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A Variational Model for Non Linear Elastic Plates

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By adopting the variational point of view, the constitutive equations of a non linear elastic plate are deduced under kinematical constraints on the admissible deformations.

1. Introduction

A classical approach to the study of thin structures in elasticity consists in starting from three-dimensional models and deducing the behaviour of two-dimensional or one-dimensional thin elastic bodies by passing to the limit when one or two dimensions go to zero.
Though the two-dimensional linear model of an elastic plate has exhaustively been studied from different points of view (see [1], [3], [8], [9]), to our knowledge it doesn't exist a rigorous theory which permits to deduce a non linear model of elastic plate as a limit (for instance, in the sense of [3]) of non linear three-dimensional thin elastic bodies.
The most difficulty which arises in treating these problems is that, without some kinematical constraints on the deformations, no information about the compactness of minimizers of the energy functional of the three-dimensional elastic bodies can be expected and therefore it is very difficult to formulate any reasonable conjecture about the "limit" functional, that is the energy functional of the limit plate.
The simplest method to avoid this difficulty is that of Kirchhoff, which consists in the linearization of the Green-St.Venant energy functional

$$\int_\Omega \Big(\mu|E|^2 + \frac{\lambda}{2}(\mathrm{tr}\ E)^2\Big)dx \tag{1.1}$$

(where $E = \frac{1}{2}({}^T D\varphi D\varphi - I)$, φ being the deformation), thus obtaining the classical functional

$$\int_\Omega \Big(\mu|e(u)|^2 + \frac{\lambda}{2}(\mathrm{tr}\ e(u))^2\Big)dx \tag{1.2}$$

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with $u(x) = \varphi(x) - x$. When one of the dimensions of Ω goes to zero and μ, λ become greater and greater in a suitable way (see [1]) the limit functional can be calculated explicitly.
In this paper we adopt another point of view: we start again, roughly speaking, from the Green-St.Venant energy functional, but we impose a different kinematical constraint on the deformation φ, which will be assumed sufficiently regular.
More precisely, if Σ_ε is the reference configuration of the three-dimensional elastic body having Σ as middle surface, we assume, following Podio-Guidugli [14] that "material fibers orthogonal to the middle surface before loading remain approximately orthogonal to it after loading"; this restriction takes the form $E(\varphi)(\underline{n}) = 0$ in Σ_ε, where $\underline{n}$ is the normal unit vector to Σ.
In the two dimensional case this is enough to obtain the required compactness properties of the minimizing sequences of the energy functional, while in the case $n = 3$ (the elastic plate) another internal constraint which takes into account some symmetry properties of the deformation must be assumed.
The method we used here is closely related to Γ-convergence, already used in similar situations; nevertheless, for the reader's convenience, all results are stated and proven without using any specific knowledge about Γ-limits. For a more precise setting of the problem in this framework we refer to [1].

2. The bidimensional case: notations and statements

Let us set $\Sigma = [0,1]$ and $\Sigma_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \Sigma,\ |x_2| \le \varepsilon\}$, which will be viewed as the reference configuration of a two dimensional elastic body. For every $\varphi \in L^2(\Sigma_\varepsilon; \mathbb{R}^2)$ we set

$$\tilde{\varphi}(x_1) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varphi(x_1, x_2)\, dx_2$$

and for every $\varphi \in C^2(\Sigma_\varepsilon; \mathbb{R}^2)$ we denote by C the right Cauchy-Green strain tensor:

$$C = {}^T D\varphi D\varphi$$

and by E the Green-St.Venant strain tensor:

$$E = \frac{1}{2}(C - I).$$

We denote by e_i, $i = 1,2$ the canonical basis in $\mathbb{R}^2$ and we define:

$$\mathcal{A}_\varepsilon(\Sigma_\varepsilon) := \{\varphi \in C^2(\Sigma_\varepsilon; \mathbb{R}^2) : E(\varphi)(e_2) = 0, \det D\varphi > 0, \varphi(0, x_2) = (0, x_2)\ \}.$$

For every $\varepsilon > 0$ and $\varphi \in C^2(\Sigma_\varepsilon; \mathbb{R}^2)$ we set:

$$F_\varepsilon(\varphi) = \begin{cases} \int_{\Sigma_\varepsilon} W(E(\varphi))\, dx & \text{if } \varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon) \\ +\infty & \text{otherwise,} \end{cases}$$

where the deformation energy density is supposed of the form:

$$W(E) = \mu |E|^2 + \frac{\lambda}{2}(\mathrm{tr}\ E)^2; \tag{2.1}$$

$\lambda, \mu \in L^\infty(\Sigma, [0, +\infty))$, $\mu(x_1) \geq \bar{\mu} > 0$ a.e. on $[0,1]$ are called the Lamè moduli of the material; we set also $\Lambda(x_1) = (\mu + \frac{\lambda}{2})(x_1)$.

Let $f \in C^0(\Sigma_1; \mathbb{R}^2)$ and $g \in C^0(\Sigma_1; \mathbb{R}^2)$ be given functions, corresponding to the applied load system; we set from now on $f_0 = f_{|\Sigma}$ and $g_0 = g_{|\Sigma}$.

For every $\varepsilon > 0$ we define the functionals $G_\varepsilon : \ L^2(\Sigma_\varepsilon; \mathbb{R}^2) \to \mathbb{R} \cup \{+\infty\}$:

$$G_\varepsilon(\varphi) = \frac{1}{\varepsilon^3} \left[F_\varepsilon(\varphi) - \varepsilon^2 \int_{\Sigma_\varepsilon} \langle f, \varphi \rangle dx - \varepsilon^3 \int_{\Sigma_\varepsilon \cap \{x_2 = \varepsilon\}} \langle g, \varphi \rangle dx_1 \right].$$

The presence of the rescaling factor $\frac{1}{\varepsilon^3}$ is essential in proving some compactness properties on the admissible deformations, as we shall see in proposition 3.2.

Since F_ε is not polyconvex neither quasiconvex (we refer to [9] and [11] for more details) we cannot expect that G_ε has an absolute minimum since it is not l.s.c. in the weak topology of $H^1(\Sigma_\varepsilon, \mathbb{R}^2)$. A sequence $\varphi^\varepsilon \in L^2(\Sigma_\varepsilon; \mathbb{R}^2)$ will be called a minimizing sequence for G_ε if $\lim_{\varepsilon \to 0} \{G_\varepsilon \varphi^\varepsilon - \inf G_\varepsilon\} = 0$; our goal is to study the asymptotic behaviour of such sequences and to our aim we are led to introduce the set of limiting admissible deformations

$$\mathcal{A}(\Sigma) := \{v \in H^2(\Sigma; \mathbb{R}^2) : |\dot{v}(x_1)|^2 = 1 \ \text{ for every } x_1 \in [0,1], v(0) = \underline{0}, \dot{v}(0) = \underline{e}_1\}.$$

For every $v \in L^2(\Sigma; \mathbb{R}^2)$ let us define:

$$F_0(v) = \begin{cases} \dfrac{2}{3} \displaystyle\int_\Sigma \Lambda(x_1) \, |\ddot{v}|^2 \, dx_1 & \text{if } v \in \mathcal{A}(\Sigma) \\ +\infty & \text{otherwise,} \end{cases}$$

and also the functional $G_0 : \ L^2(\Sigma; \mathbb{R}^2) \to \mathbb{R} \cup \{+\infty\}$:

$$G_0(v) = F_0(v) - 2 \int_\Sigma \langle f_0, v \rangle dx_1 - \int_\Sigma \langle g_0, v \rangle dx_1,$$

which will be the limit functional. Indeed the main result of this section is the following:

Theorem 2.1. *Let $(\varphi^\varepsilon)_{\varepsilon > 0} \subset L^2(\Sigma_\varepsilon; \mathbb{R}^2)$ be a minimizing sequence for G_ε. Then the sequence $(\tilde{\varphi}^\varepsilon)_{\varepsilon > 0}$ is compact in $L^2(\Sigma; \mathbb{R}^2)$ and, if v_0 is one of its limit points, then $v_0 \in \mathcal{A}(\Sigma)$, $\tilde{\varphi}^\varepsilon \to \ v_0$ in $H^1(\Sigma; \mathbb{R}^2)$ and*

$$G_0(v_0) = \min \{G_0(v) : v \in \mathcal{A}(\Sigma)\} = \lim_{\varepsilon \to 0} G_\varepsilon(\varphi^\varepsilon).$$

3. Some preliminary lemmas

We shall prove in the following lemma that the deformations φ that belong to the set $\mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ admit a particular representation.

Lemma 3.1. *If $\varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$, then there exist $v \in C^2(\Sigma; \mathbb{R}^2)$, $w \in C^1(\Sigma; \mathbb{R}^2)$ such that:*

$$\varphi(x_1, x_2) = v(x_1) + x_2 w(x_1), \ |\dot{v}(x_1)| \neq 0 \ \text{ for every } x_1 \in [0,1],$$

$$w(x_1) = \frac{1}{|\dot{v}(x_1)|}(-\dot{v}_2(x_1), \dot{v}_1(x_1)) \ \text{ for every } x_1 \in [0,1], \ v(0) = \underline{0}, \ w(0) = \underline{e}_2.$$

Moreover, if $v \in C^2(\Sigma; \mathbb{R}^2) \cap \mathcal{A}(\Sigma)$, the deformation field φ defined by $\varphi(x_1, x_2) = v(x_1) + x_2 w(x_1)$, where $w = (-\dot{v}_2, \dot{v}_1)$, belongs to $\mathcal{A}_\varepsilon(\Sigma_\varepsilon)$.

