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Γ-convergence for a Class of Functionals with Deviating Argument

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The paper deals with the problem of Γ-convergence of functionals involving nonlocal transformation of argument, in particular, the argument deviation. The property of Γ-convergence of the functionals is related with various types of convergence of inner superposition (composition) operators. The study is conducted in Orlicz spaces. Γ-convergence of functionals with impulsive constraints is also studied as one of the applications.

Keywords: Calculus of Variations, Γ-convergence, Functional Differential Equation, Orlicz Space.

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1. Introduction

When dealing with minimization problem for the given functional $I\colon X \to \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{+\infty\}$, X being some Banach space of admissible functions, it is important to know the behavior of its solutions (i.e. minimum values and points of minima) with respect to small perturbation of parameters of the problem. In other words, we are to study the problem of Γ-convergence of the functionals I_ν to I.
This paper is concerned with minimization problem for functional with deviating argument defined by the relationship

$$I(\boldsymbol{x}) = \int_0^1 f(t, \boldsymbol{x}(h_1(t)), \ldots, \boldsymbol{x}(h_k(t)), \dot{\boldsymbol{x}}(g_1(t)), \ldots, \dot{\boldsymbol{x}}(g_l(t)))\, dt, \qquad (V_\infty)$$
$$\boldsymbol{x}(t) = 0,\ \dot{\boldsymbol{x}}(t) = 0, \quad \text{whenever } t \notin [0,1].$$

where $t \in [0,1]$, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is a vector function, $\{k,l\} \subset \mathbb{N}$, $h_i\colon [0,1] \to \mathbb{R}$, $g_j\colon [0,1] \to \mathbb{R}$ and $f\colon [0,1] \times \mathbb{R}^{nk} \times \mathbb{R}^{nl} \to \bar{\mathbb{R}}$. The problem of continuous dependence of its solutions on

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parameters can be stated as follows: suppose we have a sequence of perturbed functionals of the same type

$$I_\nu(\boldsymbol{x}) = \int_0^1 f_\nu(t, \boldsymbol{x}(h_1^\nu(t)), \ldots, \boldsymbol{x}(h_k^\nu(t)), \dot{\boldsymbol{x}}(g_1^\nu(t)), \ldots, \dot{\boldsymbol{x}}(g_l^\nu(t)))\, dt, \qquad (V_\nu)$$

$$\boldsymbol{x}(t) = 0,\ \dot{\boldsymbol{x}}(t) = 0, \quad \text{whenever } t \notin [0,1],$$

$h_i^\nu\colon [0,1] \to \mathbb{R}$, $g_j^\nu\colon [0,1] \to \mathbb{R}$, $f_\nu\colon [0,1] \times \mathbb{R}^{nk} \times \mathbb{R}^{nl} \to \bar{\mathbb{R}}$ tending in certain sense to h_i, g_j and f respectively as $\nu \to \infty$. One is interested in finding out whether in this case the solutions to the minimization problems (V_ν) (minimum values of the functionals I_ν and the points of their minima $\boldsymbol{x}_\nu$) approach the solution to the minimization problem (V_∞) (respectively, minimum value of I and the points $\boldsymbol{x}$ of minimum of the latter).

The concept of Γ-convergence was introduced in the 1970s by E. De Giorgi (see, e.g., [1]) and afterwards received significant contributions by many authors. In the sequel we mainly follow its recent developments in control theory by G. Dal Maso and G. Buttazzo [2,3].

It happens that the standard Sobolev space $W^{1,p}((0,1);\mathbb{R}^n)$, being defined as the space of equivalent classes of vector functions, is not well adapted to the study of nonlocal problems of the above type, and it is much more convenient to conduct the study in the spaces of absolutely continuous vector functions. In this paper we use the spaces of absolutely continuous functions with the derivatives in some Orlicz spaces, which allow also to consider the functionals with rapidly (e.g. exponentially) growing integrands. After reiterating with minor changes and generalizations for this case the standard auxiliary results for classical functionals (without transformation of the argument) in section 4, we start with introducing the inner superposition (composition) operators which play the key role in the further discussion and study various types of their convergence in section 5. In the last section we relate the properties of convergence of inner superposition operators with Γ-convergence of the functionals with deviating argument thus solving the above-posed problem of stability of solutions. Developing the technique introduced in the paper we consider also some generalizations, e.g. for the functionals with impulsive constraints, which most often appear in the optimal control problems for impulsive functional differential equations.

2. Notation and preliminaries

Recall that the Fenchel conjugate $I'\colon X' \to \mathbb{R} \cup \{\pm\infty\}$ for the functional $I\colon X' \to \mathbb{R} \cup \{\pm\infty\}$ is defined by the formula

$$I'(x') := \sup_{x \in X}(\langle x, x'\rangle - I(x)).$$

For the general properties of the conjugate the reader is referred to [4, p. 35]. In what follows we also use the Fenchel conjugate in finite-dimensional space referring to it as the Legendre transformation.

Let $L^p(0,1)$ stand for the standard Lebesgue space of functions integrable on the interval $(0,1)$ with the power $1 \le p < \infty$ (or essentially bounded when $p = \infty$). In the sequel we make an extensive use of the space $L^p((0,1);\mathbb{R}^n)$ (further on denoted by L^p_n for the sake of brevity) of $\mathbb{R}^n$-valued vector functions with the components in $L^p(0,1)$, equipped by its usual norm. By AC^p_n denote the space of vector functions with absolutely continuous components and with the derivatives in L^p_n, equipped with the norm $||\boldsymbol{x}||_{AC^p_n} := |\boldsymbol{x}(0)| + ||\dot{\boldsymbol{x}}||_p$, where $|\cdot|$ stands for the norm in $\mathbb{R}^n$ (upon the necessity to emphasize the dimension of the space, the latter will be also denoted by $|\cdot|_n$). Also by $C_n[0,1]$ (C_n, for short) we denote the space of all $\mathbb{R}^n$-valued vector functions with continuous components.

When dealing with rapidly increasing (e.g. exponential) nonlinearities, it is often convenient to use Orlicz spaces. While introducing them here we follow the lines of [5, chapter 4]. Let P: $\mathbb{R} \to [0,+\infty)$ be the Young function (i.e. even, convex, lower semicontinuous with $P(0) = 0$). Of course, the trivial situations $P(x) \equiv 0$ and $P(x) \equiv +\infty$ are excluded. In the sequel we denote by L^P_n the Orlicz space of vector functions $\boldsymbol{x}$: $[0,1] \to \mathbb{R}^n$, the components of which are from the Orlicz space $L^P(0,1)$ generated by $P(x)$. Equip L^P_n with the usual Luxembourg norm

$$||\boldsymbol{x}||_P := \inf\left\{\lambda > 0 : \quad \int_0^1 P(|\boldsymbol{x}(\tau)|/\lambda)\, d\tau \leq 1\right\}.$$

Recall that the Lebesgue spaces L^p_n are particular examples of the Orlicz spaces introduced. The corresponding space of absolutely continuous vector functions AC^P_n is introduced in the same way as AC^p_n. In what follows we will use as well the notion of the subspace $\overset{\circ}{AC}{}^P_n$ (resp. $\overset{\circ}{AC}{}^p_n$) of AC^P_n (resp. AC^p_n) consisting of the functions satisfying the restriction $\boldsymbol{x}(0) = \boldsymbol{x}(1) = 0$.

Consider the associated Young function P' obtained by the Legendre transform of P [5, p. 122]. It generates the associate Orlicz space $L^{P'}_n$ which is closed in the topological dual $(L^P_n)'$ and coincides with the latter if and only if $P(x)$ is a Δ_2-function, that is, for each $\lambda > 0$ there is $\alpha > 0$ such that $P(\lambda x) \leq \alpha P(x)$. For the particular case $L^P_n = L^p_n$ one gets $L^{P'}_n = L^{p'}_n$, where $1/p + 1/p' = 1$. Furthermore, L^P_n is reflexive if both P and P' are Δ_2-functions [5, pp. 124-125]. Another important property of Orlicz spaces generated by Δ_2-functions is their separability.

Note that $AC^P_n \subset C_n$ (as sets). Using the standard arguments involving Young's inequality and the estimate for the Orlicz norm of a characteristic function of the set (formula (4.40) in [5]) one can easily show that the latter immersion is compact provided that $P^{-1}(x)/x \to 0$ when $x \to +\infty$. This surely holds when P' is a Δ_2-function. It is also worth noting that the inclusions $L^\infty_n \subset L^P_n \subset L^1_n$ hold for any Young function $P(x)$.

3. Basic facts about Γ-convergence

Throughout this section let (X, τ) be a topological space with a topology τ, I_ν: $X \to \bar{\mathbb{R}}$, $\nu \in \mathbb{N}$ be a sequence of functionals, I: $X \to \bar{\mathbb{R}}$.

Definition 3.1. I is called the (sequential) $\Gamma^-(\tau)$-limit of I_ν over X (written $I = \Gamma^-(\tau)\lim_\nu I_\nu$), if

$$I = \Gamma^-(\tau)\liminf{}_\nu I_\nu = \Gamma^-(\tau)\limsup{}_\nu I_\nu, \quad \text{where}$$
$$\Gamma^-(\tau)\liminf{}_\nu I_\nu(x) := \inf\{\liminf{}_\nu I_\nu(x_\nu): \quad x_\nu \xrightarrow{\tau} x\},$$
$$\Gamma^-(\tau)\limsup{}_\nu I_\nu(x) := \inf\{\limsup{}_\nu I_\nu(x_\nu): \quad x_\nu \xrightarrow{\tau} x\}.$$

From now on we restrict ourselves to the following 3 situations:

(C1) X is a Banach space with τ the topology of its norm.

