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Polishness of Weak Topologies Generated by Gap and Excess Functionals

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Let (X, d) be a separable complete metric space and $CL(X)$ the family of all nonempty closed subsets of X. We show that the finite Hausdorff topology on $CL(X)$ is Polish. The finite Hausdorff topology is measurably compatible on $CL(X)$ [1], i.e. its Borel field coincides with the Effros sigma algebra. So Polishness of this topology can be useful for measurable multifunctions with values in $CL(X)$ equipped with the finite Hausdorff topology. Polishness of other weak topologies generated by a family of gap and excess funtionals is also proved.

1. Introduction

Let (X, d) be a separable complete metric space and $CL(X)$ the hyperspace of X, i.e. the space of all nonempty closed subsets of X. We are interested to find a class of Polish topologies on $CL(X)$. In [2] it is shown that the Wijsman topology generated by the metric d is Polish and in [6] is proved that also the Wijsman topology generated by a metric ϱ topologically equivalent to d is Polish.

We are going to show that, if Δ is a subfamily of $CLB(X)$ containing the singletons and separable with respect to the induced Hausdorff metric , then the weak topology τ_Δ^G generated by the family of gap functionals determined by Δ as well as the weak topology τ_Δ^{GE} generated by the family of gap and excess functionals determined by Δ are Polish. As a corollary we obtain that the finite Hausdorff topology for a separable complete metric space is Polish.

It is known [2] that the Borel fields determined by the above mentioned topologies τ_Δ^G and τ_Δ^{GE} are equal to the Effros sigma algebra, so Polishness of these topologies can be useful in the study of measurable multifunctions with values in $CL(X)$.

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2. Preliminaries

In the sequel (X, d) will be a metric space, $CL(X)$ the family of all nonempty closed subsets of X, $CLB(X)$ the family of all nonempty closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. The open (resp. closed) ball with center x and radius ε will be denoted by $S(x,\varepsilon)$ (resp. $B(x,\varepsilon)$). The distance of x from a nonempty set A is defined as

$$d(x, A) = \inf\{d(x, a) : \ a \in A\}.$$

The open (resp. closed) ε-enlargement of A is the set

$$S(A, \varepsilon) = \{x \in X : \ d(x, A) < \varepsilon\} \quad (B(A, \varepsilon) = \{x \in X : \ d(x, A) \leq \varepsilon\}).$$

Given two nonempty sets A, B we define the gap between them as

$$D(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$$

and the excess of A over B as

$$e(A, B) = \sup\{d(a, B) : \ a \in A\}.$$

If Δ is a nonempty subfamily of $CL(X)$, then by τ_Δ^G we mean the weak topology on $CL(X)$ generated by the family of gap functionals

$$\{D(., B) : \ B \in \Delta\}$$

and by τ_Δ^{GE} we mean the weak topology on $CL(X)$ generated by the family of gap and excess functionals

$$\{D(., B) : \ B \in \Delta\} \cup \{e(., B) : \ B \in \Delta\}.$$

If $\Delta = K(X)$ then $\tau_{K(X)}^{GE}$ is just the finite Hausdorff topology ([2]) which will be denoted by τ_{fH}. The finite Hausdorff topology when restricted to $K(X)$ agrees with the Hausdorff metric topology. Some properties as well as applications of the finite Hausdorff topology can be found in [3].

If Δ contains the singleton subsets of X, then τ_Δ^G and τ_Δ^{GE} are Hausdorff and admissible and they are unchanged if Δ is replaced by $\overline{\Delta}$, the closure of Δ in the Hausdorff metric [2, Chapter 4]. Of course τ_Δ^G and τ_Δ^{GE} are completely regular as weak topologies. We finish these preliminaries with the following Lemma,which will be used later. By $C(Y, Z)$ we mean the space of the continuous functions from Y to Z.

