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A Resolution of Simons' Maximal Monotonicity Problem

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We present a complete answer to Simons' question about strong maximal monotonicity of subdifferentials of convex functions.

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1. Introduction

Rockafellar's maximal monotonicity theorem [5], [4] is one of the fundamental results of convex analysis. It says that if f is a proper convex lower semicontinuous function on a real Banach space X with topological dual X', then subdifferential of f is maximal monotone, that is if $(y, y^*) \in X \times X'$ and for every $x \in X, x^* \in \partial f(x)$ one has $\langle y^* - x^*, y - x \rangle \geq 0$, then $y^* \in \partial f(y)$.
By our knowledge there exist at least five different proofs of this fact in the literature, the original one of Rockafellar [5] and the others by Taylor [8], Borwein [1], Simons [6], Luc [2]. The proof given in the latter paper via Zagrodny's approximate mean value theorem [9] seems to be the simplest.

In [6], [7] Simons generalized maximal monotonicity by replacing y or y^* above by appropriate convex sets. Specifically, he proved the following important results which subsume maximal monotonicity:

(i) Let B be a nonempty weakly compact convex subset of X and $b^* \in X' \setminus \partial f(B)$. Then there exist $y \in X$ and $y^* \in \partial f(y)$ such that $\langle y^* - b^*, b - y \rangle > 0$, for all $b \in B$;
(ii) Let B^* be a nonempty weakly* compact convex subset of X' and $b \in X$ with $\partial f(b) \cap B^* = \emptyset$. Then there exist $y \in X$ and $y^* \in \partial f(y)$ such that $\langle y^* - b^*, b - y \rangle > 0$, for all $b^* \in B^*$;

Simple proofs of these results were given in [3] by using a "region type" generalization of Zagrodny's mean value theorem.

The following question was asked by Simons in his lecture delivered at the CIRM Conference held from 22 to 26 June 1992 at Marseille, France (see [7]). If B is a nonempty weakly

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compact convex subset of X and B^* is a nonempty weakly* compact convex subset of X' with $\partial f(B) \cap B^* = \emptyset$, does there exist $y \in X$ and $y^* \in \partial f(y)$ such that $\langle y^* - b^*, b - y \rangle \geq 0$, for all $b \in B, b^* \in B^*$.

The aim of the present note is to give a complete answer to this question. Namely, we show that the answer is yes in the case where X is one-dimensional and the answer is no in higher dimensions. We thank an anonymous referee for useful suggestions which lead to a shortened proof of Theorem 2.2.

2. The one-dimensional case

In this section we suppose that f is a proper convex lower semicontinuous function on $\mathbb{R}$ with values in the extended real line $\mathbb{R} \cup \{+\infty\}$. We shall need the following result, which is a one-dimensional version of Lemma 2.1 of [2].

Lemma 2.1. *Assume that g is a weakly lower semicontinuous function on $\mathbb{R}$ with values in $\mathbb{R} \cup \{+\infty\}, x_0 < a$ and $g(x_0) < g(a)$. Then there exist $y < a$ and $z^* \in \partial g(y)$ such that $z^* > 0$.*

In the above lemma the subdifferential can be any one which satisfies certain natural requirements (see [3]). It is sufficient to note that when g is convex, the subdifferential is understood in the sense of convex analysis, that is $\partial g(x) = \{x^* \in X' : \langle x^*, v \rangle \leq g(x+v) - g(x) \text{ for all } v \in X\}$.

Now we present the main result of this section.

Theorem 2.2. *Let f be a proper convex lower semicontinuous function on $\mathbb{R}$ with values in the extended real line. Let $[a,b]$ and $[c,d]$ be two segments in $\mathbb{R}$ such that $\partial f(x) \cap [c,d] = \emptyset$ for all $x \in [a,b]$. Then there exist $y \in \mathbb{R} \setminus [a,b]$ and $y^* \in \partial f(y)$ such that $(y^* - x^*)(x - y) > 0$ for all $x \in [a,b], x^* \in [c,d]$.*