Proof. Let φ be in $C^2(\Sigma_\varepsilon; \mathbb{R}^2)$. The condition $({}^T D\varphi D\varphi - I) \cdot e_2 = 0$ yields:

$$\langle \varphi_{x_1}, \varphi_{x_2} \rangle = 0 \tag{3.1}$$

$$|\varphi_{x_2}|^2 = 1 \tag{3.2}$$

Deriving (3.1) with respect to x_2 and (3.2) with respect to x_1, we obtain: $\langle \varphi_{x_1 x_2}, \varphi_{x_2} \rangle + \langle \varphi_{x_1}, \varphi_{x_2 x_2} \rangle = 0$ and $\langle \varphi_{x_2 x_1}, \varphi_{x_2} \rangle = 0$.
It results therefore $\langle \varphi_{x_1}, \varphi_{x_2 x_2} \rangle = 0$ and also $\langle \varphi_{x_2}, \varphi_{x_2 x_2} \rangle = 0$, because $|\varphi_{x_2}|^2 = 1$. Having in mind the condition $\langle \varphi_{x_1}, \varphi_{x_2} \rangle = 0$, we conclude that $\varphi_{x_2 x_2} = 0$, that is:

$$\varphi(x_1, x_2) = v(x_1) + x_2 w(x_1),$$

for some $v, w : \Sigma \to \mathbb{R}^2$.
We remark that $v \in C^2(\Sigma; \mathbb{R}^2)$, because $\varphi \in C^2(\Sigma_\varepsilon; \mathbb{R}^2)$ and that from (3.1) and (3.2) it follows:

$$|w(x_1)|^2 = 1 \ \text{ for every } x_1 \in [0,1] \tag{3.3}$$

$$\langle \dot{v}(x_1), w(x_1) \rangle = 0 \ \text{ for every } x_1 \in [0,1] \tag{3.4}$$

We define now: $w^\perp = (w_2, -w_1)$.
The constraint $\det D\varphi > 0$, that is: $\langle \dot{v} + x_2 \dot{w}, w^\perp \rangle > 0$, implies in particular that $\langle \dot{v}, w^\perp \rangle > 0$ and $|\dot{v}(x_1)| \neq 0$ for every $x_1 \in [0,1]$. We can therefore set by (3.3) and (3.4):

$$w(x_1) = \frac{(-\dot{v}_2(x_1), \dot{v}_1(x_1))}{|\dot{v}(x_1)|}$$

for every $x_1 \in [0,1]$. □

In the following for every φ which belongs to $\mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ we shall write:

$$\varphi(x_1, x_2) = v(x_1) + x_2 w(x_1),$$

with v, w such as in lemma 3.1 and we observe also that if $\varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ we have $\tilde{\varphi} = v$.

In the following proposition some compactness properties on the sequences (v^ε) and (w^ε) are proven, which shall permit us to demonstrate the main theorem and also provide a "a posteriori" justification of the choice of the means $\tilde{\varphi}^\varepsilon$.

Proposition 3.2. *Assume that* $(\varphi^\varepsilon)_{\varepsilon>0} \subset \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ *and* $F_\varepsilon(\varphi^\varepsilon) \le C\varepsilon^3$ *for some* $C > 0$. *Then:*

i) $(v^\varepsilon)_{\varepsilon>0}$ *is compact in* $H^1(\Sigma;\mathbb{R}^2)$ *and* $(w^\varepsilon)_{\varepsilon>0}$ *is relatively weakly compact in* $H^1(\Sigma;\mathbb{R}^2)$*;*

ii) *if* $v^\varepsilon \to v$ *in* $L^2(\Sigma,\mathbb{R}^2)$, *then* $v \in \mathcal{A}(\Sigma)$ *and* $w^\varepsilon \rightharpoonup (-\dot v_2, \dot v_1)$ *in* $H^1(\Sigma;\mathbb{R}^2)$*;*

iii) *the sequence* $(\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle)_{\varepsilon>0}$ *is bounded in* $L^2(\Sigma;\mathbb{R}^2)$ *and, if* $v^\varepsilon \to v$ *in* $L^2(\Sigma,\mathbb{R}^2)$, *then* $\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle \rightharpoonup \langle \dot v, \dot w\rangle$ *in* $L^2(\Sigma)$.

Proof. First of all, we observe that if $\varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$, then:

$$(\mathrm{tr}\ E(\varphi))^2 = |E(\varphi)|^2 .$$

We get therefore:

$$\begin{aligned} F_\varepsilon(\varphi^\varepsilon) &= \int_{\Sigma_\varepsilon} \left[\mu\, |E(\varphi^\varepsilon)|^2 + \frac{\lambda}{2}(\mathrm{tr}\ E(\varphi^\varepsilon))^2\right] dx \\ &= \int_{\Sigma_\varepsilon} \Lambda(x_1)\, |E(\varphi^\varepsilon)|^2 dx \\ &= \int_{\Sigma_\varepsilon} \frac{1}{4}\Lambda(x_1) \left|{}^T D\varphi^\varepsilon D\varphi^\varepsilon - I\right|^2 dx \\ &= \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\, |U^\varepsilon + I|^2 \cdot |U^\varepsilon - I|^2 dx, \end{aligned} \tag{3.5}$$

where $U^\varepsilon = \sqrt{{}^T D\varphi^\varepsilon D\varphi^\varepsilon}$ is defined as the unique symmetric and positive definite matrix such that $(U^\varepsilon)^2 = {}^T D\varphi^\varepsilon D\varphi^\varepsilon$.
By the polar factorization of an invertible matrix, for every $\varepsilon > 0$ we may find a positive rotation R^ε such that $D\varphi^\varepsilon = R^\varepsilon U^\varepsilon$, from which it follows:

$$F_\varepsilon(\varphi^\varepsilon) = \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1) |{}^T R_\varepsilon D\varphi^\varepsilon - I|^2 |{}^T R_\varepsilon D\varphi^\varepsilon + I|^2 dx.$$

It is easy to prove that there exists $C > 0$ such that $|{}^T R_\varepsilon D\varphi^\varepsilon + I|^2 \ge C$ for every $\varepsilon > 0$, therefore we get:

$$\begin{aligned} F_\varepsilon(\varphi^\varepsilon) &\ge C \cdot \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1) \left|{}^T R_\varepsilon D\varphi^\varepsilon - I\right|^2 dx_1 dx_2 \\ &= C \cdot \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\, |D\varphi^\varepsilon - R^\varepsilon|^2 dx_1 dx_2. \end{aligned}$$

By applying the Cayley-Hamilton theorem to the matrix $U^\varepsilon = \sqrt{{}^T D\varphi^\varepsilon D\varphi^\varepsilon}$ we can determine the matrix R^ε. We obtain:

$$R^\varepsilon = \begin{pmatrix} \dfrac{(\varphi_1^\varepsilon)_{x_1}}{|\varphi_{x_1}^\varepsilon|} & w_1^\varepsilon \\ \dfrac{(\varphi_2^\varepsilon)_{x_1}}{|\varphi_{x_1}^\varepsilon|} & w_2^\varepsilon \end{pmatrix}$$

and by the uniqueness of the rotation R^ε we conclude that:

$$R^\varepsilon = \begin{pmatrix} w_2^\varepsilon & w_1^\varepsilon \\ -w_1^\varepsilon & w_2^\varepsilon \end{pmatrix}.$$

We get now:

$$\begin{aligned} C\varepsilon^3 &\geq F_\varepsilon(\varphi^\varepsilon) \\ &\geq \frac{c}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\Big(|\dot v_1^\varepsilon + x_2\dot w_1^\varepsilon - w_2^\varepsilon|^2 + |\dot v_2^\varepsilon + x_2 \dot w_2^\varepsilon + w_1^\varepsilon|^2 \Big)dx \\ &\geq \frac{c}{4}\bar\mu \int_{\Sigma_\varepsilon} \Big(|\dot v_1^\varepsilon - w_2^\varepsilon|^2 + {x_2}^2\,|\dot w_1^\varepsilon|^2 + |\dot v_2^\varepsilon + w_1^\varepsilon|^2 + {x_2}^2\,|\dot w_2^\varepsilon|^2\Big)dx. \end{aligned} \tag{3.6}$$

By integrating with respect to x_2, we obtain : $\int_\Sigma |\dot{w^\varepsilon}|^2\,dx_1 \leq C$, for some $C > 0$; since $w^\varepsilon(0) = \underline{e}_2$ for every $\varepsilon > 0$, the sequence $(w^\varepsilon)_{\varepsilon>0}$ is relatively weakly compact in $H^1(\Sigma;\mathbb{R}^2)$ and then there exists a subsequence, which we always denote with w^ε, such that $w^\varepsilon \rightharpoonup w = (w_1, w_2)$ in $H^1(\Sigma;\mathbb{R}^2)$.
By using inequality (3.6) we get $\dot v_1^\varepsilon \to w_2$ and $\dot v_2^\varepsilon \to -w_1$ in $L^2(\Sigma;\mathbb{R}^2)$, therefore (v^ε) is relatively compact in the strong topology of $H^1(\Sigma;\mathbb{R}^2)$ and if $v^\varepsilon \to v$ in $L^2(\Sigma,\mathbb{R}^2)$ we have $\dot v_1 = w_2$ and $\dot v_2 = -w_1$, then $\ddot v =: (\ddot v_1, \ddot v_2) = (\dot w_2, -\dot w_1) \in L^2(\Sigma,\mathbb{R}^2)$ and $v \in H^2(\Sigma;\mathbb{R}^2)$.
We observe now that $|\dot v(x_1)|^2 = 1$ for every $x_1 \in [0,1]$, since w^ε converges, up to a subsequence, to $(-\dot v_2, \dot v_1)$ uniformly on Σ and $|w^\varepsilon(x_1)|^2 = 1$ for every $x_1 \in [0,1]$.
To prove iii) we observe that:

$$\begin{aligned} F_\varepsilon(\varphi^\varepsilon) &= \int_{\Sigma_\varepsilon} \frac{1}{4}\Lambda(x_1)\left|{}^T D\varphi^\varepsilon D\varphi^\varepsilon - I\right|^2 dx \\ &= \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\Big(|\dot v^\varepsilon + x_2 \dot w^\varepsilon|^2 - 1\Big)^2 dx \\ &= \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\Big(|\dot v^\varepsilon|^2 - 1 + 2x_2\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle + {x_2}^2\,|\dot w^\varepsilon|^2\Big)^2 dx \\ &= \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\Big[(\,|\dot v^\varepsilon|^2 - 1 + {x_2}^2\,|\dot w^\varepsilon|^2)^2 + 4{x_2}^2\,|\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle|^2\Big] dx \\ &\geq \int_{\Sigma_\varepsilon} \Lambda(x_1)\cdot {x_2}^2 |\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle|^2 dx \\ &\geq \bar\mu \int_{\Sigma_\varepsilon} {x_2}^2 |\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle|^2 dx. \end{aligned}$$

By the hypotheses $F_\varepsilon(\varphi^\varepsilon) \leq C\varepsilon^3$ we obtain now:

$$\frac{2}{3}\bar\mu \int_\Sigma |\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle|^2 dx_1 \leq C. \tag{3.7}$$