(C2) X is a reflexive Banach space with separable topological dual X', τ is the weak topology.

(C3) $X = V'$ is the topological dual of a separable Banach space V, τ is the $*$-weak topology.

In the case (C1) we speak about strong Γ^--limits, in the cases (C2) and (C3) about weak and $*$-weak Γ^--limits, respectively. The reference to the topology τ will be omitted in the sequel whenever it cannot cause misunderstanding.

Recall that the functional I_0: $X \to \bar{\mathbb{R}}$ is said to be *coercive*, when $I_0(x) \to +\infty$ as $||x||_X \to \infty$. The sequence $\{I_\nu\}$ is then called *equicoercive*, if $I_\nu(x) \geq I_0(x)$ for each $\nu \in \mathbb{N}$. We will make an extensive use of following criterion for the Γ^--convergence which is the reformulation of the propositions 1.2 and 1.4 from [6].

Proposition 3.2. *In the case* (C1) $I = \Gamma^- \lim_\nu I_\nu$, *if and only if two conditions are satisfied simultaneously:*

$$\forall\{x_\nu\} \subset X,\ x \in X,\ x_\nu \to x \ \Rightarrow\ I(x) \leq \liminf_\nu I_\nu(x_\nu); \tag{1}$$

$$\forall x \in X,\ \exists\{x_\nu\} \subset X,\ x_\nu \to x \textit{ such that } I(x) = \lim_\nu I_\nu(x_\nu). \tag{2}$$

The same is true in the cases (C2) *and* (C3) *under the additional requirement that the sequence* $\{I_\nu\}$ *be equicoercive.*

The proposition below gives the well-known main property of Γ^--limits (see [2], corollary 7.20 and theorem 7.8, corollary 1.25).

Proposition 3.3. *Let* $I = \Gamma^- \lim_\nu I_\nu$ *and* $\{x_\nu\} \subset X$ *be such that*

$$\lim_\nu I_\nu(x_\nu) = \lim_\nu(\inf_X I_\nu).$$

I. *If* $x_\nu \to x$, *then* x *provides global minimum for* I *on* X, *while*

$$\lim_\nu(\inf_X I_\nu) = \min_X I.$$

II. *If* $\{I_\nu\}$ *is equicoercive,* $I \not\equiv +\infty$, *then* I *is coercive, while there exists a subsequence* $x_{\nu_\mu} \to x$, *the above being valid. Moreover, if* I *has unique point of minimum* x, *then* $x_\nu \to x$.

Remark 3.4. In particular, for x_ν one can take the points of global minima of the respective I_ν over X.

It is well-known that the weak Γ^--convergence of a sequence of functionals in a reflexive Banach space with separable dual is closely related to the strong Γ^--convergence of their Fenchel conjugates. To state the exact assertion to be used in the sequel, recall first that the functional I: $X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *proper*, if $I(x) \not\equiv \pm\infty$, and *lower semicontinuous*, if $x_\nu \to x$ in X implies $I(x) \leq \liminf_\nu I(x_\nu)$.

Proposition 3.5. *Let X be a reflexive Banach space with the separable dual X', while the functionals I, I_ν, I_0: $X \to \bar{\mathbb{R}}$ and their Fenchel conjugates I', I'_ν, I'_0: $X' \to \mathbb{R}\cup\{\pm\infty\}$ satisfy the following assumptions:*

(i) *I and all I_ν are convex and lower semicontinuous;*

(ii) *$I_\nu(x) \geq I_0(x)$ for all $x \in X$, where I_0 is coercive, I'_0 being proper;*

(iii) *$\lim_\nu I'_\nu(x') = I'(x')$ for all $x' \in X'$, the sequence $\{I'_\nu\}$ being equicontinuous.*

Then $I_\nu \to I$ Γ^--weakly in X.

Remark 3.6. In the condition (iii) it is enough to require that $\lim_\nu I'_\nu(x') = I'(x')$ only for each $x' \in D'$, where $\operatorname{cl} D' = X'$, whenever $\{I'_\nu\}$ is equilipschitzian.

Proof. The condition (ii) implies, by virtue of the proposition 5.9 of [2], that the pointwise convergence of I'_ν to I' over X', provided by (iii), is equivalent to the Γ^--convergence of I'_ν to I' in the strong (normed) topology of X'. From condition (i) follows that the second conjugates $I'' = I$ and $I''_\nu = I_\nu$. Thus, using the fact that all I'_ν are proper and the theorem 3.2.4 of [7], one yields the desired result. □

4. Auxiliary lemmata

Recall that g: $[0,1]\times\mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be a Carathéodory function, if $g(t, \boldsymbol{y})$ is continuous in $\boldsymbol{y} \in \mathbb{R}^n$ for a.e. $t \in [0,1]$ and Lebesgue measurable in $t \in [0,1]$ for all $\boldsymbol{y} \in \mathbb{R}^n$. Consider its Legendre transform

$$g'(t, \boldsymbol{y}') := \sup_{\boldsymbol{y}\in\mathbb{R}^n} (\boldsymbol{y}\boldsymbol{y}' - g(t, \boldsymbol{y})).$$

Introduce the functional J: $L^P_n \to \mathbb{R} \cup \{\pm\infty\}$ by the relationship

$$J(\boldsymbol{y}) = \int_0^1 g(t, \boldsymbol{y}(t))\, dt. \tag{3}$$

The statement below which gives the representation of its Fenchel conjugate, is the slight variation of the proposition 2.1 in [8, chapter IX].

Lemma 4.1. *Let $g(t, \boldsymbol{y})$ be a Carathéodory function, while the functional J given by (3) be proper in L^P_n. Then the conjugate J': $(L^P_n)' \to \mathbb{R} \cup \{\pm\infty\}$ of the latter admits representation $J'(\boldsymbol{y}') = \int_0^1 g'(t, \boldsymbol{y}'(t))\, dt$.*

Proof. According to the conditions, there exists $\boldsymbol{y}_0 \in L^P_n$ such that $J(\boldsymbol{y}_0) \neq \pm\infty$. Given $\boldsymbol{y}' \in (L^P_n)'$, consider the functions

$$g'_j(t) := \sup_{|\boldsymbol{y}|\leq j|\boldsymbol{y}_0(t)|} (\boldsymbol{y}\boldsymbol{y}'(t) - g(t,\boldsymbol{y})).$$

One observes that the sequence $\{g'_j\}$ is nondecreasing and convergent for a.e. $t \in [0,1]$ to $g'(t,\boldsymbol{y}'(t))$. Moreover, $g'_j(t) \geq \boldsymbol{y}_0(t)\boldsymbol{y}'(t) - g(t,\boldsymbol{y}_0(t))$ for all $j \in \mathbb{N}$, the right hand side being integrable due to the original assumption. According to the Krasnosel'skiĭ-Ladyzhenskiĭ lemma on measurable selections [5, theorem 6.2], for each $j \in \mathbb{N}$ there is a measurable vector function $\boldsymbol{y}_j\colon [0,1] \to \mathbb{R}^n$, such that $|\boldsymbol{y}_j(t)| \leq j|\boldsymbol{y}_0(t)|$ and

$$g'_j(t) = \boldsymbol{y}_j(t)\boldsymbol{y}'(t) - g(t,\boldsymbol{y}_j(t)).$$

Noting that $g'(t,\boldsymbol{y}'(t))$ is measurable as a limit of measurable functions and bounded from below by an integrable function, it is easy to conclude that

$$\begin{aligned}\int_0^1 g'(t,\boldsymbol{y}'(t))\,dt &= \sup_j \left(\int_0^1 (\boldsymbol{y}_j(t)\boldsymbol{y}'(t) - g(t,\boldsymbol{y}_j(t)))\,dt\right)\\ &\leq \sup_{x\in L^P_n} \left(\int_0^1 \big(\boldsymbol{y}(t)\boldsymbol{y}'(t) - g(t,\boldsymbol{y}(t))\big)\,dt\right) = J'(\boldsymbol{y}').\end{aligned}$$

The analogous upper estimate $J'(\boldsymbol{y}') \leq \int_0^1 g'(t,\boldsymbol{y}'(t))\,dt$ follows from Young's inequality. □

Consider the sequence of functionals $J_\nu\colon L^P_n \to \mathbb{R}\cup\{\pm\infty\}$ defined by the formula

$$J_\nu(\boldsymbol{y}) = \int_0^1 g_\nu(t,\boldsymbol{y}(t))\,dt. \tag{4}$$

Another auxiliary assertion to be used in the sequel concerns its pointwise convergence to a functional $J\colon L^P_n \to \mathbb{R}\cup\{\pm\infty\}$ given by the relationship (3), and follows closely lemma 3.1 of [6].

Lemma 4.2. *Let g_ν, g, g_0, $g^0\colon [0,1]\times\mathbb{R}^n \to \bar{\mathbb{R}}$ be Carathéodory functions satisfying the assumptions*

(i) *For a.e. $t \in [0,1]$, for all $\boldsymbol{y} \in \mathbb{R}^n$ holds one has $g^0(t,\boldsymbol{y}) \leq g_\nu(t,\boldsymbol{y}) \leq g_0(t,\boldsymbol{y})$, $g_\nu(t,\cdot)$ and $g(t,\cdot)$ being convex, while $g^0(\cdot,\boldsymbol{y})$, $g_0(\cdot,\boldsymbol{y}) \in L^1(0,1)$;*

(ii) *$g_\nu(\cdot,\boldsymbol{y}) \rightharpoonup g(\cdot,\boldsymbol{y})$ weakly in $L^1(0,1)$.*

Then $J_\nu(\boldsymbol{y}) \to J(\boldsymbol{y})$ for each $\boldsymbol{y} \in C_n$.