Proof. Let h be a convex function on $\mathbb{R}$ defined by $h(x) := d(a-x)$ if $x \leq a$ and $h(x) := c(a-x)$ if $x > a$. Consider the function $g(x) := f(x) + h(x)$. This function is proper convex, lower semicontinuous. Moreover, $\partial g(x) \subseteq \partial f(x) + \partial h(x) \subseteq \partial f(x) + [-d, -c]$. It follows from the condition of the theorem that $0 \notin \partial g([a,b])$. Consequently, the segment $[a,b]$ does not contain any minimum of g, that is there exists a point $x_0 \in \mathbb{R}$ such that $g(x_0) < \inf_{x \in [a,b]} g(x)$. We assume $x_0 < a$. The case $x_0 > b$ is similar. Then $g(x_0) < g(a)$. By Lemma 2.1, there exist $y < a$ and $z^* \in \partial g(y)$ such that $z^* > 0$. For $x < a$, one has $\partial h(x) = \{-d\}$, therefore one can find $y^* \in \partial f(y)$ such that $z^* = y^* - d$. Hence $y* > d$. Now, for $x \in [a,b]$ and $x^* \in [c,d]$ we have $x > y$ and $y^* > x^*$. Consequently, $(y^* - x^*)(x - y) > 0$ as requested. □

It is worthwhile noticing that Corollary 2.2 of [3] which is a generalization of Lemma 2.1 to the case where g is defined on a general Banach space and a is replaced by a weakly compact convex set can also be used to prove Theorem 2.2 above.

3. A counterexample

We are going to construct a differentiable convex function f on $\mathbb{R}^2$ and two convex compact sets B, B^* in $\mathbb{R}^2$ for which the answer to Simons' question is "no". First we define a function h on $\mathbb{R}$ by

$$h(x) = \begin{cases} -1 & \text{if } x < -2, \\ (x+2)^2 - 1 & \text{if } x \in [-2,-1], \\ 2(x+1) & \text{if } x \in [-1,1], \\ x^2 + 3 & \text{if } x > 1. \end{cases}$$

It is obvious that h is a differentiable convex function on $\mathbb{R}$. Now define f on $\mathbb{R}^2$ by $f(x,y) = h(x) + y/2$. Furthermore, let B^* be the unit ball in $\mathbb{R}^2$ (equiped with the Euclidean norm) and B the rectangle $[-1,1]\times[-20,20]$. Since for $(x,y) \in B$ the derivative of f is a constant vector equal to $(2,1/2) \notin B^*$, the hypothesis of Simons' question is satisfied. We wish to show that there exists no point $(a,b) \in \mathbb{R}^2$ such that

$$\langle f'(a,b) - (u,v), (x,y) - (a,b)\rangle > 0 \tag{3.1}$$

for all $(x,y) \in B, (u,v) \in B^*$.
In fact, let us rewrite the above inequality in the following form

$$\langle f'(a,b), (x,y) - (a,b)\rangle > ||(x,y) - (a,b)|| \tag{3.2}$$

for every $(x,y) \in B$. Direct calculation shows that

$$\langle f'(a,b), (x,y) - (a,b)\rangle = \begin{cases} (y-b)/2 & \text{if } a < -2, \\ 2(a+2)(x-a) + (y-b)/2 & \text{if } a \in [-2,-1], \\ 2(x-a) + (y-b)/2 & \text{if } a \in [-1,1], \\ 2a(x-a) + (y-b)/2 & \text{if } a > 1. \end{cases}$$

By taking $x = -1$ and $y = 0$ in the above formula we see that for $a > 1$,

$$\langle f'(a,b), (x,y) - (a,b)\rangle = -2a(1+a) - b/2 < -b/2,$$

while

$$||(x,y) - (a,b)|| = \sqrt{(1+a)^2 + b^2} > \sqrt{4 + b^2},$$

which shows that (3.2) does not hold. In the other cases we have the following estimate

$$\langle f'(a,b), (x,y) - (a,b)\rangle < 5 + (y-b)/2$$

for every $(x,y) \in B$ because

$$\begin{aligned} \max_{a\in[-2,-1],x\in[-1,1]} 2(a+2)(x-a) &\leq \max_{a\in[-2,-1]} 2(a+2)(1-a) \\ &\leq \max_{a\in\mathbb{R}} 2(a+2)(1-a) = 4.5, \end{aligned}$$

$$\max_{a\in[-2,-1],x\in[-1,1]} 2(x-a) \ = \ 4.$$

This and (3.2) imply that

$$5+(y-b)/2 \ > \ \sqrt{1+(y-b)^2}$$

for every $y \in [-20,20]$. Consequently, $y-b \in [(10-\sqrt{388})/3, (10+\sqrt{388})/3] \subseteq [-4,10]$ for all $y \in [-20,20]$. Such a number b does not exist, hence neither does (a,b) satisfying (3.1) exist.

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