Recalling that $\dot v^\varepsilon \to \dot v$ in $L^2(\Sigma;\mathbb{R}^2)$ and $\dot w^\varepsilon \rightharpoonup \dot w$ in $L^2(\Sigma;\mathbb{R}^2)$, it is easy to verify that $\langle \dot v^\varepsilon, \dot w^\varepsilon\rangle \rightharpoonup \langle \dot v, \dot w\rangle$ in the sense of distributions and therefore by using (3.7) we get iii). $\square$

4. Proof of the main result

Proposition 4.1. *For every $v \in L^2(\Sigma;\mathbb{R}^2)$, for every sequence $(\varphi^\varepsilon)_{\varepsilon>0}$ such that $\tilde{\varphi}^\varepsilon = v^\varepsilon \to v$ in $L^2(\Sigma;\mathbb{R}^2)$ we have:*

$$\liminf_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon) \geq G_0(v). \tag{4.1}$$

Proof. We may suppose $G_\varepsilon(\varphi^\varepsilon) \leq C$, since otherwise (4.1) is obvious. Then we have $\varphi^\varepsilon \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ and:

$$\begin{aligned}
F_\varepsilon(\varphi^\varepsilon) &\leq C\varepsilon^3 + \varepsilon^2 \int_{\Sigma_\varepsilon} \langle f, \varphi^\varepsilon\rangle dx + \varepsilon^3 \int_{\Sigma_\varepsilon \cap \{x_2=\varepsilon\}} \langle g, \varphi^\varepsilon\rangle dx_1 \\
&\leq C\Big(\varepsilon^3 + \varepsilon^2 \int_{\Sigma_\varepsilon} |\varphi^\varepsilon|^2\, dx + \varepsilon^2 \int_{\Sigma_\varepsilon} |f|^2\, dx \\
&\qquad + \varepsilon^3 \int_{\Sigma_\varepsilon \cap \{x_2=\varepsilon\}} |\varphi^\varepsilon(x_1,\varepsilon)|^2 + \varepsilon^3 \int_{\Sigma_\varepsilon \cap \{x_2=\varepsilon\}} |g|^2\Big) \\
&\leq C(\varepsilon^3 + \varepsilon^2 \int_{\Sigma_\varepsilon} |v^\varepsilon + x_2 w^\varepsilon|^2 dx + \varepsilon^3 \int_\Sigma |v^\varepsilon + \varepsilon w^\varepsilon|^2\, dx_1) \\
&\leq C(\varepsilon^3 + \varepsilon^3 \int_\Sigma |v^\varepsilon|^2 dx_1 dx_2 + \varepsilon^5 \int_\Sigma |w^\varepsilon|^2 dx_1 dx_2) \\
&\leq C\varepsilon^3,
\end{aligned}$$

because $v^\varepsilon \to v$ in $L^2(\Sigma;\mathbb{R}^2)$, $|w^\varepsilon|^2 = 1$ on Σ and f and g are continuous; hence by proposition 3.2 it follows that $v \in \mathcal{A}(\Sigma)$. As in proposition 3.2 we get now:

$$\begin{aligned}
\frac{1}{\varepsilon^3} F_\varepsilon(\varphi^\varepsilon) &= \frac{1}{\varepsilon^3} \int_{\Sigma_\varepsilon} \frac{1}{4}(\mu + \frac{\lambda}{2})(x_1) \left| {}^T D\varphi^\varepsilon D\varphi^\varepsilon - I \right|^2 dx \\
&= \frac{1}{\varepsilon^3} \int_{\Sigma_\varepsilon} \Lambda(x_1) \cdot {x_2}^2\, |\langle \dot{v}^\varepsilon, \dot{w}^\varepsilon\rangle|^2 dx \\
&= \frac{2}{3} \int_\Sigma \Lambda(x_1)\, |\langle \dot{v}^\varepsilon, \dot{w}^\varepsilon\rangle|^2\, dx_1 \\
&\geq \frac{2}{3} \int_\Sigma \Lambda(x_1)[\, |\langle \dot{v}, \dot{w}\rangle|^2 + \langle \dot{v}, \dot{w}\rangle \cdot (\langle \dot{v}^\varepsilon, \dot{w}^\varepsilon\rangle - \langle \dot{v}, \dot{w}\rangle)] dx_1.
\end{aligned}$$

Recalling that $(\langle \dot{v}^\varepsilon, \dot{w}^\varepsilon\rangle)_{\varepsilon>0}$ converges weakly in $L^2(\Sigma;\mathbb{R}^2)$ to $\langle \dot{v}, \dot{w}\rangle$, we get finally:

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} F_\varepsilon(\varphi^\varepsilon) &\geq \frac{2}{3} \int_\Sigma \Lambda(x_1)\, |\langle \dot{v}, \dot{w}\rangle|^2\, dx_1 \\
&= \frac{2}{3} \int_\Sigma \Lambda(x_1)\, |\ddot{v}|^2\, dx_1,
\end{aligned}$$

where the last equality follows from the fact that $\langle \dot{w}, \ddot{v}\rangle = 0$ by proposition 3.2 and $\langle \dot{v}, \ddot{v}\rangle = 0$ because $|\dot{v}|^2 = 1$ on Σ.

We can now conclude:

$$\begin{aligned}\liminf_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon) &= \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3}\Big[F_\varepsilon(\varphi^\varepsilon) - \varepsilon^2 \int_{\Sigma_\varepsilon} \langle f, v^\varepsilon + x_3 w^\varepsilon\rangle dx \\ &\quad - \varepsilon^3 \int_{\Sigma_\varepsilon\cap\{x_2=\varepsilon\}} \langle g, v^\varepsilon + \varepsilon w^\varepsilon\rangle dx_1\Big] \\ &\geq \frac{2}{3}\int_\Sigma \Lambda(x_1)\,|\ddot{v}|^2\,dx_1 - 2\int_\Sigma \langle f_0, v\rangle dx_1 - \int_\Sigma \langle g_0, v\rangle dx_1.\end{aligned}$$

□

We are going now to prove that the limit functional G_0 is as good as possible, with respect to the convergence of the means introduced above.

Proposition 4.2. *For every $v \in L^2(\Sigma;\mathbb{R}^2)$ there exists a sequence $(\varphi^\varepsilon)_{\varepsilon>0} \subset \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ such that*

$$v^\varepsilon = \tilde{\varphi}^\varepsilon \to v \text{ in } L^2(\Sigma;\mathbb{R}^2) \text{ and } \lim_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon) = G_0(v).$$

Proof. For every $v \in L^2(\Sigma;\mathbb{R}^2)$ we set:

$$G^+(v) = \inf\{\limsup_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon) : \varphi^\varepsilon \in L^2(\Sigma_\varepsilon,\mathbb{R}^2),\ v^\varepsilon = \tilde{\varphi}^\varepsilon \to v \text{ in } L^2(\Sigma;\mathbb{R}^2)\}.$$

It can be proven that the infimum is actually a minimum by the growth condition on the functional, therefore by proposition 2.1 we need only prove:

$$G^+(v) \leq G_0(v) \tag{4.2}$$

for every $v \in L^2(\Sigma;\mathbb{R}^2)$.
We observe first that inequality 4.2 is obvious when v doesn't belong to $\mathcal{A}(\Sigma)$ since for such a v we have $G_0(v) = +\infty$.
When $v \in C^2 \cap \mathcal{A}(\Sigma)$ we define for every $\varepsilon > 0$

$$\varphi^\varepsilon(x_1,x_2) = v(x_1) + x_2 w(x_1),$$

where $w = (-\dot{v}_2, \dot{v}_1)$.
We get now:

$$\begin{aligned}F_\varepsilon(\varphi^\varepsilon) &= \int_{\Sigma_\varepsilon} \frac{1}{4}\Lambda(x_1)\left|{}^T D\varphi^\varepsilon D\varphi^\varepsilon - I\right|^2 dx \\ &= \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\Big(\,|\dot{v} + x_2\dot{w}|^2 - 1\Big)^2 dx \\ &= \frac{1}{4}\int_{\Sigma_\varepsilon} \Lambda(x_1)\Big(2x_2\langle\dot{v},\dot{w}\rangle + {x_2}^2\,|\dot{w}|^2\Big)^2 dx \\ &= \frac{1}{4}\int_\Sigma \Lambda(x_1)[\frac{8}{3}\varepsilon^3\,|\langle\dot{v},\dot{w}\rangle|^2 + \frac{2}{5}\varepsilon^5|\dot{w}|^4]dx_1,\end{aligned}$$

and finally, since $v \in C^2(\Sigma;\mathbb{R}^2)$:

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^3} F_\varepsilon(\varphi^\varepsilon) = \frac{2}{3}\int_\Sigma \Lambda(x_1)\,|\ddot{v}|^2 dx_1,$$

which yields $G^+(v) \le G_0(v)$ for every $v \in C^2(\Sigma)$ and then recalling that G^+ and G_0 are weakly l.s.c. in $L^2(\Sigma,\mathbb{R}^2)$ and that C^2 is dense in $L^2(\Sigma,\mathbb{R}^2)$ inequality 4.2 is proven.

$\square$

Proof of Theorem 2.1 If $m^\varepsilon = \inf\{G_\varepsilon(\varphi) : \varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)\}$ and u^* denotes the identity function in $\mathbb{R}^2$ we get:

$$m^\varepsilon \le G_\varepsilon(u^*) = -\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon}\langle f, u^*\rangle dx - \int_{\Sigma_\varepsilon\cap\{x_2=\varepsilon\}}\langle g, u^*\rangle dx_1 \le C(f,g)$$

and therefore:

$$\begin{aligned} F_\varepsilon(\varphi^\varepsilon) &\le \varepsilon^3 m^\varepsilon + \varepsilon^2\int_{\Sigma_\varepsilon}\langle f,\varphi^\varepsilon\rangle dx + \varepsilon^3\int_{\Sigma_\varepsilon\cap\{x_2=\varepsilon\}}\langle g,\varphi^\varepsilon\rangle dx_1 + \varepsilon^4 \\ &\le \tilde{C}(f,g)\varepsilon^3. \end{aligned}$$

The same argument described in proposition 3.2 shows that the sequence $(\tilde{\varphi}^\varepsilon)$ is compact in $H^1(\Sigma;\mathbb{R}^2)$ and the limit point v_0 belongs to $H^2(\Sigma;\mathbb{R}^2)$.
We have therefore only to prove:

$$G_0(v) \ge G_0(v_0) \ \text{ for every } v \in \mathcal{A}(\Sigma). \tag{4.3}$$

By prop.4.2 there exists a sequence $(\psi_\varepsilon)_{\varepsilon>0}$ such that $\psi_\varepsilon \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$, $\tilde{\psi}_\varepsilon \to v$ in $H^1(\Sigma;\mathbb{R}^2)$ and

$$\lim_{\varepsilon\to 0} G_\varepsilon(\psi_\varepsilon) = G_0(v) \qquad (4.4).$$

Let us now observe that $G_\varepsilon(\psi_\varepsilon) \ge G_\varepsilon(\varphi^\varepsilon) - \varepsilon$, from the hypotheses on (φ^ε); hence by prop.4.1 we have:

$$\liminf_{\varepsilon\to 0} G_\varepsilon(\psi_\varepsilon) \ge \liminf_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon) \ge G_0(v_0).$$

Finally by (4.4) we get (4.3). $\square$

5. Comparison with the linear approach

We approach in this section the same constrained problem by using the linear strain measure instead of the non linear one (for the unconstrained case we refer to [3]).
We recall that, if u denotes the displacement, that is $\varphi(x) = x + u(x)$, the linearized strain tensor is defined as:

$$e(u) = \frac{1}{2}({}^T Du + Du), \tag{5.1}$$

and it is easy to verify that $E(u) = e(u) + \frac{1}{2}{}^T DuDu$.