Proof. First observe that as $g^0(t,\cdot)$ and $g_0(t,\cdot)$ are finite, then $g_\nu(t,\cdot)$ and $g(t,\cdot)$ are equilipschitzian according to corollary 2.4 of [8, chapter I], that is, from $|\boldsymbol{y}_1| < R$, $|\boldsymbol{y}_2| < R$ follows that for a.e. $t \in [0,1]$

$$|g_\nu(t,\boldsymbol{y}_1) - g_\nu(t,\boldsymbol{y}_2)| \leq l_R(t)|\boldsymbol{y}_1 - \boldsymbol{y}_2|, \quad \text{where}$$

$$l_R(t) = \frac{\max_{|\boldsymbol{y}|\le R_0} g_0(t,\boldsymbol{y}) - \min_{|\boldsymbol{y}|\le R_0} g^0(t,\boldsymbol{y})}{R_0 - R}, \; R_0 > R.$$

Obviously $l_R \in L^1(0,1)$. It is also easy to show that $g(t,\boldsymbol{y})$ has the same upper and lower bounds, and therefore, it has the same estimate for a Lipschitz constant.
For $\boldsymbol{y} \in C_n$ consider the finite ε-net $\{\boldsymbol{y}_i\}_{i=1}^m \subset \mathbb{R}^n$ of $\boldsymbol{y}([0,1])$ and the respective open balls $B_\varepsilon(\boldsymbol{y}_i)$: $|\boldsymbol{y}-\boldsymbol{y}_i| < \varepsilon$ covering the latter. Let $\{h_i(t)\}$ stand for the partition of unity on $\boldsymbol{y}^{-1}(B_\varepsilon(\boldsymbol{y}_i))$ and denote $\Omega_i := [0,1] \cap \boldsymbol{y}^{-1}(B_\varepsilon(\boldsymbol{y}_i))$. Also let R stand for the radius of the ball covering all $\boldsymbol{y}([0,1])$. The estimate

$$|J_\nu(\boldsymbol{y}) - J(\boldsymbol{y})| \le \sum_{i=1}^m \left| \int_{\Omega_i} (g_\nu(t,\boldsymbol{y}(t)) - g(t,\boldsymbol{y}(t)))h_i(t)\,dt \right| \le$$
$$\sum_{i=1}^m \Bigg(\left| \int_{\Omega_i} (g_\nu(t,\boldsymbol{y}(t)) - g_\nu(t,\boldsymbol{y}_i))h_i(t)\,dt \right| + \left| \int_{\Omega_i} (g_\nu(t,\boldsymbol{y}_i) - g(t,\boldsymbol{y}_i))h_i(t)\,dt \right| +$$
$$\left| \int_{\Omega_i} (g(t,\boldsymbol{y}(t)) - g(t,\boldsymbol{y}_i))h_i(t)\,dt \right| \Bigg) \le (2l_R + 1)\varepsilon$$

for sufficiently large ν shows the statement. □

5. Convergence of sequences of inner superposition operators

In this section we discuss various notions of continuous convergence of a sequence of mappings A_ν: $X \to Y$ between two Banach spaces. In the sequel we will be particularly interested in studying the convergence of sequence of linear inner superposition operators $\{S_{g^\nu}\}$ acting on some space of vector functions $\boldsymbol{x}$: $[0,1] \to \mathbb{R}^n$ and defined formally by the relationships

$$(S_{g^\nu}\boldsymbol{x})(t) = \begin{cases} \boldsymbol{x}(g^\nu(t)), & g^\nu(t) \in [0,1], \\ 0, & g^\nu(t) \notin [0,1], \end{cases} \tag{5}$$

where g_ν: $[0,1] \to \mathbb{R} \cup \{\pm\infty\}$. Denote by S_g the operator

$$(S_g\boldsymbol{x})(t) = \begin{cases} \boldsymbol{x}(g(t)), & g(t) \in [0,1], \\ 0, & g(t) \notin [0,1], \end{cases} \tag{6}$$

with g: $[0,1] \to \mathbb{R} \cup \{\pm\infty\}$. From now on we assume that both $g(t)$ and all $g^\nu(t)$ are almost everywhere finite and measurable. For the sake of brevity we denote $g^0 := g$.
On the σ-algebra of measurable subsets $e \subset [0,1]$ define for all $\nu \in \mathbb{N} \cup \{0\}$ the functions $\mu_{g^\nu}(e) := \operatorname{meas}(g^\nu)^{-1}(e)$ and, when the latter are absolutely continuous with respect to the Lebesgue measure, the respective Radon-Nikodym derivatives $\frac{d\mu_{g^\nu}}{dm}$. The methods of the explicit calculation of the latter in various particular cases are given in [9, pp. 21-23]. In an analogous way, for each $\tau \in [0,1]$, $\nu \in \mathbb{N} \cup \{0\}$ consider the set functions $\mu_{g^\nu}(\tau,e) := \operatorname{meas}(g^\nu)^{-1}(e) \cap [0,\tau]$ and their Radon-Nikodym derivatives

$$\frac{d\mu_{g^\nu}(\tau)}{dm}(t) := \lim_{\varepsilon\to+0} \frac{\mu_{g^\nu}(\tau,[t-\varepsilon,t+\varepsilon])}{2\varepsilon}, \qquad \text{for a.e. } t \in (0,1).$$

Further on we as usual omit the superscript if $\nu = 0$. To calculate the above functions one can use the relationship

$$\frac{d\mu_{g^\nu}(\tau)}{dm}(t) = \frac{d}{dt}\,\mathrm{meas}\,(g^\nu)^{-1}([0,t]) \cap [0,\tau].$$

It is also worth noting that $\frac{d\mu_{g^\nu}(1)}{dm} \equiv \frac{d\mu_{g^\nu}}{dm}$. Also for every $\tau \in [0,1]$ consider the sets

$$\mathcal{E}_\nu(\tau) := (g^\nu)^{-1}([0,\tau]).$$

Further on we will write $\mathcal{E}(\tau)$ instead of $\mathcal{E}_0(\tau)$ and omit the reference to τ when $\tau = 1$. Let from now on $\mathcal{F}(X,Y)$ denote the set of all mappings between two Banach spaces X and Y, $\mathcal{L}(X,Y)$ stand for the space of linear bounded operators acting in the same spaces. If $X = Y$, we write as usual $\mathcal{F}(X)$, $\mathcal{L}(X)$ respectively.

5.1. Pointwise and continuous convergence

Definition 5.1. The sequence of maps $A_\nu \in \mathcal{F}(X,Y)$ is said to converge continuously to $A \in \mathcal{F}(X,Y)$, $A_\nu \overset{C}{\to} A$, if $\{x_\nu\} \subset X$, $x_\nu \to x \in X \Rightarrow A_\nu x_\nu \to Ax \in Y$.

One notes that whether A_ν are continuous or not, $A_\nu \overset{C}{\to} A$ implies that A is continuous (see Kuratowski [10], Chapter II, § 20, Section VII). Moreover, if all A_ν are linear, then it also implies their boundedness as well as the linearity and boundedness of the limit operator A. Furthermore, it is well-known that for linear operators between two Banach spaces continuous convergence is equivalent to the *strong (pointwise)* convergence.
Turn now to the particular case of a sequence of inner superposition operators. At first, we study the strong (pointwise) and, therefore, continuous convergence of the sequence $S_{g^\nu}\colon C_n \to L^P_n$ to the $S_g\colon C_n \to L^P_n$. Observe that for these operators to be well-defined by the formulae (5) and (6) respectively it suffices to require the measurability of all g_ν and g. The assertion below extends the analogous one for Lebesgue space (see theorem 2.1 in § 4.2 of [9]).

Proposition 5.2. *Let the Young function P satisfy the Δ_2-condition. The sequence of operators $S_{g^\nu}\colon C_n \to L^P_n$ defined by (5) converges pointwise to $S_g\colon C_n \to L^P_n$ defined by (6) if and only if the following two assumptions hold as $\nu \to \infty$:*
(i) *$g^\nu \to g$ in measure on $\mathcal{E}$;*
(ii) $\mathrm{meas}\,\mathcal{E}_\nu \Delta \mathcal{E} \to 0$, *where Δ stands for symmetric difference between the sets.*

Remark 5.3. The same result is valid for $L^P_n = L^\infty_n$ if the assumptions are replaced by
(i) $g^\nu \to g$ in L^∞ norm on $\mathcal{E}$;
(ii) $\mathrm{meas}\,\mathcal{E}_\nu \Delta \mathcal{E} = 0$ starting from some $\nu \in \mathbb{N}$.

Proof. The necessity has been proved in the theorem 2.1 in § 4.2 of [9]. To prove sufficiency recall that according to the same theorem $S_{g^\nu}\boldsymbol{x} \to S_g\boldsymbol{x}$ in measure for any $\boldsymbol{x} \in C_n$ under the conditions (i) and (ii). To show the norm convergence note that

$$||S_{g^\nu}\boldsymbol{x} - S_g\boldsymbol{x}||_P = \inf \lambda,$$

where infimum is taken over all $\lambda > 0$ such that

$$\int\limits_{\mathcal{E}_\nu \setminus \mathcal{E}} P\left(\left|\left(S_{g^\nu}\boldsymbol{x}\right)(t)\right|/\lambda\right)\, dt + \int\limits_{\mathcal{E}\setminus \mathcal{E}_\nu} P\left(\left|\left(S_{g}\boldsymbol{x}\right)(t)\right|/\lambda\right)\, dt +$$
$$\int\limits_{\mathcal{E}\cap \mathcal{E}_\nu} P\left(\left|\left(S_{g^\nu}\boldsymbol{x}\right)(t) - \left(S_{g}\boldsymbol{x}\right)(t)\right|/\lambda\right)\, dt \leq 1.$$

For each fixed $\lambda > 0$ the first two integrals in this sum tend to zero when $\nu \to \infty$ due to the the boundedness by the value $P(\max_{[0,1]} |\boldsymbol{x}(t)|/\lambda)\operatorname{meas} \mathcal{E}\Delta\mathcal{E}_\nu$, the last one tending to zero by virtue of the Lebesgue monotone convergence theorem, which shows the statement. □

For the sake of completeness we mention here that the strong convergence of the sequence of inner superposition operators acting between two Lebesgue spaces has been studied in [11].