Lemma 2.1. *Let (Y, e) be a separable metric space and (Z, f) a metric space. Then the uniformity $\mathcal{U}$ of the uniform convergence on compact subsets of Y on every equicontinuous family $\mathcal{F} \subset C(Y, Z)$ has a countable base, and so it is metrizable.*

Proof. For every compact subset $K \subset Y$ and $\varepsilon > 0$ denote by

$$U_{K,\varepsilon} = \{(g, h) \in \mathcal{F} \times \mathcal{F} : \ f(g(y), h(y)) < \varepsilon \text{ for every } y \in K\}.$$

Let E be a countable dense set in Y and $\mathcal{E}$ the family of the finite subsets of E. Then the family

$$\{U_{B,\frac{1}{n}} : \ B \in \mathcal{E}, \ n \in Z^+\}$$

is a base of $\mathcal{U}$. (Let $K \subset Y$ be a compact set in Y and $\varepsilon > 0$. The compactness of K and the equicontinuity of $\mathcal{F}$ imply that there is $\delta > 0$ such that whenever $e(x,y) < \delta$ and $x \in K$ then $f(g(x), g(y)) < \frac{\varepsilon}{3}$ for every $g \in \mathcal{F}$. Let $\{x_1, ..., x_n\}$ be a finite set in K such that $K \subset S(\{x_1, ..., x_n\}, \frac{\delta}{2})$. For every $i \in \{1, 2, ..., n\}$ let $e_i \in E \cap S(x_i, \frac{\delta}{2})$. Then $K \subset S(\{e_1, ..., e_n\}, \delta)$. It easy to verify that $U_{\{e_1,...,e_n\},\frac{1}{k}} \subset U_{K,\varepsilon}$ where $k \in Z^+$ is such that $\frac{1}{k} < \frac{\varepsilon}{3}$.) □

3. Weak topologies generated by gap functionals

Let (X, d) be a metric space and Δ a nonempty subfamily of $CL(X)$. In this section we will study the weak topology τ_Δ^G on $CL(X)$. It is known that if $\Delta = CL(X)$ we obtain the proximal topology [5], if $\Delta = CLB(X)$ we obtain the bounded proximal topology [4] and if Δ is the family of all nonempty closed bounded and convex sets in a normed linear space we have the slice topology [2].
We start with the following proposition which generalizes some results from [5] and [4].

Proposition 3.1. *Let (X, d) be a metric space and Δ a subfamily of $CL(X)$ containing the singletons. If $(CL(X), \tau_\Delta^G)$ is first countable, then X is separable.*

Proof. Notice that the family of τ_Δ^G neighbourhoods of X is the same as in the lower Vietoris topology. So by Proposition 4.3 in [8] we have that X is separable. □

Theorem 3.2. *Let (X, d) be a metric space and Δ a subfamily of $CL(X)$ containing the singletons. The following are equivalent:*
(1) *$(CL(X), \tau_\Delta^G)$ is metrizable;*
(2) *$(CL(X), \tau_\Delta^G)$ is second countable;*
(3) *There is a countable subfamily $\Sigma \subset \Delta$ such that $\tau_\Delta^G = \tau_\Sigma^G$ on $CL(X)$.*

Proof. (1) ⇒ (2) It is sufficient to prove that $(CL(X), \tau_\Delta^G)$ is separable. Let E be a countable dense set in X (see Proposition 3.1). It is easy to verify that the finite subsets of E form a dense set in $(CL(X), \tau_\Delta^G)$.
(2) ⇒ (3) is clear.
(3) ⇒ (1) $(CL(X), \tau_\Delta^G)$ is completely regular, Hausdorff and has a countable base. So $(CL(X), \tau_\Delta^G)$ is metrizable. □

We are now going to prove the following result:

Theorem 3.3. *Let (X, d) be a complete metric space. Let $\Delta \subset CL(X)$ be separable with respect to the induced Hausdorff metric H. Suppose moreover Δ contains the singletons. Then $(CL(X), \tau_\Delta^G)$ is a Polish space.*

The proof of Theorem 3.3 is based on the following remarks and lemmas.