Neglecting the second order term we shall determine a limit functional ${F_0}^{lin}$, defined on a suitable class of functions $X^{lin}(\Sigma)$, which is different from F_0.
To deal with this case, we introduce the set of admissible displacements:

$$\mathcal{A}^{lin}(\Sigma_\varepsilon) := \{u \in C^2(\Sigma_\varepsilon; \mathbb{R}^2) : e(u) \cdot (e_2) = 0,\ u(0, x_2) = (0, 0)\},$$

and we define then W, F_ε and G_ε as in section 2, with $e = e(u)$ at the place of $E = E(\varphi)$. Finally we introduce the limit functionals. We set

$$\mathcal{A}^{lin}(\Sigma) := \{z \in H^2(\Sigma; \mathbb{R}^2) :\ z_1(x_1) \equiv 0,\ z_2(0) = 0,\ \dot{z}_2(0) = 0\ \}.$$

For every $z \in H^2(\Sigma; \mathbb{R})$ we define:

$${F_0}^{lin}(z) = \begin{cases} \dfrac{2}{3} \displaystyle\int_\Sigma \Lambda(x_1)\, |\ddot{z}_2|^2\, dx_1 & \text{if } z \in \mathcal{A}^{lin}(\Sigma) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$${G_0}^{lin}(z) = \begin{cases} {F_0}^{lin}(z_2) - 2 \displaystyle\int_\Sigma f_2 z_2 dx_1 - \int_\Sigma g_2 z_2 dx_1 & \text{if } z \in \mathcal{A}^{lin}(\Sigma) \\ +\infty & \text{otherwise.} \end{cases}$$

We observe that the condition $e(u)(e_2) = 0$ is equivalent to:

$$\begin{cases} \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2} = 0 \\ \dfrac{\partial u_2}{\partial x_2} = 0, \end{cases}$$

thus obtaining the following representation for the displacements u in $\mathcal{A}^{lin}(\Sigma_\varepsilon)$:

Lemma 5.1. *If $u \in \mathcal{A}^{lin}(\Sigma_\varepsilon)$, then there exist $h,\ u_2 \in C^2(\Sigma; \mathbb{R}^2)$, such that:*

$$u(x_1, x_2) = (h(x_1) - x_2 \dot{u}_2(x_1),\ u_2(x_1)),$$

$h(0) = 0,\ u_2(0) = 0,\ \dot{u}_2(0) = 0.$

Let us now define the means as in (2.2); if $u \in \mathcal{A}^{lin}(\Sigma_\varepsilon)$ we get : $\tilde{u} = (h, u_2)$.
We give now a compactness result analogous to proposition 3.2:

Proposition 5.2. *Assume that $(u^\varepsilon)_{\varepsilon>0} \subset \mathcal{A}^{lin}(\Sigma_\varepsilon)$ and $F_\varepsilon(u^\varepsilon) \le C\varepsilon^3$ for some $C > 0$. Then, up to a subsequence, $\tilde{u}^\varepsilon \to z$ in $L^2(\Sigma, \mathbb{R}^2)$ and $z \in \mathcal{A}^{lin}(\Sigma)$. Moreover $h^\varepsilon \to 0$ in $H^1(\Sigma, \mathbb{R}^2)$ and $u_2^\varepsilon \rightharpoonup z_2$ in $H^2(\Sigma, \mathbb{R}^2)$.*

Proof. If $u^\varepsilon \in \mathcal{A}^{lin}(\Sigma_\varepsilon)$, it results:

$$e(u^\varepsilon) = \begin{pmatrix} \dot{h}^\varepsilon - x_2 \ddot{u}_2^\varepsilon & 0 \\ 0 & 0 \end{pmatrix},$$

hence $(\text{tr } e(u^\varepsilon))^2 = |e(u^\varepsilon)|^2$ and

$$\begin{aligned} C\varepsilon^3 \geq F_\varepsilon(\varphi^\varepsilon) &= \int_{\Sigma_\varepsilon} \left[\mu\, |e(u^\varepsilon)|^2 + \frac{\lambda}{2}(\text{tr } e(u^\varepsilon))^2\right] dx \\ &= \int_{\Sigma_\varepsilon} \Lambda(x_1) \left|\dot{h}^\varepsilon - x_2 \ddot{u}_2^\varepsilon\right|^2 dx \\ &= \int_{\Sigma_\varepsilon} \Lambda(x_1) \Big(\left|\dot{h}^\varepsilon\right|^2 + {x_2}^2\, |\ddot{u}_2^\varepsilon|^2 dx \Big) \\ &\geq \bar{\mu} \int_\Sigma \Big(2\varepsilon \left|\dot{h}^\varepsilon\right|^2 + \frac{2}{3}\, |\ddot{u}_2^\varepsilon|^2 \Big) dx_1. \end{aligned}$$

In particular $\dot{h}^\varepsilon$ converges to 0 in $L^2(\Sigma; \mathbb{R}^2)$. Since $h^\varepsilon(0) = 0$ we deduce that $h^\varepsilon \to 0$ in $H^1(\Sigma; \mathbb{R}^2)$ and finally, recalling that $u_2^\varepsilon(0) = \dot{u}_2^\varepsilon(0) = 0$ we get that (u_2^ε) is weakly compact in $H^2(\Sigma, \mathbb{R}^2)$. □

Proposition 5.3. *For every $z \in L^2(\Sigma; \mathbb{R}^2)$, for every sequence $(u^\varepsilon)_{\varepsilon>0}$ such that $\tilde{u}_2^\varepsilon \to z$ in $L^2(\Sigma, \mathbb{R}^2)$ we have:*

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon) \geq {G_0}^{lin}(z).$$

Proof. We may suppose that $G_\varepsilon(u^\varepsilon) \leq C$ for every $\varepsilon > 0$, otherwise the thesis is obvious and as in proposition 4.1 we have: $F_\varepsilon(u^\varepsilon) \leq \tilde{C}\varepsilon^3$ for every $\varepsilon > 0$ and for a suitable $\tilde{C} > 0$, hence, by proposition 5.2, $h^\varepsilon \to 0$ in $H^1(\Sigma; \mathbb{R}^2)$ and $u_2^\varepsilon \rightharpoonup z_2$.
We have now:

$$\begin{aligned} \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} F_\varepsilon(u^\varepsilon) &\geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\Sigma_\varepsilon} \Lambda(x_1) {x_2}^2\, |\ddot{u}_2^\varepsilon|^2 dx \\ &\geq \frac{2}{3} \int_\Sigma \Lambda(x_1)\, |\ddot{z}_2|^2\, dx_1, \end{aligned}$$

and finally:

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon) \geq {F_0}^{lin}(z_2) - 2\int_\Sigma f_2 z_2 dx_1 - \int_\Sigma g_2 z_2 dx_1.$$

□

Proposition 5.4. *For every $z \in L^2(\Sigma; \mathbb{R}^2)$ there exists a sequence*

$$(u^\varepsilon)_{\varepsilon>0} \subset \mathcal{A}^{lin}(\Sigma_\varepsilon)$$

such that

$$\tilde{u}_2^\varepsilon = u_2^\varepsilon \to z \text{ in } L^2(\Sigma; \mathbb{R}^2) \text{ and } \lim_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon) = {G_0}^{lin}(z).$$

Proof. If $z \notin \mathcal{A}^{lin}(\Sigma)$ it is easy to find a sequence $u^\varepsilon \in L^2(\Sigma_\varepsilon, \mathbb{R}^2)$ such that $\tilde{u}^\varepsilon \to z$ in $L^2(\Sigma; \mathbb{R}^2)$ and $G_\varepsilon(u^\varepsilon) = +\infty$, thus proving the thesis. When $z \in \mathcal{A}^{lin}(\Sigma)$ the proof is analogous to that of proposition 4.2, choosing:

$$u^\varepsilon(x_1, x_2) = (-x_2 \dot{z}(x_1), z(x_1))$$

for every $(x_1, x_2) \in \Sigma_\varepsilon$ and for every $\varepsilon > 0$. □

Remark 5.5. By arguing as in the previous section it is now easy to prove that minimizing sequences of the functionals G_ε are compact in the sense of theorem 2.1, that every limit point is a minimizer of the functional G_0^{lin}, while a direct computation shows that $G_0^{lin} < G_0$. We observe that in [1] it has been proven that the "limit" of the sequence of functionals

$$G_\varepsilon^{\#}(u) = \begin{cases} G_\varepsilon^{lin}(u) & \text{if } u \in H^1(\Sigma_\varepsilon, \mathbb{R}^2) \\ +\infty & \text{elsewhere in } L^2(\Sigma_\varepsilon, \mathbb{R}^2) \end{cases}$$

is ${G_0}^{lin}$ that is, roughly speaking, the constraints don't modify the limit.