5.2. *Weak pointwise convergence*

Definition 5.4. The sequence of maps $A_\nu \in \mathcal{F}(X,Y)$ is said to converge weakly pointwise to $A \in \mathcal{F}(X,Y)$, if for all $x \in X$ $A_\nu x \rightharpoonup Ax \in Y$.

This type of convergence is clearly strictly weaker than the pointwise convergence unless Y is a finite-dimensional space. However, for a sequence of linear operators it still implies their boundedness as well the linearity and the boundedness of the limit operator A. One notes also that the weak pointwise convergence of the linear operators A_ν to A is equivalent to the weak pointwise convergence of their adjoints A'_ν: $Y' \to X'$ to A': $Y' \to X'$.
We discuss now the case of inner superposition operators.

Proposition 5.5. *Let the Young functions P and Q satisfy the Δ_2-condition, while $\Phi(x) = P(Q^{-1}(x))$ be the Young function with the Legendre transform Φ'. The sequence of operators S_{g^ν}: $L^P_n \to L^Q_n$ defined by* (5) *converges weakly pointwise to S_g: $L^P_n \to L^Q_n$ defined by* (6), *if*

(i) $\exists A : \left\|\frac{d\mu_{g^\nu}}{dm}\right\|_{\Phi'} \leq A$ *for all* $\nu \in \mathbb{N}\cup\{0\}$;

(ii) $\operatorname{meas}\mathcal{E}_\nu(\tau)\Delta\mathcal{E}(\tau) \to 0$ *when* $\nu\to\infty$ *for any* $\tau\in[0,1]$.

Remarks 5.6.

1. Under the conditions of this proposition necessarily $L^P_n \subset L^Q_n$.
2. The same result holds when instead of Q its Legendre transform Q' satisfies the Δ_2-condition. In this case weak convergence should be replaced by the $*$-weak one.

Proof. It is enough to show that

(I) The sequence S_{g^ν} has uniformly bounded norms;

(II) $S_{g^\nu}\boldsymbol{x} \rightharpoonup S_g\boldsymbol{x}$ for all $\boldsymbol{x}\in X_0$, where $\operatorname{span} X_0$ is dense in L^P_n.

Clearly, (I) follows from (i). One should demonstrate also that (ii) implies (II). For this purpose take

$$X_0 = \left\{\chi_{[0,\tau]}\boldsymbol{e}_j \mid \tau\in[0,1], j = 1,\ldots,n\right\} \subset L^P_n,$$

where $\chi_{[0,\tau]}$ stands for the characteristic function of the interval $[0,\tau]$, $\boldsymbol{e}_j$ is the j-th coordinate unit vector in $\mathbb{R}^n$. Observe that $\operatorname{cl\,span} X_0 = L_n^P$, because P is a Δ_2-function. Taking into account that

$$S_{g^\nu}\chi_{[0,\tau]}\boldsymbol{e}_j = \chi_{\mathcal{E}_\nu(\tau)}\boldsymbol{e}_j \quad \text{and} \quad S_g\chi_{[0,\tau]}\boldsymbol{e}_j = \chi_{\mathcal{E}(\tau)}\boldsymbol{e}_j,$$

and that for any $f \in L^{Q'}(0,1)$

$$\left|\int_0^1 \chi_{\mathcal{E}_\nu(\tau)}(t)f(t)\,dt - \int_0^1 \chi_{\mathcal{E}(\tau)}(t)f(t)\,dt\right| \le \left|\int_{\mathcal{E}_\nu(\tau)\setminus\mathcal{E}(\tau)} f(t)\,dt\right| + \left|\int_{\mathcal{E}(\tau)\setminus\mathcal{E}_\nu(\tau)} f(t)\,dt\right| \to 0,$$

according to (ii), one shows (II). □

It is clear from the proof that condition (ii) of the above proposition is also necessary for the weak pointwise convergence of the sequence of inner superposition operators between two Orlicz spaces. In fact, the choice $\boldsymbol{x} = \chi_{[0,\tau]}\boldsymbol{e}_j$ implies then $\chi_{\mathcal{E}_\nu(\tau)} \rightharpoonup \chi_{\mathcal{E}(\tau)}$ in $L^Q(0,1)$, wherefrom (ii) follows. On the contrary, condition (i) is only sufficient. For instance, if $P \equiv Q$, then $L^{\Phi'}(0,1) = L^\infty(0,1)$, while the uniform essential boundedness of $\frac{d\mu_{g^\nu}}{dm}$ is generally speaking not necessary for the uniform boundedness of the operators S_{g^ν}. However for the particular case of Lebesgue spaces we can improve at this point proposition 5.5.

Corollary 5.7. *The sequence of operators S_{g^ν}: $L_n^p \to L_n^q$, $1 < q \le p < \infty$, defined by* (5)*, converges weakly pointwise to S_g: $L_n^p \to L_n^q$ defined by* (6) *if and only if*

(i) $\exists A : \left\|\frac{d\mu_{g^\nu}}{dm}\right\|_{p/(p-q)} \le A$;

(ii) $\operatorname{meas}\mathcal{E}_\nu(\tau)\Delta\mathcal{E}(\tau) \to 0$ *when* $\nu \to \infty$ *for any* $\tau \in [0,1]$.

Proof. One observes that if $L_n^P = L_n^p$, $L_n^Q = L_n^q$, $1 < q \le p < \infty$, then $L_n^{\Phi'} = L_n^{p/(p-q)}$. It remains to note that the condition (i) is also necessary for the uniform boundedness of the operators S_{g^ν} (which is an immediate consequence of theorem 1 of [12]). □

5.3. Weak continuous convergence

Definition 5.8. The sequence of maps $A_\nu \in \mathcal{F}(X,Y)$ is said to converge weakly continuously to $A \in \mathcal{F}(X,Y)$, $A_\nu \xrightarrow{C_{\sigma}} A$, if $\{x_\nu\} \subset X$, $x_\nu \rightharpoonup x \in X \Rightarrow A_\nu x_\nu \rightharpoonup Ax \in Y$.

Obviously this type of convergence implies weak pointwise convergence. The reverse however is not true even for linear operators. Consider for example, the sequence of linear operators A_ν: $L^2(0,1) \to L^2(0,1)$, $(A_\nu x)(t) = x(t)\sin\nu\pi t$. It converges weakly pointwise to the zero operator, while does not converge in the weak continuous sense. To show the latter, take the sequence $x_\nu(t) = \sin\nu\pi t$ (to be more precise, the restrictions of the above functions to the interval $[0,1]$), and observe that $A_\nu x_\nu$ does not converge weakly to zero in $L^2(0,1)$. One also notes that due to the statement of Kuratowski ([10], Chapter II, § 20, Section VII) from $A_\nu \xrightarrow{C_{\sigma}} A$ follows the weak continuity of A (i. e. A maps any weakly convergent sequence into a weakly convergent one).

For linear operators it is obvious that for $A_\nu \xrightarrow{C_{\mathfrak{T}}} A$ in $\mathcal{L}(X,Y)$ it is sufficient that $A'_\nu \to A'$ strongly (pointwise) in $\mathcal{L}(Y',X')$. If, in addition, X is reflexive and Y is uniformly convex (in particular, Hilbert) space, then this condition is also necessary. To show this, assume $A_\nu \xrightarrow{C_{\mathfrak{T}}} A$ in $\mathcal{L}(X,Y)$ and consider an arbitrary subsequence of a sequence of operators $\{A_\nu\}$ (for brevity we will use the same index ν). As $A'_\nu y' \rightharpoonup A'y'$ for each $y' \in Y'$, then $||A'y'||_{X'} \leq \liminf_\nu ||A'_\nu y'||_{X'}$. On the other hand, according to the Hahn-Banach theorem there is such a sequence $\{x_\nu\} \in X$, that $||x_\nu||_X = 1$ and $< x_\nu, A'_\nu y' >= ||A'_\nu y'||_{X'}$. Extracting a weakly converging subsequence $x_{\nu_\mu} \rightharpoonup x$, $||x||_X \leq 1$, and recalling that the continuous convergence of a sequence of operators implies the continuous convergence of any its subsequence to the same limit ([10], Chapter II, § 20, Section VII), one sees that $||A'_{\nu_\mu} y'||_{X'} \to \langle x, A'y' \rangle \leq ||A'y'||_{X'}$ and hence $A'_{\nu_\mu} y' \to A'y'$. Therefore, the whole weakly convergent sequence $\{A'_\nu y'\} \subset X'$ is compact and thus $A'_\nu y' \to A'y'$.

Now return to the particular case of inner superposition operators acting between two Orlicz spaces.