Suppose $\Delta \subset CL(X)$ contains the singletons. A net $\{A_\sigma : \sigma \in \Omega\}$ in $CL(X)$ τ^G_Δ-converges to $A \in CL(X)$ if and only if for every $B \in \Delta$ the net $\{D(A_\sigma, B) : \sigma \in \Omega\}$ converges to $D(A, B)$. Consider Δ as a metric space equipped with the induced Hausdorff metric H. Under the identification $A \leftrightarrow D(A, .)$, where $D(A, .)$ is defined on Δ, we can consider $(CL(X), \tau^G_\Delta)$ as a topological subspace of $C_p(\Delta, R)$, the space of the continuous real valued functions defined on Δ equipped with the topology of pointwise convergence. For every $A, B, C \in CL(X)$ we have

$$|D(A, B) - D(A, C)| \le H(B, C),$$

so for every $A \in CL(X)$ $D(A, .)$ is a Lipschitz continuous function with constant 1 ([2], Proposition 3.2.5.)

Lemma 3.4. *Let (X, d) be a metric space. Suppose Δ is a subfamily of $CL(X)$ separable with respect to the induced Hausdorff metric H and containing the singletons. Then $\overline{CL(X)}$ in $C_p(\Delta, R)$ is Polish, where $\overline{CL(X)}$ is the closure of $CL(X)$ in $C_p(\Delta, R)$.*

Proof. The family $\{D(A, .) : A \in CL(X)\}$ of the continuous functions defined on (Δ, H) is equicontinuous, so $\overline{CL(X)}$ is equicontinuous. By Lemma 2.5.2 in [2] $\overline{CL(X)}$ is second countable. On $\overline{CL(X)}$ the topology of pointwise convergence and the topology of uniform convergence on compact sets coincide. By Lemma 2.1 the uniformity of uniform convergence on compact sets on $\overline{CL(X)}$ is metrizable and of course it is complete, since R is complete. □

Lemma 3.5. *Let (X, d) be a metric space. Suppose Δ is a subfamily of $CL(X)$ containing the singletons. Let $F \in \overline{CL(X)} \subset C_p(\Delta, R)$. Put $A = \{x \in X : F(\{x\}) = 0\}$. Then $F = D(A, .)$ if and only if $A \neq \emptyset$ and for every $B \in \Delta$, $D(A, B) \le F(B)$.*

Proof. $\Rightarrow$ This part is easy.
$\Leftarrow$ Let $B \in \Delta$. We show that $F(B) \le D(A, B)$. Let $\varepsilon > 0$. Pick $a \in A$ with $D(a, B) < D(A, B) + \frac{\varepsilon}{3}$. $F \in \overline{CL(X)}$, so there is $C \in CL(X)$ such that

$$|F(B) - D(C, B)| < \frac{\varepsilon}{3} \text{ and } |F(\{a\}) - D(C, \{a\})| < \frac{\varepsilon}{3}.$$

So

$$F(B) < D(C, B) + \frac{\varepsilon}{3} < D(C, \{a\}) + D(\{a\}, B) + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + D(A, B) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Since $\varepsilon > 0$ was arbitrary, we have $F(B) \le D(A, B)$. □

Proof of Theorem 3.3. The assumptions of Theorem 3.3 imply that X is separable. Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X and $\{B_1, B_2, ..., B_n, ...\}$ a countable dense set in (Δ, H). For every $n, k \in Z^+$ define

$$\Omega^1(n, k) = \{F \in \overline{CL(X)} : \{x \in X : F(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and}$$

$$d(x_k, \{x \in X : F(\{x\}) < \frac{1}{2^{n+1}}\}) < F(\{x_k\}) + \frac{1}{2^n}\};$$

$$\Omega^2(n,k) = \{F \in \overline{CL(X)} : \{x \in X : F(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and}$$

$$D(\{x \in X : F(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) < F(B_k) + \frac{1}{2^n}\}.$$

We show that $CL(X) = \bigcap_{n,k} (\Omega^1(n,k) \cap \Omega^2(n,k))$.