6. The elastic plate

In this section Σ will be an open bounded subset of $\mathbb{R}^2$ having Lipschitz boundary, Γ a subset of $\partial\Sigma$ having non zero one-dimensional Hausdorff measure; we set for every $\varepsilon > 0$

$$\Sigma_\varepsilon := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Sigma,\ |x_3| < \varepsilon\},$$

which will be considered as the reference configuration of a three-dimensional elastic body. As in section 2 if $\varphi \in L^2(\Sigma_\varepsilon; \mathbb{R}^3)$ we set:

$$\tilde{\varphi}(x_1, x_2) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varphi(x_1, x_2, x_3) dx_3$$

and if $\varphi \in C^2(\overline{\Sigma_\varepsilon}; \mathbb{R}^3)$ is a deformation we may define the right Cauchy-Green strain tensor $C = {}^T D\varphi D\varphi$ and the Green-St.Venant strain tensor: $E = \frac{1}{2}(C - I)$.
For every $a, b, c \in \mathbb{R}^3$ we will denote by $(a|b|c)$ the matrix whose columns are a, b, c; by $\mathbb{S}^3_+$ we mean the set of all symmetric, positive definite 3×3 matrices.
We denote by e_i, $i = 1, 2, 3$ the canonical basis in $\mathbb{R}^3$ and we define the class of admissible deformations of $\overline{\Sigma_\varepsilon}$:

$$\mathcal{A}_\varepsilon(\Sigma_\varepsilon) := \{\varphi \in C^2(\overline{\Sigma_\varepsilon}, \mathbb{R}^3) : E(\varphi)(\underline{e}_3) = 0,\ \det D\varphi > 0,$$
$$\varphi_{x_1} \wedge \varphi_{x_1 x_3} = \varphi_{x_2 x_3} \wedge \varphi_{x_2} \text{ in } \overline{\Sigma_\varepsilon},\ \varphi(x) = x \text{ on } \Gamma \times [-\varepsilon, \varepsilon]\}.$$

It is worth noticing that this class contains an additional constraint with respect to the two dimensional case. As we shall prove in proposition 7.2, however, this additional constraint is automatically satisfied by the admissible limit deformations if the metric associated to the coordinate system is euclidean. Moreover, in remark 7.3 we exhibit a class of deformations that satisfy this constraint and also allow us to consider the bidimensional case just studied as a particular case of this, more general, problem.
For every $\varepsilon > 0$ and $\varphi \in L^2(\Sigma; \mathbb{R}^3)$ we define:

$$F_\varepsilon(\varphi) = \begin{cases} \int_{\Sigma_\varepsilon} W(E(\varphi)) dx & \text{if } \varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon) \\ +\infty & \text{otherwise,} \end{cases}$$

where $W(E) = \mu|E|^2 + \frac{\lambda}{2}(\text{tr } E)^2$, $\lambda, \mu \in L^\infty(\Sigma, [0, +\infty))$ being the Lamè moduli of the material and $\mu = \mu(x_1, x_2) \geq \bar{\mu} > 0$ a.e. on Σ.

As in the previous section let $f, g \in C^0(\overline{\Sigma}_1; \mathbb{R}^3)$ be given functions, corresponding to the applied loads and let f_0, g_0 be their restrictions to the "middle plane" Σ.
For every $\varepsilon > 0$ and for every $\varphi \in L^2(\Sigma_\varepsilon, \mathbb{R}^3)$ we set now

$$G_\varepsilon(\varphi) = \frac{1}{\varepsilon^3}\left[F_\varepsilon(\varphi) - \varepsilon^2 \int_{\Sigma_\varepsilon} \langle f, \varphi\rangle dx - \varepsilon^3 \int_{\Sigma_\varepsilon \cap \{x_3=\varepsilon\}} \langle g, \varphi\rangle dx_1 dx_2\right]$$

and as in section 2 a sequence $\varphi^\varepsilon \in L^2(\Sigma_\varepsilon, \mathbb{R}^3)$ will be called a minimizing sequence for G_ε if

$$\lim_{\varepsilon\to 0}\Big[G_\varepsilon(\varphi^\varepsilon) - \inf G_\varepsilon(\varphi)\Big] = 0.$$

In order to study the asymptotic behaviour of such a sequence we introduce now the set of limiting admissible deformations:

$$\mathcal{A}(\Sigma) := \{v \in H^1(\Sigma; \mathbb{R}^3) : |v_{x_1}|^2 = 1,\ |v_{x_2}|^2 = 1,\ \langle v_{x_1}, v_{x_2}\rangle = 0 \text{ on } \Sigma,$$

$$w = v_{x_1} \wedge v_{x_2} \in H^1(\Sigma; \mathbb{R}^3),\ v(x_1, x_2) = (x_1, x_2, 0),\ w(x_1, x_2) = \underline{e}_3 \text{ on } \Gamma\}.$$

For every $v \in L^2(\Sigma; \mathbb{R}^3)$ let us define:

$$F_0(v) = \begin{cases} \dfrac{2}{3}\displaystyle\int_\Sigma \Big[(\mu + \frac{\lambda}{2})H^2 - 2\mu K\Big] dx_1 dx_2 & \text{if } v \in \mathcal{A}(\Sigma) \\ +\infty & \text{otherwise,} \end{cases}$$

where $K = \langle v_{x_1}, w_{x_1}\rangle\langle v_{x_2}, w_{x_2}\rangle - |\langle v_{x_2}, w_{x_1}\rangle|^2$ and $H = -\langle Dv, Dw\rangle$ represent respectively the gaussian and the mean curvature. We define also the functional:

$$G_0(v) = F_0(v) - 2\int_\Sigma \langle f_0, v\rangle dx_1 - \int_\Sigma \langle g_0, v\rangle dx_1.$$

We shall prove the following:

Theorem 6.1. *Let $(\varphi^\varepsilon)_{\varepsilon>0} \subset L^2(\Sigma_\varepsilon, \mathbb{R}^3)$ be a minimizing sequence for G_ε. Then the sequence $(\tilde{\varphi}^\varepsilon)_{\varepsilon>0}$ is compact in $L^2(\Sigma; \mathbb{R}^3)$ and, if v_0 is one of its limit points, then $v_0 \in \mathcal{A}(\Sigma)$ (in particular $v_{0x_1} \wedge v_{0x_2} \in H^1(\Sigma; \mathbb{R}^3)$), $\tilde{\varphi}^\varepsilon \to v_0$ in $H^1(\Sigma; \mathbb{R}^3)$ and*

$$G_0(v_0) = \min\{G_0(v) : v \in \mathcal{A}(\Sigma)\} = \lim_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon).$$

7. Some preliminary lemmas

We prove now the analogous of lemma 3.1.

Lemma 7.1. *If $\varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ then there exist $v \in C^2(\Sigma; \mathbb{R}^3)$, $w \in C^1(\Sigma; \mathbb{R}^3)$ such that $\varphi(x_1, x_2, x_3) = v(x_1, x_2) + x_3 w(x_1, x_2)$, $|w|^2 = 1$, $w = \dfrac{v_{x_1} \wedge v_{x_2}}{|v_{x_1} \wedge v_{x_2}|}$,*

$v_{x_1} \wedge w_{x_1} = w_{x_2} \wedge v_{x_2}$ for every $(x_1, x_2) \in \Sigma$, $v(x_1, x_2) = (x_1, x_2, 0)$ and $w(x_1, x_2) = (0, 0, 1)$ for every $(x_1, x_2) \in \Gamma$.
Moreover, if $v \in C^2(\Sigma; \mathbb{R}^3) \cap \mathcal{A}(\Sigma)$, the deformation field φ defined by $\varphi(x_1, x_2, x_3) = v(x_1, x_2) + x_3 w(x_1, x_2)$, where $w = v_{x_1} \wedge v_{x_2}$, belongs to $\mathcal{A}_\varepsilon(\Sigma_\varepsilon)$.

Proof. Let φ be in $C^2(\Sigma_\varepsilon; \mathbb{R}^3)$. The condition $({}^T D\varphi D\varphi) \cdot (e_3) = 0$ implies:

$$\langle \varphi_{x_1}, \varphi_{x_3} \rangle = 0, \tag{7.1}$$
$$\langle \varphi_{x_2}, \varphi_{x_3} \rangle = 0, \tag{7.2}$$
$$|\varphi_{x_3}|^2 = 1. \tag{7.3}$$

Deriving again (7.1) and (7.2) with respect to x_3 we obtain now:

$$\langle \varphi_{x_1 x_3}, \varphi_{x_3} \rangle + \langle \varphi_{x_1}, \varphi_{x_3 x_3} \rangle = 0 \tag{7.4}$$
$$\langle \varphi_{x_2 x_3}, \varphi_{x_3} \rangle + \langle \varphi_{x_2}, \varphi_{x_3 x_3} \rangle = 0. \tag{7.5}$$

Deriving equation (7.3) with respect to x_1 and x_2 we have that:

$$\langle \varphi_{x_3 x_1}, \varphi_{x_3} \rangle = 0$$
$$\langle \varphi_{x_2 x_3}, \varphi_{x_3} \rangle = 0,$$

therefore it follows from (7.4) and (7.5) that $\langle \varphi_{x_1}, \varphi_{x_3 x_3} \rangle = 0$ and $\langle \varphi_{x_2}, \varphi_{x_3 x_3} \rangle = 0$.
From (7.3) it results also $\langle \varphi_{x_3}, \varphi_{x_3 x_3} \rangle = 0$; because of (7.1), (7.2) and the orientation-preserving condition $\det D\varphi > 0$ it must be: $\varphi_{x_3 x_3} \equiv 0$, that is $\varphi(x_1, x_2, x_3) = v(x_1, x_2) + x_3 w(x_1, x_2)$, for some $v,\ w;\ \Sigma \to \mathbb{R}^3$, with:

$$\begin{aligned} \langle v_{x_1}, w \rangle &= 0 \\ \langle v_{x_2}, w \rangle &= 0 \\ |w(x_1, x_2)|^2 &= 1 \quad \forall\ (x_1, x_2) \in \Sigma \end{aligned} \tag{7.6}$$

We remark that $v \in C^2(\Sigma; \mathbb{R}^3)$, because $\varphi \in C^2(\Sigma_\varepsilon; \mathbb{R}^3)$.
We have now: $\det\ D\varphi = \langle (v_{x_1} + x_3 w_{x_1}), (v_{x_2} + x_3 w_{x_2}) \wedge w \rangle$. If $\det\ D\varphi > 0$, for $x_3 = 0$ we obtain in particular $\langle v_{x_1} \wedge v_{x_2}, w \rangle > 0$.
By (7.6) we can therefore define:

$$w = \frac{v_{x_1} \wedge v_{x_2}}{|v_{x_1} \wedge v_{x_2}|}.$$

We have now that: $\varphi_{x_1} \wedge \varphi_{x_1 x_3} = (v_{x_1} + x_3 w_{x_1}) \wedge w_{x_1} = v_{x_1} \wedge w_{x_1}$ and $w_{x_2} \wedge (v_{x_2} + x_3 w_{x_2}) = w_{x_2} \wedge v_{x_2}$. Finally, the boundary conditions on v and w are easily deduced from the boundary conditions on φ. □

We observe that if $\varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ and $\varphi(x_1, x_2, x_3) = v(x_1, x_2) + x_3 w(x_1, x_2)$, then $\tilde{\varphi} = v$.
In the following if $\varphi \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ we shall write $\varphi(x_1, x_2, x_3) = v(x_1, x_2) + x_3 w(x_1, x_2)$, with v, w such as in lemma 7.1.