Proposition 5.9. *Let both the Young functions P, Q and their Legendre transforms P', Q' satisfy the Δ_2-condition, while $\Phi(x) = P(Q^{-1}(x))$ be the Young function with the Legendre transform Φ'. If the following requirements are satisfied*

(i) $\exists A > 0$: $||\frac{d\mu_{g^\nu}}{dm}||_{\Phi'} \leq A$ *for all* $\nu \in \mathbb{N} \cup \{0\}$;

(ii) $\left\|\frac{d\mu_{g^\nu}(\tau)}{dm}\right\|_{P'} \to \left\|\frac{d\mu_g(\tau)}{dm}\right\|_{P'}$ *for each* $\tau \in [0,1]$;

(iii) $\operatorname{meas} \mathcal{E}_\nu(\tau) \Delta \mathcal{E}(\tau) \to 0$ *when* $\nu \to \infty$ *for each* $\tau \in [0,1]$,

then $S_{g^\nu} \xrightarrow{C_{\mathfrak{T}}} S_g$ *in* $\mathcal{L}(L_n^P, L_n^Q)$. *Moreover, if* $L_n^P = L_n^p$, $L_n^Q = L_n^q$, $1 < q \leq p < +\infty$, *then also the reverse is true.*

Proof. We will show that the adjoint operators S'_{g^ν}: $L_n^{Q'} \to L_n^{P'}$ converge strongly (pointwise) to S'_g: $L_n^{Q'} \to L_n^{P'}$. For this purpose it is enough to prove that

(I) The sequence S_{g^ν} has uniformly bounded norms;

(II) $S'_{g^\nu}\boldsymbol{y}' \to S'_g\boldsymbol{y}'$ for all $\boldsymbol{y} \in X_0 \subset L_n^{Q'}$, where X_0 is defined in the proof of the proposition 5.5.

Obviously (I) holds due to the fulfillment of the condition (i). Now note that the operators S_{g^ν} converge to S_g weakly pointwise due to the proposition 5.5, and thus

$$S'_{g^\nu}\chi_{[0,\tau]}\boldsymbol{e}_j = \frac{d\mu_{g^\nu}(\tau)}{dm}\boldsymbol{e}_j \rightharpoonup \frac{d\mu_g(\tau)}{dm}\boldsymbol{e}_j = S'_g\chi_{[0,\tau]}\boldsymbol{e}_j,$$

which together with (ii) implies (II).

It is clear that the conditions (ii) and (iii) are also necessary for the weak continuous convergence of the operators S_{g^ν} to S_g. However, condition (i) is necessary only in the case of Lebesgue spaces, which can be proved with reference to theorem 1 of [12]. □

5.4. Strong continuous and uniform convergence

Definition 5.10. The sequence of maps $A_\nu \in \mathcal{F}(X,Y)$ is said to converge strongly continuously to $A \in \mathcal{F}(X,Y)$, $A_\nu \overset{C}{\rightharpoonup} A$, if $\{x_\nu\} \subset X$, $x_\nu \rightharpoonup x \in X \Rightarrow A_\nu x_\nu \to Ax \in Y$.

Of course, the strong continuous convergence implies all the other types of convergences discussed before. The reference to Kuratowski ([10], Chapter II, § 20, Section VII) also shows that from $A_\nu \overset{C}{\rightharpoonup} A$ follows the strong continuity of A (i. e. A maps any weakly convergent sequence into a norm convergent one). In particular, if X is reflexive, then A is necessarily compact and continuous. Furthermore, one can easily show (see again [10, p. 109]) that in the case of reflexive X the strong continuous convergence implies the *uniform* convergence.

It has been shown (see Theorem 3.3 in § 1.3 of [9]) that an inner superposition operator acting between two Orlicz spaces cannot be compact and continuous unless it is identically zero. Hence one concludes that the sequence of inner superposition operators defined on a reflexive Orlicz space cannot converge strongly continuously to an inner superposition operator different from zero.

Finally we point out that if the sequence $\{A_\nu\}$ is compact in totality (i. e. $\cup_\nu A_\nu(B)$ is compact in Y for any bounded $B \in X$), then $A_\nu \overset{C_\sigma}{\rightharpoonup} A$ implies $A_\nu \overset{C}{\rightharpoonup} A$.

5.5. Condition T

In the sequel we need also the following property of the sequence of mappings $\{A_\nu\} \subset \mathcal{F}(X,Y)$, closely related to the property of weak continuous convergence:

$$A_\nu x_\nu \rightharpoonup y \in Y \Rightarrow \exists x \in X : \ x_\nu \rightharpoonup x. \tag{T}$$

The statement below concerns the verification of this condition for the sequence of operators $\{S_{g^\nu}\}$ defined by the expression (5) and acting in some Orlicz space.

Proposition 5.11. *Assume that the Young function P and its Legendre transform P' satisfy the Δ_2-condition. Let for the operator S_g: $L_n^P \to L_n^P$ and the sequence of operators S_{g^ν}: $L_n^P \to L_n^P$ the following conditions hold:*

(i) *There is a measurable function γ: $g(\mathcal{E}) \to \mathcal{E}$ satisfying $\gamma(g(t)) = t$ for a.e. $t \in \mathcal{E}$, while*

$$e \subset g(\mathcal{E}),\ \mathrm{meas}\, e = 0 \ \Rightarrow\ \mathrm{meas}\, \gamma^{-1}(e) = 0;$$

(ii) *There are numbers a, A such that for all $\nu \in \mathbb{N}$ a.e. on $[0,1]$ holds*

$$0 < a \le \frac{d\mu_{g^\nu}}{dm} \le A,\ 0 < a \le S_g \frac{d\mu_g}{dm} \le A;$$

(iii) *Conditions* (ii) *and* (iii) *of proposition* 5.9 *hold.*

Then the sequence $\{S_{g^\nu}\}$ satisfies property (T).

Remark 5.12. The condition (i) is usually referred to as the ω-condition (see [9, pp. 42–44]).

Proof. Let $S_{g^\nu}\boldsymbol{x}_\nu \rightharpoonup \boldsymbol{y} \in L_n^P$ for some $\{\boldsymbol{x}_\nu\} \subset L_n^P$. Thus there is a constant $C > 0$ satisfying $||S_{g^\nu}\boldsymbol{x}_\nu||_P \le C$. It is easy to show that condition (ii) implies the uniform

boundedness from below of the operator norms of S_{g^ν} and thus the uniform boundedness of the sequence $\{\boldsymbol{x}_\nu\}$.
On the other hand, it is easy to show by the method used in [13], that due to the conditions (i) and (ii) the operator S_g is continuously invertible in L_n^P. Set then $\boldsymbol{x} = S_g^{-1}\boldsymbol{y}$.
According to the conditions $S_{g^\nu} \xrightarrow{C_g} S_g$ in L_n^P. Furthermore, since $\{\boldsymbol{x}_\nu\}$ is bounded and thus weakly precompact, $(S_{g^\nu} - S_g)\boldsymbol{x}_\nu \rightharpoonup 0$. Minding that $S_{g^\nu}\boldsymbol{x}_\nu \rightharpoonup S_g\boldsymbol{x}$, we obtain $S_g\boldsymbol{x}_\nu \rightharpoonup S_g\boldsymbol{x}$, which implies $\boldsymbol{x}_\nu \rightharpoonup \boldsymbol{x}$ due to the continuous invertibility of S_g. □

6. Γ-convergence for functionals with transformed argument

We start this section with a simple abstract statement which relates various types of convergence of mappings introduced in section 5 with the weak Γ^--convergence of functionals with transformed argument. Let X, Y be reflexive Banach spaces with separable duals and consider the functionals I, I_ν: $Y \to \bar{\mathbb{R}}$. Recall that the linear operator A: $X \to Y$ is called *correctly solvable*, if $||Ax||_Y \geq C||x||_X$ for all $x \in X$ and for some constant $C > 0$.

Proposition 6.1. *Assume that the sequence I_ν be equicoercive and equicontinuous (in norm), while $I_\nu \to I$ Γ^--weakly over Y. If the sequence of linear uniformly in ν correctly solvable operators A_ν: $X \to Y$ with dense images in Y satisfies $A_\nu \xrightarrow{C_g} A$ in $\mathcal{L}(X, Y)$ and condition (T), then $I_\nu(A_\nu(\cdot)) \to I(A(\cdot))$ Γ^--weakly over X.*

Remark 6.2. It will be clear from the proof, that if in addition every A_ν is surjective, then the statement is true without the requirement of equicontinuity of I_ν.

