1. (cf. [3] Theor. 2). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces with topological duals $\mathcal{X}^*$ and $\mathcal{Y}^*$ respectively. Let S be a convex cone in $\mathcal{X}$, and let $A : \mathcal{X} \to \mathcal{Y}$ be a weakly continuous linear map with adjoint $A^* : \mathcal{Y}^* \to \mathcal{X}^*$. If $A(S)$ is weakly closed, then the following are equivalent conditions on $b \in \mathcal{Y}$:*

(a) *The equation $Ax = b$ has a solution x in S.*
(b) $A^* y^* \in S^* \Rightarrow \langle b, y^* \rangle \geq 0$.

Theorem 5.2. *Let $\mathcal{X}$ be a Banach space with topological dual $\mathcal{X}^*$.*

(a) *If x_n converges to x in the weak topology $\sigma(\mathcal{X}, \mathcal{X}^*)$, then $||x_n||$ is bounded and* $\liminf ||x_n|| \geq ||x||$.
(b) *The unit sphere U in $\mathcal{X}^*$ is compact in the weak* topology.*
(c) *If $\mathcal{X}$ is separable, then the weak* topology of U is metrizable.*
(d) *If $\mathcal{X}$ is separable, then a convex subset K of $\mathcal{X}^*$ is closed in the weak* topology if and only if*

$$(x_n^* \in K \quad \text{and} \quad \langle x, x_n^* \rangle \to \langle x, x^* \rangle \quad \forall x \in \mathcal{X}) \Rightarrow x^* \in K.$$

Theorem 5.2 (b) is the so-called *Alaoglu* (or *Banach-Alaoglu-Bourbaki*) theorem. For a proof of Theorem 5.2 see e.g. [2] or [6].

Lemma 5.3. *Let $(X, \mathcal{B}, \mu)$ be as in* Section 3. *Let $\{\varphi_n\}$ and φ be σ-finite measures on $(X, \mathcal{B})$ such that*

$$\langle \varphi_n, u \rangle \to \langle \varphi, u \rangle \quad \forall u \in C_0(X), \tag{5.1}$$

where $\langle \varphi, u \rangle := \int u d\varphi$. Suppose, in addition, that every φ_n is absolutely continuous (a.c.) with respect to μ. Then

(a) *φ is a.c. with respect to μ.*

Moreover (by the Radon-Nikodym theorem), let u_n and u be nonnegative measurable functions such that

$$\varphi_n(B) = \int_B u_n d\mu, \quad \text{and} \quad \varphi(B) = \int_B u d\mu \quad \forall B \in \mathcal{B}.$$

(b) *If (for a given* $1 \leq p \leq \infty$*)* $u_n \in L_p$ $\forall n$, *and* $\liminf_n ||u_n||_p \leq M$ *for some constant* M*, then* u *is in* L_p*.*

For a proof of Lemma 5.3 see [8].

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