We will now prove that the internal constraint $v_{x_1} \wedge w_{x_1} = w_{x_2} \wedge v_{x_2}$ is satisfied by a wide class of functions. Indeed we have the following:

Proposition 7.2. *Let $v : \Sigma \to \mathbb{R}^3$ a C^2 function such that:*

$$|v_{x_1}|^2 = 1, \; |v_{x_2}|^2 = 1, \; \langle v_{x_1}, v_{x_2} \rangle = 0. \tag{7.7}$$

Then, if $w = v_{x_1} \wedge v_{x_2}$, we have:

$$v_{x_1} \wedge w_{x_1} = w_{x_2} \wedge v_{x_2}.$$

Proof. By the conditions (7.7) we get:

$$0 = \langle v_{x_1}, v_{x_1 x_1} \rangle = \langle v_{x_2}, v_{x_2 x_2} \rangle = \langle v_{x_1}, v_{x_1 x_2} \rangle = \langle v_{x_2}, v_{x_1 x_1} \rangle,$$

hence there exist $\alpha, \; \beta, \; \gamma, \; \delta, \; \theta \; : \Sigma \to \mathbb{R}$ such that:

$$\begin{aligned} v_{x_1 x_1} &= \alpha v_{x_2} + \beta v_{x_1} \wedge v_{x_2} \\ v_{x_2 x_2} &= \gamma v_{x_1} + \delta v_{x_1} \wedge v_{x_2} \\ v_{x_1 x_2} &= \theta v_{x_1} \wedge v_{x_2}. \end{aligned}$$

We get now:

$$w_{x_1} = (v_{x_1} \wedge v_{x_2})_{x_1} = \beta v_{x_1} - \theta v_{x_2}, \tag{7.8}$$
$$w_{x_2} = (v_{x_1} \wedge v_{x_2})_{x_2} = -\theta v_{x_1} - \delta v_{x_2}. \tag{7.9}$$

We get now from (7.8) and (7.9):

$$v_{x_1} \wedge w_{x_1} = \theta v_{x_1} \wedge v_{x_2} = w_{x_2} \wedge v_{x_2}.$$

□

Broadly speaking, the previous result shows that, if the metric associated to the coordinate system is euclidean, then the internal constraint is always satisfied.

Remark 7.3. It is interesting to observe that also the functions of the type:

$$v(x_1, x_2, x_3) = (v_1(x_1), x_2, v_2(x_1))$$

verify $v_{x_1} \wedge w_{x_1} = w_{x_2} \wedge v_{x_2}$, because this allows us to consider the case studied in the previous section as a particular case of the threedimensional problem. In particular if we take $\Sigma = (0,1) \times (\alpha, \beta)$ simple computations show that:

$$F_0(v) = \frac{2}{3} \int_{\Sigma} \Lambda(x_1, x_2) \, |\langle v_{x_1}, w_{x_1} \rangle|^2 dx_1 dx_2$$

and, since in this case $\langle v_{x_1}, w_{x_1} \rangle = |\ddot{v}|^2$ we have:
$F_0(v) = \frac{2}{3} \int_0^1 \tilde{\Lambda}(x_1) \, |\ddot{v}|^2 dx_1$, where $\tilde{\Lambda}(x_1) = \int_\alpha^\beta \Lambda(x_1, x_2) dx_2$.

Lemma 7.4. *Let a^ε, b^ε, c^ε: $\Sigma \to \mathbb{R}^3$ be such that: $|a^\varepsilon|^2 = |b^\varepsilon|^2 = 1$, $\langle a^\varepsilon, b^\varepsilon\rangle = 0$, $c^\varepsilon = a^\varepsilon \wedge b^\varepsilon$. If $c^\varepsilon \to c$ in $L^2(\Sigma;\mathbb{R}^2)$, then $a^{\varepsilon_j} \to a$ and $b^{\varepsilon_j} \to b$ in $L^2(\Sigma;\mathbb{R}^2)$ for some subsequences $(a^{\varepsilon_j})_j$ and $(b^{\varepsilon_j})_j$ of $(a^\varepsilon)_\varepsilon$ and $(b^\varepsilon)_\varepsilon$.*

Proof. It holds for every $\varepsilon > 0$:

$$a^\varepsilon = b^\varepsilon \wedge (a^\varepsilon \wedge b^\varepsilon).$$

There exists a subsequence (b^{ε_j}) of (b^ε) such that $b^{\varepsilon_j} \rightharpoonup b$ and $a^\varepsilon \wedge b^\varepsilon \to c$, therefore a^{ε_j} converges weakly to $b \wedge c$ in $L^2(\Sigma;\mathbb{R}^2)$. Moreover it results $a = b \wedge c$ and also: $a \wedge b = (b \wedge c) \wedge b = c$.

We have now that for every measurable subset $A \subset \Sigma$ $a^{\varepsilon_j} \rightharpoonup a$ in $L^2(A)$ so that by the lower semicontinuity of the norm we deduce:

$$\int_A |a|^2 dx \leq \liminf \int_A |a^{\varepsilon_j}|^2 dx = |A|.$$

It results therefore:

$$\frac{1}{|A|}\int_A |a|^2 dx \leq 1$$

and also

$$\frac{1}{|A|}\int_A |b|^2 dx \leq 1$$

for every $A \in \Sigma$, A measurable.

This implies $|a|^2 \leq 1$ and $|b|^2 \leq 1$ a.e. on Σ.

We have now that:

$$1 = |a \wedge b| \leq |a| \cdot |b| \leq 1,$$

that is $|a| = |b| = 1$ and $\langle a, b\rangle = 0$.

We get finally

$$\int_\Sigma |a^{\varepsilon_j} - a|^2 \, dx_1 dx_2 = \int_\Sigma (|a^{\varepsilon_j}|^2 + |a|^2 - 2\langle a^{\varepsilon_j}, a\rangle) dx_1 dx_2 = 2\int_\Sigma (1 - \langle a^{\varepsilon_j}, a\rangle),$$

hence $a^{\varepsilon_j} \to a$ in $L^2(\Sigma;\mathbb{R}^2)$. Analogously we obtain $b^{\varepsilon_j} \to b$. □

Proposition 7.5. *Assume that $\varphi^\varepsilon \in \mathcal{A}_\varepsilon(\Sigma_\varepsilon)$ and $F_\varepsilon(\varphi^\varepsilon) \leq C\varepsilon^3$. Then:*

i) *$(v^\varepsilon)_\varepsilon$ is relatively compact in $H^1(\Sigma;\mathbb{R}^3)$ and (w^ε) is relatively weakly compact in $H^1(\Sigma;\mathbb{R}^3)$;*

ii) *if $v^\varepsilon \to v$ in $L^2(\Sigma,\mathbb{R}^3)$ then $v \in \mathcal{A}(\Sigma)$;*

iii)

$$\langle v^\varepsilon_{x_i}, w^\varepsilon_{x_i}\rangle \rightharpoonup \langle v_{x_i}, w_{x_i}\rangle, \; i = 1, 2$$

$$\langle v^\varepsilon_{x_1}, w^\varepsilon_{x_2}\rangle + \langle w^\varepsilon_{x_1}, v^\varepsilon_{x_2}\rangle \rightharpoonup \langle v_{x_1}, w_{x_2}\rangle + \langle w_{x_1}, v_{x_2}\rangle$$

in $L^2(\Sigma)$.

Proof. Exactly as in proposition 3.2 by applying the polar factorization of the invertible matrix $D\varphi^\varepsilon$ we get:

$$F_\varepsilon(\varphi^\varepsilon) \geq C \int_{\Sigma_\varepsilon} |D\varphi^\varepsilon - R^\varepsilon|^2 dx,$$

where R^ε is the rotation such that:

$$D\varphi^\varepsilon = R^\varepsilon \sqrt{{}^T D\varphi^\varepsilon D\varphi^\varepsilon}.$$

We represent now R^ε as: $R^\varepsilon = (a^\varepsilon | b^\varepsilon | c^\varepsilon)$, with $|a^\varepsilon|^2 = |b^\varepsilon|^2 = |c^\varepsilon|^2 = 1$, $c^\varepsilon = a^\varepsilon \wedge b^\varepsilon$ and we observe that the matrix :

$${}^T R^\varepsilon D\varphi^\varepsilon = \begin{pmatrix} \langle a^\varepsilon, \varphi^\varepsilon_{x_1} \rangle & \langle a^\varepsilon, \varphi^\varepsilon_{x_2} \rangle & \langle a^\varepsilon, w^\varepsilon \rangle \\ \langle b^\varepsilon, \varphi^\varepsilon_{x_1} \rangle & \langle b^\varepsilon, \varphi^\varepsilon_{x_2} \rangle & \langle b^\varepsilon, w^\varepsilon \rangle \\ \langle a^\varepsilon, w^\varepsilon \rangle & \langle b^\varepsilon, w^\varepsilon \rangle & \langle c^\varepsilon, w^\varepsilon \rangle \end{pmatrix}$$

belongs to $\mathbb{S}^3_+$ as square root of a symmetric and positive definite matrix.
Since ${}^T R^\varepsilon D\varphi^\varepsilon = \sqrt{{}^T D\varphi^\varepsilon D\varphi^\varepsilon}$ we get $\langle a^\varepsilon, w^\varepsilon \rangle = 0$, $\langle b^\varepsilon, w^\varepsilon \rangle = 0$ and $\langle c^\varepsilon, w^\varepsilon \rangle = 1$, therefore $c^\varepsilon = w^\varepsilon$ for every $\varepsilon > 0$.
The symmetry condition yields:

$$\langle a^\varepsilon, \varphi^\varepsilon_{x_2} \rangle = \langle b^\varepsilon, \varphi^\varepsilon_{x_1} \rangle \text{ for every } \varepsilon > 0.$$