Proof. From proposition 3.2 follows that

$$y_\nu \rightharpoonup y \text{ in } Y \Rightarrow I(y) \leq \liminf_\nu I_\nu(y_\nu), \tag{α}$$

$$\exists\{\bar{y}_\nu\} \subset Y,\, \bar{y}_\nu \rightharpoonup y \text{ in } Y \Rightarrow I(y) = \lim_\nu I_\nu(\bar{y}_\nu) \tag{β}$$

for every $y \in Y$. The weak continuous convergence of the operators A_ν to A and the property (α) imply

$$x_\nu \rightharpoonup x \text{ in } X \Rightarrow I(Ax) \leq \liminf_\nu I_\nu(Ax_\nu). \tag{α_1}$$

On the other hand, given $x \in X$, according to (β) there is a sequence $y_\nu \rightharpoonup Ax$ satisfying $I(Ax) = \lim_\nu I_\nu(y_\nu)$. Using the density of the images of each A_ν in Y, one finds a sequence $\{\bar{x}_\nu\} \subset X$, such that $||A_\nu\bar{x}_\nu - y_\nu||_Y \to 0$ when $\nu \to \infty$. Obviously $A_\nu\bar{x}_\nu \rightharpoonup Ax$. Moreover, the estimate

$$|I(Ax) - I_\nu(A_\nu\bar{x}_\nu)| \leq |I(Ax) - I_\nu(y_\nu)| + |I_\nu(A_\nu\bar{x}_\nu) - I_\nu(y_\nu)|$$

together with condition (T) provide that $\bar{x}_\nu \rightharpoonup x$ and

$$I(Ax) = \lim_\nu I_\nu(A_\nu\bar{x}_\nu). \tag{β_1}$$

To verify the equicoercivity of $I_\nu(A_\nu(\cdot))$, it is enough to check according to proposition 7.7 of [2] that the sublevels of the latter are uniformly bounded in X. This can be observed as an immediate consequence of the uniform correct solvability of A_ν. To conclude the proof, it remains to refer to proposition 3.2. □

One should point out that the conditions on the sequence $\{A_\nu\}$ used in the above result are in general not necessary for the Γ^--convergence of the respective functionals with transformed argument. To illustrate this consider a particular example of a functional J and the sequence of its perturbations J_ν

$$J(x) = \int_0^1 (S_g x)^2 (t)\, dt, \qquad J_\nu(x) = \int_0^1 (S_{g^\nu} x)^2 (t)\, dt,$$

where $x \in L^2(0,1)$, S_{g^ν} and S_g are the inner superposition operators as introduced by the relationships (5) and (6) respectively. Changing the variable under the integration sign and applying proposition 3.5 with lemmata 4.1 (representation of Fenchel conjugate) and 4.2 (pointwise convergence of the conjugates) one obtains that for J_ν to converge Γ^--weakly to J on $L^2(0,1)$ it is sufficient that

$$\exists a : \frac{d\mu_{g^\nu}}{dm} \geq a > 0 \quad \text{and} \quad \left(\frac{d\mu_{g^\nu}}{dm}\right)^{-1} \rightharpoonup \left(\frac{d\mu_g}{dm}\right)^{-1} \quad \text{weakly in } L^1(0,1) \tag{7}$$

which is certainly much less than required by proposition 6.1.

6.1. Functionals with deviating argument

Now return to the original problem (V_∞) and its perturbations (V_ν). The Legendre transforms of f_ν, f in the last variables are given by the expressions

$$f'(t, \boldsymbol{u}, \boldsymbol{v}') := \sup_{\boldsymbol{v} \in \mathbb{R}^{nl}} (\boldsymbol{v}\boldsymbol{v}' - f(t, \boldsymbol{u}, \boldsymbol{v})),$$
$$f'_\nu(t, \boldsymbol{u}, \boldsymbol{v}') := \sup_{\boldsymbol{v} \in \mathbb{R}^{nl}} (\boldsymbol{v}\boldsymbol{v}' - f_\nu(t, \boldsymbol{u}, \boldsymbol{v})),$$

Introduce the following set of conditions.

(P0) The Young functions P, Q and their Legendre transforms P', Q' satisfy the Δ_2-condition.

(Q1) f, f_ν: $[0,1] \times \mathbb{R}^{nk} \times \mathbb{R}^{nl} \to \bar{\mathbb{R}}$ are Carathéodory functions, such that $f_\nu(t, \boldsymbol{u}, \cdot)$, $f(t, \boldsymbol{u}, \cdot)$ are convex, while for a.e. $t \in [0,1]$, for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^{nk} \times \mathbb{R}^{nl}$, $\boldsymbol{v}' \in \mathbb{R}^{nl}$ and for some $\bar{\boldsymbol{v}} \in \mathbb{R}^{nl}$, $a_j \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l a_j^2 > 0$, $b \in L^1(0,1)$ holds

$$f_\nu(t, \boldsymbol{u}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_l) \geq \sum_{j=1}^l a_j P(|\boldsymbol{v}_j|) + b(t),$$
$$f_\nu(t, \boldsymbol{u}, \bar{\boldsymbol{v}}) \leq c(t, \boldsymbol{u}),$$

where c: $[0,1] \times \mathbb{R}^{nk} \to \bar{\mathbb{R}}$ is some Carathéodory function, $c(\cdot, \boldsymbol{u}) \in L^1(0,1)$.

(Q2) For a.e. $t \in [0,1]$, for each $\{\boldsymbol{u}_1, \boldsymbol{u}_2\} \subset \mathbb{R}^{nk}$, $\boldsymbol{v} \in \mathbb{R}^{nl}$ holds

$$|f_\nu(t, \boldsymbol{u}_1, \boldsymbol{v}) - f_\nu(t, \boldsymbol{u}_2, \boldsymbol{v})| \leq \omega(t, |\boldsymbol{u}_1 - \boldsymbol{u}_2|_{nk}),$$

where ω: $[0,1]\times[0,+\infty)\to[0,+\infty)$ is a Carathéodory function nondecreasing in the second variable, with $\omega(t,0)\equiv 0$, $\omega(\cdot,u)\in L^1(0,1)$ for each $u>0$.

(Q3) For any $u\in L^Q_{nk}$ the functionals

$$J_\nu(\boldsymbol{v})=\int_0^1 f_\nu(t,\boldsymbol{u}(t),\boldsymbol{v}(t))\,dt \text{ and } J(\boldsymbol{v})=\int_0^1 f(t,\boldsymbol{u}(t),\boldsymbol{v}(t))\,dt$$

are proper in the space L^P_{nl};

(Q4) $f'_\nu(\cdot,\boldsymbol{u},\boldsymbol{v}')\rightharpoonup f'(\cdot,\boldsymbol{u},\boldsymbol{v}')$ weakly in $L^1(0,1)$ for all $\{\boldsymbol{u},\boldsymbol{v}'\}\in\mathbb{R}^{nk}\times\mathbb{R}^{nl}$.

In the sequel we denote for the sake of brevity $h_i^0:=g_i$, $g_j^0:=g_j$. The notations below refer to $\nu\in\mathbb{N}\cup\{0\}$. Define the inner superposition operators $T_{g_i^\nu}$: $C_n\to L^Q_n$, $i=1,\dots,k$ and $S_{g_j^\nu}$: $L^P_n\to L^P_n$, $j=1,\dots,l$ by the relationships

$$(T_{h_i^\nu}\boldsymbol{x})(t)=\begin{cases}\boldsymbol{x}(h_i^\nu(t)), & h_i^\nu(t)\in[0,1],\\ 0, & h_i^\nu(t)\notin[0,1],\end{cases}\qquad (S_{g_j^\nu}\boldsymbol{x})(t)=\begin{cases}\boldsymbol{x}(g_j^\nu(t)), & g_j^\nu(t)\in[0,1],\\ 0, & g_j^\nu(t)\notin[0,1].\end{cases}$$

Also denote

$$T^\nu_{\boldsymbol{h}}:\ \boldsymbol{x}\in C_n\mapsto\left(T_{h_1^\nu}\boldsymbol{x},\dots,T_{h_k^\nu}\boldsymbol{x}\right)\in L^Q_{nk},\quad S^\nu_{\boldsymbol{g}}:\ \boldsymbol{x}\in L^P_n\mapsto\left(S_{g_1^\nu}\boldsymbol{x},\dots,S_{g_l^\nu}\boldsymbol{x}\right)\in L^P_{nl}.$$

Let for each $\tau\in[0,1]$, $i=1,\dots,k$, $j=1,\dots,l$,

$$\mathcal{E}_i^\nu\ :=\ (h_i^\nu)^{-1}([0,1]),\quad E_j^\nu(\tau)\ :=\ \left(g_j^\nu\right)^{-1}([0,\tau]).$$

Again we omit the reference to τ when $\tau=1$ and to ν when $\nu=0$.
For the above-introduced operators consider the following assumptions.

(R1) For any $j=1,\dots,l$ the functions g_j, g_j^ν: $[0,1]\to\mathbb{R}\cup\{\pm\infty\}$ are almost everywhere finite and measurable, while

$$\exists(a,A):\ 0<a\le\frac{d\mu_{g_j^\nu}}{dm}(t)\le A \text{ a.e. on } [0,1],\quad \nu\in\mathbb{N}\cup\{0\} \text{ and}$$
$$\left\|\frac{d\mu_{g_j^\nu}(\tau)}{dm}\right\|_{P'}\to\left\|\frac{d\mu_{g_j}(\tau)}{dm}\right\|_{P'}\quad\text{as } \nu\to\infty.$$

(R2) $\operatorname{meas}E_j^\nu=1$, while there are measurable functions γ_j^ν: $\mathbb{R}\to[0,1]$, $j=1,\dots,l$, $\nu\in\mathbb{N}\cup\{0\}$ such that $e\subset[0,1]$, $\operatorname{meas}e=0$ implies $\operatorname{meas}\left(\gamma_j^\nu\right)^{-1}(e)=0$, and

$$\gamma_j^\nu(g_j^\nu(t))=t \text{ for a.e. } t\in E_j^\nu.$$

(R3) For all $i=1,\dots,k$ the functions h_i, h_i^ν: $[0,1]\to\mathbb{R}\cup\{\pm\infty\}$ are almost everywhere finite and measurable, while $h_i^\nu\to h_i$ in measure on $\mathcal{E}_i$.

(R4) meas $\mathcal{E}_i^\nu \Delta \mathcal{E}_i \to 0$ for each $i = 1, \ldots, k$ and meas $E_j^\nu(\tau) \Delta E_j(\tau) \to 0$ for any $\tau \in [0,1]$, $j = 1, \ldots, l$.