Recalling that $b^\varepsilon = w^\varepsilon \wedge a^\varepsilon$ we get:

$$\langle \varphi^\varepsilon_{x_2} - \varphi^\varepsilon_{x_1} \wedge w^\varepsilon, a^\varepsilon \rangle = 0.$$

We observe first of all that the vector:

$$\varphi^\varepsilon_{x_2} - \varphi^\varepsilon_{x_1} \wedge w^\varepsilon = (v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon) + x_3(w^\varepsilon_{x_2} - w^\varepsilon_{x_1} \wedge w^\varepsilon)$$

is orthogonal to a^ε and to w^ε and also that there exists $\sigma = \sigma(x_1, x_2)$ such that:

$$\varphi^\varepsilon_{x_2} - \varphi^\varepsilon_{x_1} \wedge w^\varepsilon = [1 + x_3\sigma(x_1, x_2)] \cdot (v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon). \tag{7.10}$$

In fact the vectors $v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon$ and $w^\varepsilon_{x_2} - w^\varepsilon_{x_1} \wedge w^\varepsilon$ are both orthogonal to w^ε and also satisfy:

$$(v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon) \wedge (w^\varepsilon_{x_2} - w^\varepsilon_{x_1} \wedge w^\varepsilon) = 0. \tag{7.11}$$

To prove (7.11) we observe first of all that:

$$0 = \langle w^\varepsilon, v^\varepsilon_{x_2} \rangle_{x_1} = \langle w^\varepsilon_{x_1}, v^\varepsilon_{x_2} \rangle + \langle w^\varepsilon, v^\varepsilon_{x_2 x_1} \rangle$$

$$0 = \langle w^\varepsilon, v^\varepsilon_{x_1} \rangle_{x_2} = \langle w^\varepsilon_{x_2}, v^\varepsilon_{x_1} \rangle + \langle w^\varepsilon, v^\varepsilon_{x_1 x_2} \rangle.$$

Since v^ε belong to $C^2(\Sigma_\varepsilon; \mathbb{R}^3)$ we have:

$$\langle w^\varepsilon_{x_1}, v^\varepsilon_{x_2} \rangle = \langle w^\varepsilon_{x_2}, v^\varepsilon_{x_1} \rangle.$$

We get now:

$$(v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon) \wedge (w^\varepsilon_{x_2} - w^\varepsilon_{x_1} \wedge w^\varepsilon) = v^\varepsilon_{x_2} \wedge w^\varepsilon_{x_2} - w^\varepsilon_{x_1}\langle v^\varepsilon_{x_2}, w^\varepsilon\rangle + w^\varepsilon\langle v^\varepsilon_{x_2}, w^\varepsilon_{x_1}\rangle +$$

$$v^\varepsilon_{x_1}\langle w^\varepsilon, w^\varepsilon_{x_2}\rangle - w^\varepsilon\langle v^\varepsilon_{x_1}, w^\varepsilon_{x_2}\rangle + w^\varepsilon_{x_1}\langle v^\varepsilon_{x_1} \wedge w^\varepsilon, w^\varepsilon\rangle - w^\varepsilon\langle v^\varepsilon_{x_1} \wedge w^\varepsilon, w^\varepsilon_{x_1}\rangle =$$

$$v^\varepsilon_{x_2} \wedge w^\varepsilon_{x_2} - w^\varepsilon\langle v^\varepsilon_{x_1} \wedge w^\varepsilon, w^\varepsilon_{x_1}\rangle = v^\varepsilon_{x_2} \wedge w^\varepsilon_{x_2} - w^\varepsilon\langle w^\varepsilon_{x_1} \wedge v^\varepsilon_{x_1}, w^\varepsilon\rangle =$$

$$v^\varepsilon_{x_2} \wedge w^\varepsilon_{x_2} - w^\varepsilon_{x_1} \wedge v^\varepsilon_{x_1} = 0$$

and (7.11) is proven.
Because of the orthogonality between $\varphi^\varepsilon_{x_2} - \varphi^\varepsilon_{x_1} \wedge w^\varepsilon$ and a^ε and w^ε, we get that $v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon$ is orthogonal to a^ε and w^ε. Moreover it holds: $v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon \neq 0$, because, if $v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon = 0$, it would be:

$$\det D\varphi^\varepsilon{}_{|x_3=0} = \langle v^\varepsilon_{x_1}, v^\varepsilon_{x_2} \wedge w^\varepsilon\rangle = -\langle v^\varepsilon_{x_1}, w^\varepsilon \wedge v^\varepsilon_{x_2}\rangle = -|v^\varepsilon_{x_1}|^2 < 0,$$

a contradiction. We have therefore:

$$b^\varepsilon = \frac{v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon}{|v^\varepsilon_{x_2} - v^\varepsilon_{x_1} \wedge w^\varepsilon|}$$

and also:

$$a^\varepsilon = b^\varepsilon \wedge w^\varepsilon = \frac{v^\varepsilon_{x_1} + v^\varepsilon_{x_2} \wedge w^\varepsilon}{|v^\varepsilon_{x_1} + v^\varepsilon_{x_2} \wedge w^\varepsilon|}.$$

We get now as in proposition 3.2:

$$\begin{aligned} C\varepsilon^3 \geq & \int_{\Sigma_\varepsilon} \Big(\left|v^\varepsilon_{x_1} + x_3 w^\varepsilon_{x_1} - a^\varepsilon\right|^2 + \left|v^\varepsilon_{x_2} + x_3 w^\varepsilon_{x_2} - b^\varepsilon\right|^2 \\ & \quad + |w^\varepsilon - c^\varepsilon|^2 \Big) dx \\ = & \int_{\Sigma_\varepsilon} \Big(\left|v^\varepsilon_{x_1} - a^\varepsilon\right|^2 + {x_3}^2 \left|w^\varepsilon_{x_1}\right|^2 + \left|v^\varepsilon_{x_2} - b^\varepsilon\right|^2 + {x_3}^2 \left|w^\varepsilon_{x_2}\right|^2 \Big) dx \\ = & \, 2\varepsilon \int_\Sigma (\left|v^\varepsilon_{x_1} - a^\varepsilon\right|^2 + \left|v^\varepsilon_{x_2} - b^\varepsilon\right|^2) dx_1 dx_2 + \frac{2}{3}\varepsilon^3 \int_\Sigma |Dw|^2 dx_1 dx_2. \end{aligned}$$

We have $\int_\Sigma |w^\varepsilon|^2 dx_1 dx_2 + \int_\Sigma |Dw^\varepsilon|^2 dx_1 dx_2 \leq C$, for some $C > 0$; there exist therefore $w \in H^1(\Sigma; \mathbb{R}^3)$ and a subsequence of (w^ε), always indicated by (w^ε), such that w^ε converges weakly in $H^1(\Sigma; \mathbb{R}^3)$ to w.
In particular, w^ε converges strongly to w in $L^2(\Sigma, \mathbb{R}^3)$; therefore $c^\varepsilon \to w$ in $L^2(\Sigma, \mathbb{R}^3)$.
By using Lemma 7.4 we obtain $v^\varepsilon_{x_1} \to a$ and $v^\varepsilon_{x_2} \to b$ in $L^2(\Sigma, \mathbb{R}^3)$, possibly passing to a subsequence, with $|a|^2 = |b|^2 = 1$ and $\langle a, b\rangle = 0$, therefore (v^ε) is relatively compact in $H^1(\Sigma; \mathbb{R}^3)$, with $|v_{x_1}|^2 = 1$, $|v_{x_2}|^2 = 1$ and $\langle v_{x_1}, v_{x_2}\rangle = 0$ a.e. on Σ.

To prove iii) we observe that:

$$
\begin{aligned}
F_\varepsilon(\varphi^\varepsilon) &= \int_{\Sigma_\varepsilon} \Big[\mu(x_1,x_2)\,|E(\varphi^\varepsilon)|^2 + \frac{\lambda}{2}(x_1,x_2)(\mathrm{tr}\ E(\varphi^\varepsilon))^2\Big]dx \\
&\geq \bar{\mu}\int_{\Sigma_\varepsilon} |E(\varphi^\varepsilon)|^2 dx \\
&= \frac{\bar{\mu}}{4}\int_{\Sigma_\varepsilon} [(\,|\varphi^\varepsilon_{x_1}|^2-1)^2 + (\,|\varphi^\varepsilon_{x_2}|^2-1)^2 + 2\,|\langle\varphi^\varepsilon_{x_1},\varphi^\varepsilon_{x_2}\rangle|^2\,]dx \\
&= \frac{\bar{\mu}}{4}\int_{\Sigma_\varepsilon} \Big\{\Big(|v^\varepsilon_{x_1}+x_3 w^\varepsilon_{x_1}|^2-1\Big)^2 + \Big(|v^\varepsilon_{x_2}+x_3 w^\varepsilon_{x_2}|^2-1\Big)^2 \\
&\qquad + 2\Big[\langle v^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle + x_3(\langle v^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle+\langle v^\varepsilon_{x_2},w^\varepsilon_{x_1}\rangle) \\
&\qquad + x_3^2\langle w^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle\Big]^2\Big\}dx \\
&= \frac{\bar{\mu}}{4}\int_{\Sigma_\varepsilon} \Big\{\Big[\,|v^\varepsilon_{x_1}|^2-1+{x_3}^2\,|w^\varepsilon_{x_1}|^2\,\Big]^2 + \Big[\,|v^\varepsilon_{x_2}|^2-1+{x_3}^2\,|w^\varepsilon_{x_2}|^2\,\Big]^2 \\
&\qquad + 4{x_3}^2|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_1}\rangle|^2 + 4{x_3}^2|\langle v^\varepsilon_{x_2},w^\varepsilon_{x_2}\rangle|^2 + 2\Big[\langle v^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle \\
&\qquad + x_3^2|\langle w^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle|^2\Big]^2 + 2{x_3}^2|\langle w^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle+\langle v^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle|^2\Big\} \\
&\geq \frac{\bar{\mu}}{4}\int_{\Sigma_\varepsilon} 4{x_3}^2[\,|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_1}\rangle|^2+|\langle v^\varepsilon_{x_2},w^\varepsilon_{x_2}\rangle|^2\,] \\
&\qquad + 2x_3^2\,|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle+\langle w^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle|^2\,dx.
\end{aligned}
$$