Theorem 6.3. *Consider the functionals I, I_ν: $\overset{\circ}{AC}{}_n^P \to \bar{\mathbb{R}}$ defined by the expressions* (V_∞) *and* (V_ν) *respectively. Under the assumptions* (P0), (Q1)–(Q4), (R1)–(R4) *the following statement holds: if $\{\boldsymbol{x}_\nu\} \subset \overset{\circ}{AC}{}_n^P$ is such that*

$$\lim_\nu I_\nu(\boldsymbol{x}_\nu) = \lim_\nu (\inf_{\overset{\circ}{AC}{}_n^P} I_\nu).$$

(in particular, when each $\boldsymbol{x}_\nu$ is the point of global minimum of the respective I_ν over $\overset{\circ}{AC}{}_n^P$) and $\boldsymbol{x}_\nu \rightharpoonup \boldsymbol{x}$, then

$$\lim_\nu (\inf_{\overset{\circ}{AC}{}_n^P} I_\nu) = \min_{\overset{\circ}{AC}{}_n^P} I = I(\boldsymbol{x}).$$

Remarks 6.4.

1. If the sequence $\{\boldsymbol{x}_\nu\}$ is bounded, then it contains a weakly convergent subsequence $\{\boldsymbol{x}_{\nu_\mu}\}$ for which the statement of the theorem is valid.
2. The statement is valid in particular in $\overset{\circ}{AC}{}_n^p$, $1 < p < \infty$.

Proof.
Step 1. Consider the functionals G_ν, G: $L_{nk}^Q \times L_{nk}^P \to \bar{\mathbb{R}}$ defined by the formulae

$$G_\nu(\boldsymbol{u}, \boldsymbol{v}) = \int_0^1 f_\nu(t, \boldsymbol{u}(t), \boldsymbol{v}(t))\, dt \quad \text{and} \quad G(\boldsymbol{u}, \boldsymbol{v}) = \int_0^1 f(t, \boldsymbol{u}(t), \boldsymbol{v}(t))\, dt.$$

We will prove (see, for instance, the analogous results of G. Buttazzo and G. Dal Maso [3]) that due to conditions (P0), (Q1)-(Q4) from $\boldsymbol{u}_\nu \to \boldsymbol{u}$ in L_{nk}^Q follows

$$\boldsymbol{v}_\nu \rightharpoonup \boldsymbol{v} \text{ in } L_{nl}^P \;\Rightarrow\; G(\boldsymbol{u}, \boldsymbol{v}) \leq \liminf_\nu G_\nu(\boldsymbol{u}_\nu, \boldsymbol{v}_\nu); \tag{A}$$

$$\exists \{\boldsymbol{v}_\nu\} \subset L_{nl}^P \text{ such that } \boldsymbol{v}_\nu \rightharpoonup \boldsymbol{v} \in L_{nl}^P \text{ and } G(\boldsymbol{u}, \boldsymbol{v}) = \lim_\nu G_\nu(\boldsymbol{u}_\nu, \boldsymbol{v}_\nu), \tag{B}$$

where in (B) the the sequence $\{\boldsymbol{v}_\nu\}$ is independent of the choice of $\{\boldsymbol{u}_\nu\}$. These properties will be proved below in step 4.
Step 2. Turn now to the original functionals I_ν, I and prove properties (1), (2) of proposition 3.2. Suppose $\boldsymbol{x}_\nu \rightharpoonup \boldsymbol{x}$ in $\overset{\circ}{AC}{}_n^P$. Let then

$$\boldsymbol{v}_\nu := S_{\boldsymbol{g}^\nu} \dot{\boldsymbol{x}}_\nu,\ \boldsymbol{v} := S_{\boldsymbol{g}} \dot{\boldsymbol{x}},\ \boldsymbol{u}_\nu := T_{\boldsymbol{h}^\nu} \boldsymbol{x}_\nu,\ \boldsymbol{u} := T_{\boldsymbol{h}} \boldsymbol{x}.$$

From conditions (R1)-(R4) and proposition 5.9 follows that $\boldsymbol{v}_\nu \rightharpoonup \boldsymbol{v}$ in L_{nl}^P and $\boldsymbol{u}_\nu \to \boldsymbol{u}$ in L_{nk}^Q, and hence (A) implies that $I(\boldsymbol{x}) \leq \liminf_\nu I_\nu(\boldsymbol{x}_\nu)$, proving (1).

On the other hand, proposition 5.11 provides the validity of condition (T) for the sequence of operators $\{S_{\boldsymbol{g}^\nu}\}$. Therefore, if a sequence $\{S_{\boldsymbol{g}^\nu}\dot{\boldsymbol{x}}_\nu\}$ is weakly convergent in L^P_{nl}, then so is $\{\dot{\boldsymbol{x}}_\nu\}$ in L^P_n, and thus $\{\boldsymbol{x}_\nu\}$ is weakly convergent in $\overset{\circ}{AC}{}^P_n$. Conditions (R1) and (R2) provide the continuous invertibility of $S_{\boldsymbol{g}^\nu}$, $S_{\boldsymbol{g}}$. Hence (B) implies that there is a sequence $S_{\boldsymbol{g}^\nu}\dot{\boldsymbol{x}}_\nu = \boldsymbol{v}_\nu \rightharpoonup \boldsymbol{v}_0$ in L^P_{nl} such that given $T_{\boldsymbol{h}^\nu}\boldsymbol{x}_\nu = \boldsymbol{u}_\nu \to \boldsymbol{u}_0$ in L^Q_{nk}, the relationship holds

$$G(\boldsymbol{u}_0, \boldsymbol{v}_0) = \lim_\nu I_\nu(\boldsymbol{x}_\nu).$$

Observing by our previous discussion that $\boldsymbol{x}_\nu \rightharpoonup \boldsymbol{x}$ in $\overset{\circ}{AC}{}^P_n$, we obtain $\boldsymbol{v}_0 = S_{\boldsymbol{g}}\dot{\boldsymbol{x}}$, $\boldsymbol{u}_0 = T_{\boldsymbol{h}}\boldsymbol{x}$ and $I(x) = \lim_\nu I_\nu(\boldsymbol{x}_\nu)$, which proves (2).

Step 3. At this moment we are going to show that $I = \Gamma^-(w)\lim_\nu I_\nu$, where w stands for the weak topology of $\overset{\circ}{AC}{}^P_n$. For this purpose it is enough to prove that the sequence $\{I_\nu\}$ is equicoercive and refer to proposition 3.2. To prove equicoercivity, use conditions (Q1) and (R1), observing that

$$I_\nu(\boldsymbol{x}) \geq b_1 + \sum_{j=1}^{l} a_j \int_0^1 P\left(\left|\left(S_{g_j^\nu}\boldsymbol{x}\right)(t)\right|\right)\, dt \geq b_1 + a\sum_{j=1}^{l} a_j \int_0^1 P\left(|\boldsymbol{x}(t)|\right)\, dt \geq I_0(||\boldsymbol{x}||_P),$$

where $b_1 \in \mathbb{R}$ and $I_0(x) \to +\infty$ for $x \to +\infty$.

At last, having established that I is the (weak) Γ^--limit of the sequence $\{I_\nu\}$, it remains only to conclude the proof referring to proposition 3.3.

Step 4. We prove (A) and (B). Let $\boldsymbol{u}_\nu \to \boldsymbol{u}$ in L^Q_{nk} and consider the functionals J_ν, J: $L^P_{nl} \to \bar{\mathbb{R}}$ introduced by the formulae

$$J_\nu(\boldsymbol{v}) := G_\nu(\boldsymbol{u}, \boldsymbol{v}), \qquad J(\boldsymbol{v}) := G(\boldsymbol{u}, \boldsymbol{v}).$$

Proposition 3.5 with the help of lemmata 4.1 and 4.2 implies that $J_\nu \to J$ Γ^--weakly over L^P_{nl}. This according to proposition 3.2 provides the following two conditions:

(A′) If $\boldsymbol{v}_\nu \rightharpoonup \boldsymbol{v}$ in L^P_{nl}, then $J(\boldsymbol{u}) = G(\boldsymbol{u}, \boldsymbol{v}) \leq \liminf_\nu J_\nu(\boldsymbol{v}_\nu)$.

(B′) There is a sequence $\{\boldsymbol{v}_\nu\} \subset L^P_{nl}$, and $\boldsymbol{v} \in L^P_{nl}$ satisfying $\boldsymbol{v}_\nu \rightharpoonup \boldsymbol{v}$, such that $J_\nu(\boldsymbol{v}_\nu) \to J(\boldsymbol{v})$.