By the hypothesis $F_\varepsilon(\varphi^\varepsilon)\leq C\varepsilon^3$ it results therefore that $(\|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_1}\rangle\|_{L^2})$ $(\|\langle v^\varepsilon_{x_2},w^\varepsilon_{x_2}\rangle\|_{L^2})$ and $(\|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle+\langle w^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle\|_{L^2})$ are bounded. Since $\langle v^\varepsilon_{x_1},w^\varepsilon_{x_1}\rangle$, $\langle v^\varepsilon_{x_2},w^\varepsilon_{x_2}\rangle$ and $\langle v^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle+$ $\langle w^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle$ converge in the sense of the distribution respectively to $\langle v_{x_1},w_{x_1}\rangle$, $\langle v_{x_2},w_{x_2}\rangle$ and $\langle v_{x_1},w_{x_2}\rangle+\langle w_{x_1},v_{x_2}\rangle$, and $Dv^\varepsilon\to Dv$ and $Dw^\varepsilon\rightharpoonup Dw$ in $L^2(\Sigma;\mathbb{R}^3)$, we get the thesis. □

8. Proof of the main result

Proposition 8.1. *For every $v\in H^1(\Sigma;\mathbb{R}^3)$, for every sequence $(\varphi^\varepsilon)_{\varepsilon>0}$ such that*

$$\varphi^\varepsilon\in\mathcal{A}_\varepsilon(\Sigma_\varepsilon) \text{ and } \tilde{\varphi}^\varepsilon = v^\varepsilon\to v \text{ in } H^1(\Sigma;\mathbb{R}^3)$$

we have:

$$\liminf_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon)\geq G_0(v). \tag{8.1}$$

Proof. Exactly as in proposition 4.1 we may suppose $G_\varepsilon(\varphi^\varepsilon)\leq C$ and we get $F_\varepsilon(\varphi^\varepsilon)\leq C\varepsilon^3$, hence $v\in\mathcal{A}(\Sigma)$ by proposition 7.5.

We have now:

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}F_\varepsilon(\varphi^\varepsilon) &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\Big[\mu(x_1,x_2)\,|E(\varphi^\varepsilon)|^2+\frac{\lambda}{2}(x_1,x_2)(\mathrm{tr}\ E(\varphi^\varepsilon))^2\Big]dx\\ &= \liminf_{\varepsilon\to 0}\,(I^1_\varepsilon+I^2_\varepsilon).\end{aligned}$$

As in proposition 7.5 we get:

$$\begin{aligned}\liminf_{\varepsilon\to 0} I^1_\varepsilon &\geq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\mu}{4}\{4{x_3}^2[\,\big|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_1}\rangle\big|^2+\big|\langle v^\varepsilon_{x_2},w^\varepsilon_{x_2}\rangle\big|^2\,]\\ &\qquad\qquad + 2x_3^2\,\big|\langle v^\varepsilon_{x_1},w^\varepsilon_{x_2}\rangle+\langle w^\varepsilon_{x_1},v^\varepsilon_{x_2}\rangle\big|^2\,\}dx\\ &\geq \frac{1}{3}\int_\Sigma\mu\Big[2\Big(\,|\langle v_{x_1},w_{x_1}\rangle|^2+|\langle v_{x_2},w_{x_2}\rangle|^2\Big)\\ &\qquad\qquad + |\langle v_{x_1},w_{x_2}\rangle+\langle w_{x_1},v_{x_2}\rangle|^2\Big]dx_1dx_2.\end{aligned}$$

We have moreover:

$$\begin{aligned}I^2_\varepsilon &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{2}(x_1,x_2)(\mathrm{tr}\ E(\varphi^\varepsilon))^2dx\\ &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{8}\Big[|\varphi^\varepsilon_{x_1}|^2+|\varphi^\varepsilon_{x_2}|^2-2\Big]^2dx\\ &\geq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\Big\{\frac{\lambda}{8}\Big[|\varphi_{x_1}|^2+|\varphi_{x_2}|^2-2\Big]^2\\ &\quad + 2\Big(|\varphi_{x_1}|^2+|\varphi_{x_2}|^2-2\Big)\cdot\Big[|\varphi^\varepsilon_{x_1}|^2-|\varphi_{x_1}|^2+|\varphi^\varepsilon_{x_2}|^2-|\varphi_{x_2}|^2\Big]\Big\}dx\\ &= \liminf_{\varepsilon\to 0}\,(J^1_\varepsilon+J^2_\varepsilon).\end{aligned}$$

We have:

$$\begin{aligned}\liminf_{\varepsilon\to 0} J^1_\varepsilon &\geq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{8}\Big[2x_3\langle Dv,Dw\rangle+x_3^2\,|Dw|^2\Big]^2dx\\ &\geq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{2}{x_3}^2\,|\langle Dv,Dw\rangle|^2dx\\ &= \frac{1}{3}\int_\Sigma\lambda\,|\langle Dv,Dw\rangle|^2dx_1dx_2.\end{aligned}$$

We get finally:

$$\begin{aligned}
\liminf_{\varepsilon\to 0} J^2_\varepsilon &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{4}\Big(|\varphi_{x_1}|^2+|\varphi_{x_2}|^2-2\Big)\cdot\Big(|D\varphi^\varepsilon|^2-|D\varphi|^2\Big)dx\\
&= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{4}\Big[2x_3\langle Dv, Dw\rangle + {x_3}^2\,|Dw|^2\Big]\cdot\Big[\,|Dv^\varepsilon|^2-|Dv|^2+\\
&\qquad + 2x_3(\langle Dv^\varepsilon, Dw^\varepsilon\rangle - \langle Dv, Dw\rangle) + x_3^2(\,|Dw^\varepsilon|^2-|Dw|^2\,)\Big]dx\\
&= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Sigma_\varepsilon}\frac{\lambda}{4}\Big\{4x_3^2\langle Dv, Dw\rangle\cdot[\langle Dv^\varepsilon, Dw^\varepsilon\rangle - \langle Dv, Dw\rangle]\\
&\quad + x_3^2\,|Dw|^2\,(\,|Dv^\varepsilon|^2-|Dv|^2\,) + {x_3}^2|Dw|^2(\,|Dw^\varepsilon|^2-|Dw|^2\,)\Big\}dx.
\end{aligned}$$

The minimum limit of the first integral is 0 because $\langle Dv^\varepsilon, Dw^\varepsilon\rangle \rightharpoonup \langle Dv, Dw\rangle$ in $L^2(\Sigma)$; for the second one we observe that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_\Sigma |Dw|^2(\,|Dv^\varepsilon|^2-|Dv|^2)dx + \liminf_{\varepsilon\to 0}\int_\Sigma |Dw|^2(\,|Dw^\varepsilon|^2-|Dw|^2\,)dx\\
\geq &\liminf_{\varepsilon\to 0}\int_\Sigma |Dw|^2\,|Dv^\varepsilon|^2 dx_1dx_2 - \int_\Sigma |Dw|^2\,|Dv|^2 dx_1dx_2\\
&+\liminf_{\varepsilon\to 0}\int_\Sigma |Dw|^2\,|Dw^\varepsilon|^2 dx_1dx_2 - \int_\Sigma |Dw|^2\,|Dw|^2 dx_1dx_2 \geq 0
\end{aligned}$$

by the lower semicontinuity of the functional: $\eta\mapsto\int_\Sigma |Dw|^2\,|\eta|^2 dx_1dx_2$ with respect to the weak topology of $H^1(\Sigma)$, hence

$$\liminf_{\varepsilon\to 0} I^2_\varepsilon \geq \frac{1}{3}\int_\Sigma \lambda(\langle v_{x_1}, w_{x_1}\rangle + \langle v_{x_2}, w_{x_2}\rangle)^2 dx_1dx_2$$

and we can conclude as in proposition 4.1.

□

Proposition 8.2. *For every $v\in L^2(\Sigma;\mathbb{R}^3)$ there exists a sequence*

$$(\varphi^\varepsilon)_{\varepsilon>0}\subset\mathcal{A}_\varepsilon(\Sigma_\varepsilon)$$

such that

$$v^\varepsilon=\tilde{\varphi}^\varepsilon\to v \text{ in } L^2(\Sigma;\mathbb{R}^3) \text{ and } \lim_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon)=G_0(v).$$

Proof. Exactly as in proposition 4.2 we define for every $v\in L^2(\Sigma;\mathbb{R}^3)$ the functional $G^+(v)$. The inequality $G^+(v)\leq G_0(v)$ is obvious if v doesn't belong to $\mathcal{A}(\Sigma)$.
When $v\in C^2\bigcap\mathcal{A}(\Sigma)$ we define for every $\varepsilon>0$

$$\varphi^\varepsilon(x_1,x_2,x_3)=v(x_1,x_2)+x_3w(x_1,x_2),$$

where $w=v_{x_1}\wedge v_{x_2}$.

We obtain now:

$$F_\varepsilon(\varphi^\varepsilon) = \frac{1}{4}\int_{\Sigma_\varepsilon} \Big\{\mu\Big[\Big(2x_3\langle v_{x_1}, w_{x_1}\rangle + x_3^2\,|w_{x_1}|^2\Big)^2 + \Big(2x_3\langle v_{x_2}, w_{x_2}\rangle + x_3^2\,|w_{x_2}|^2\Big)^2 + |\langle v_{x_1} + x_3 w_{x_1}, v_{x_2} + x_3 w_{x_2}\rangle|^2\Big] + \frac{\lambda}{2}\Big[|v_{x_1} + x_3 w_{x_1}|^2 - 1 + |v_{x_2} + x_3 w_{x_2}|^2 - 1\Big]\Big\}dx.$$

With the same computations of propositions 7.5 and 8.1 we get finally:

$$\lim_{\varepsilon\to 0} G_\varepsilon(\varphi^\varepsilon) = G_0(v).$$

□

Proof of Theorem 6.1 The proof is exactly the same of Theorem 2.1.

Remark 8.3. It is possible generalize with slight modifications the previous result to the physically significant case, in which the energy becomes infinite when the volume locally vanishes.
If we suppose that the deformation energy density is of the form:

$$W(E) = \Theta(\det(2\mathrm{E}+\mathrm{I}))\cdot[\mu|E|^2 + \frac{\lambda}{2}(\mathrm{tr}\ E)^2],$$

where the function $\Theta : (0,+\infty) \to [0,+\infty)$ satisfies the conditions:
i) Θ is continuous;
ii) $\Theta(s) \geq \Theta(1) > 0$ for every $s \in (0,+\infty)$;
iii) $\lim_{s\to 0^+}\Theta(s) = +\infty$,
then all results continue to hold.

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HIER :

Leere Seite
244