To prove (A), use (A') and the lower semicontinuity of each $G_\nu(\cdot, \boldsymbol{v}_\nu)$, concluding

$$G(\boldsymbol{u}, \boldsymbol{v}) \leq \liminf_\nu \liminf_\mu G_\nu(\boldsymbol{u}_\mu, \boldsymbol{v}_\nu) \leq \liminf_\nu G_\nu(\boldsymbol{u}_\nu, \boldsymbol{v}_\nu).$$

Property (B) is proved by (B') and the estimate

$$|G(\boldsymbol{u}, \boldsymbol{v}) - G_\nu(\boldsymbol{u}_\nu, \boldsymbol{v}_\nu)| \leq |J(\boldsymbol{v}) - J_\nu(\boldsymbol{v}_\nu)| + \int_0^1 \omega(t, |\boldsymbol{u}(t) - \boldsymbol{u}_\nu(t)|_{nl})\, dt.$$

□

6.2. Nonlocal perturbations of the classical functional

In this section we apply theorem 6.3 to the analysis of the typical situation when the unperturbed functional (V_∞) has the classical form, that is, does not involve the argument deviation:

$$I(\boldsymbol{x}) = \int_0^1 \tilde{f}(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t))\, dt, \tag{$\tilde{V}_\infty$}$$

where $\tilde{f}(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)) := f(t, \boldsymbol{x}(t), \ldots, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \ldots, \dot{\boldsymbol{x}}(t))$. In this case the Euler equations written out for the functional ($\tilde{V}_\infty$) are the ordinary differential equations, as it is usual for such well studied variational problems. Thus one may formulate in other words that the purpose of this section is to analyze the stability of solutions to the classical problem of the calculus of variations with respect to the nonlocal perturbations. It is particularly interesting to note that such a setting can be regarded as the "dual" to the well-known situations, when the homogenization of the local problems leads to a nonlocal one (see for instance [14, 15, 16] and references therein).
Replace the conditions (R1)–(R4) by the following ones:

(R1′) For any $j = 1, \ldots, l$ the functions $g_j^\nu \colon [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ are almost everywhere finite and measurable, while

$$\exists (a, A):\ 0 < a \le \frac{d\mu_{g_j^\nu}}{dm}(t) \le A \text{ a.e. on } [0,1],\ \nu \in \mathbb{N} \cup \{0\} \text{ and}$$
$$\left\| \frac{d\mu_{g_j^\nu}(\tau)}{dm} \right\|_{P'} \to \tau P^{-1}(1/\tau) \text{ as } \nu \to \infty.$$

(R2′) $\operatorname{meas} E_j^\nu = 1$ and there are measurable functions $\gamma_j^\nu \colon \mathbb{R} \to [0,1]$, $j = 1, \ldots, l$, $\nu \in \mathbb{N}$ such that $e \subset [0,1]$, $\operatorname{meas} e = 0$ implies $\operatorname{meas} \left(\gamma_j^\nu\right)^{-1}(e) = 0$, and

$$\gamma_j^\nu(g_j^\nu(t)) = t \text{ for a.e. } t \in E_j^\nu.$$

(R3′) For all $i = 1, \ldots, k$ the functions $h_i^\nu \colon [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ are almost everywhere finite and measurable, while $h_i^\nu \to h$, $h(t) \equiv t$, in measure.

(R4′) $\operatorname{meas} \mathcal{E}_i^\nu \to 1$ for each $i = 1, \ldots, k$ and $\operatorname{meas} E_j^\nu(\tau) \Delta [0, \tau] \to 0$ for any $\tau \in [0,1]$, $j = 1, \ldots, l$.

Now we claim the following direct corollary from theorem 6.3.

Theorem 6.5. *Consider the functionals I, I_ν: $AC_n^P \to \bar{\mathbb{R}}$ defined by the expressions ($\tilde{V}_\infty$) and (V_ν) respectively. Under the assumptions* (P0), (Q1)–(Q4), *(R1′)–(R4′) the assertion of theorem* 6.3 *remains valid.*

6.3. Functionals with impulsive constraints

In this section we show how the technique developed can be used to tackle the problems of Γ^--convergence for a sequence of functionals defined on some space different from the

the standard space of absolutely continuous functions. Such problems most often arise in applications from optimal control theory. In particular, here we deal with an important example of Γ^--convergence of a sequence of functionals with deviating argument subject to impulsive constraints, which very often appears when analyzing continuous dependence on parameters of optimal solutions to controlled impulsive differential equations. To apply the general scheme used in the proof of theorem 6.3 we construct some special space of piecewise absolutely continuous functions with prescribed behavior at discontinuity points. Let the original functional (V_∞) be restricted by the system of impulsive constraints:

$$\Delta \boldsymbol{x}(t_i) \equiv \boldsymbol{x}(t_i) - \boldsymbol{x}(t_i - 0) = \zeta_i, \qquad i = 1, \ldots, m, \tag{C_∞}$$

where the points of discontinuity $\{t_i\}_{i=1}^m$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$ are fixed, while $\zeta_i \in \mathbb{R}^n$ are given vectors. Introduce the space $PAC_n^P[\{t_i\}_{i=1}^m]$ (for short, PAC_n^P) isomorphic to $L_n^P \times \mathbb{R}^{n+mn}$ by the relationship

$$J: \ (\boldsymbol{z}, \beta) \in L_n^P \times \mathbb{R}^{n+mn} \mapsto \boldsymbol{x} = \Lambda \boldsymbol{z} + D\beta := \int_0^t \boldsymbol{z}(\tau)\, d\tau + \omega(t)\beta \in PAC_n^P, \tag{8}$$

where $\omega(t)$ is the $n \times (n + mn)$ matrix

$$\omega(t) = \left(E_n, \chi_{[t_1,1]}E_n, \cdots, \chi_{[t_m,1]}E_n\right),$$

E_n is the unit $n \times n$ matrix, $\chi_A(t)$ is the characteristic function of the set $A \subset [0, 1]$. The inverse of (8) is determined by the equality

$$J^{-1}: \ \boldsymbol{x} \in PAC_n^P \mapsto (\boldsymbol{z}, \beta) = (\delta \boldsymbol{x}, \rho \boldsymbol{x}) := (\dot{\boldsymbol{x}}, \Delta \boldsymbol{x}) \in L_n^P \times \mathbb{R}^{n+mn}, \tag{9}$$

where $\Delta \boldsymbol{x} := \operatorname{col}\{\boldsymbol{x}(0), \Delta \boldsymbol{x}(t_1), \ldots, \Delta \boldsymbol{x}(t_m)\}$. Thus any element of PAC_n^P is uniquely representable in the form

$$\boldsymbol{x}(t) = \int_0^t \dot{\boldsymbol{x}}(\tau)\, d\tau + \boldsymbol{x}(0) + \sum_{i=1}^m \chi_{[t_i,1]} \Delta \boldsymbol{x}(t_i).$$

Introducing the norm $||\boldsymbol{x}||_{PAC} := ||\dot{\boldsymbol{x}}||_P + |\Delta \boldsymbol{x}|_{n+mn}$ one turns PAC_n^P into a Banach space. The isomorphism J given by the expressions (8) and (9) was first suggested by A. Anokhin (see [9, § 6.3]). One easily observes that PAC_n^P is exactly the desired space of piecewise absolutely continuous vector functions $\boldsymbol{x}$: $[0, 1] \to \mathbb{R}^n$ continuous from the right at the given points $\{t_i\}_{i=1}^m$ and satisfying the impulsive conditions (C_∞) at these points. It is reasonable to expect that the perturbations of the original problem affect also the set of constraints, and therefore, we have to deal with the sequence of perturbed variational problems (V_ν) restricted by additional conditions

$$\Delta \boldsymbol{x}(t_i^\nu) \equiv \boldsymbol{x}(t_i^\nu) - \boldsymbol{x}(t_i^\nu - 0) = \zeta_i^\nu, \qquad i = 1, \ldots, m, \tag{C_ν}$$

where again the points of discontinuity are ordered: $0 = t_0^\nu < t_1^\nu < \ldots < t_m^\nu < t_{m+1}^\nu = 1$. Reiterating the above construction we obtain for each functional I_ν the respective space of piecewise absolutely continuous vector functions $PAC_n^P[\{t_i^\nu\}_{i=1}^m]$ (for short, $PAC_n^P(\nu)$)

isometrically isomorphic to $L^P_n \times \mathbb{R}^{n+mn}$. These spaces are however different and so to study the problem of stability of minimizers of (V_∞) subject to (C_∞) as the problem of Γ-convergence in PAC^P_n one needs extra constructions. For this purpose denote

$$r^\nu(t) = \sum_{i=0}^{m} \left(t_i + \frac{t_{i+1} - t_i}{t^\nu_{i+1} - t^\nu_i}(t - t^\nu_i) \right) \chi_{[t^\nu_i, t^\nu_{i+1}]}(t), \qquad t \in [0,1].$$

It is easy to verify that the inner superposition operator $S_{r^\nu}\colon L^P_n \to L^P_n$ is continuous and continuously invertible, while $S_{r^\nu}\boldsymbol{x} \in PAC^P_n$ whenever $\boldsymbol{x} \in PAC^P_n(\nu)$.

Definition 6.6. The sequence of functions $\boldsymbol{x}_\nu \in PAC^p_n(\nu)$ is said to be weakly $\mathcal{P}$-convergent to $\boldsymbol{x} \in PAC^P_n$, $\boldsymbol{x}_\nu \overset{\mathcal{P}}{\rightharpoonup} \boldsymbol{x}$, if $S_{r^\nu}\boldsymbol{x}_\nu \rightharpoonup \boldsymbol{x}$ in PAC^P_n.

Denote by $\overline{PAC}^P_n$, $\overline{PAC}^P_n(\nu)$ the subspaces of PAC^P_n, $PAC^P_n(\nu)$ respectively, consisting of the functions satisfying the constraints $\boldsymbol{x}(0) = \boldsymbol{x}(1) = 0$. One can now claim the following extension of theorem 6.3 to the case of functionals with impulsive constraints.

Theorem 6.7. *Let $t^\nu_i \to t_i$ and $\zeta^\nu_i \to \zeta_i$ when $\nu \to \infty$. Then under the conditions* (P0), (Q1)–(Q4), (R1)–(R4) *if $\{\boldsymbol{x}_\nu\} \subset \overline{PAC}^P_n(\nu)$ is such that*

$$\lim_\nu I_\nu(\boldsymbol{x}_\nu) = \lim_\nu (\inf_{\overline{PAC}^P_n(\nu)} I_\nu).$$

(in particular, when each $\boldsymbol{x}_\nu$ is the point of global minimum of the respective I_ν over $\overline{PAC}^P_n(\nu)$) and $\boldsymbol{x}_\nu \overset{\mathcal{P}}{\rightharpoonup} \boldsymbol{x}$, then

$$\lim_\nu (\inf_{\overline{PAC}^P_n(\nu)} I_\nu) = \min_{\overline{PAC}^P_n} I = I(\boldsymbol{x}).$$

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