The inclusion $CL(X) \subset \bigcap_{n,k} (\Omega^1(n,k) \cap \Omega^2(n,k))$ is clear. Now let

$$F \in \bigcap_{n,k} (\Omega^1(n,k) \cap \Omega^2(n,k)).$$

To prove that $F \in CL(X)$, it is sufficient by Lemma 3.5 to show that

$$A = \{x \in X : F(\{x\}) = 0\}$$

is nonempty, and that for each $B \in \Delta$ we have $D(A, B) \leq F(B)$. Clearly

$$F \in \bigcap_{n,k} \Omega^1(n,k) \quad \text{and} \quad F_{|X} \in \overline{CL(X)} \text{ in } C_p(X, R).$$

From the proof of Theorem 2.5.4 in [2] we can see that

$$A = \{x \in X : (F_{|X})(x) = 0\} \neq \emptyset \text{ and}$$

$$F(\{x\}) = d(A, x) = D(A, x) \text{ for every } x \in X.$$

Now we prove that also for every $B \in \Delta$ $D(A, B) \leq F(B)$. First, let $B = B_l$ for some l. Suppose $D(A, B_l) > 0$. Let $\varepsilon > 0$. Let $n \in Z^+$ be such that

$$\frac{1}{2^{n-1}} < \min\{\varepsilon, D(A, B_l)\}. \text{ Then}$$

$$\begin{aligned} D(A, B_l) &\leq D(S(A, \frac{1}{2^{n+1}}), B_l) + H(A, S(A, \frac{1}{2^{n+1}})) \\ &< F(B_l) + \frac{1}{2^n} + \frac{1}{2^{n+1}} < F(B_l) + \varepsilon, \end{aligned}$$

since $F \in \Omega^2(n,k)$ and $F_{|X} = d(A, .)$, so

$$S(A, \frac{1}{2^{n+1}}) = \{x \in X : d(A, x) < \frac{1}{2^{n+1}}\} = \{x \in X : F(\{x\}) < \frac{1}{2^{n+1}}\}.$$

Since $\varepsilon > 0$ was arbitrary, we have $D(A, B_l) \leq F(B_l)$.

Now let $B \in \Delta$. We show that $D(A,B) \leq F(B) + \varepsilon$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. There is $B_l \in \Delta$ such that $H(B, B_l) < \frac{\varepsilon}{2}$. Then $|D(A,B_l) - D(A,B)| \leq H(B,B_l)$ and also $|F(B_l) - F(B)| \leq H(B,B_l)$, so

$$D(A,B) \leq D(A,B_l) + H(B,B_l) < F(B_l) + \frac{\varepsilon}{2} \leq F(B) + \varepsilon.$$

It is very easy to verify that for every $n, k \in Z^+$ $\Omega^1(n,k)$ and $\Omega^2(n,k)$ are open subsets of $\overline{CL(X)}$ in $C_p(\Delta, R)$. By Lemma 3.4 $\overline{CL(X)}$ is a Polish space, so also $(CL(X), \tau_\Delta^G)$ is a Polish space as a G_δ-subset of a Polish space. □

Theorem 3.6. *Let (X,d) be a separable complete metric space and Δ a subfamily of $CL(X)$ containing the singletons. If $(CL(X), \tau_\Delta^G)$ is metrizable, then it is Polish.*

Proof of Theorem 3.6. Theorem 3.2 and the metrizability of $(CL(X), \tau_\Delta^G)$ imply that there is a countable subfamily Σ of Δ such that $\tau_\Sigma^G = \tau_\Delta^G$. Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X. Put

$$\Gamma = \overline{\Sigma \cup \{\{x_i\} : \ i \in Z^+\}},$$

where the closure is taken with respect to the Hausdorff metric H. Then by Theorem 3.3 $(CL(X), \tau_\Gamma^G)$ is a Polish space and $\tau_\Gamma^G = \tau_\Delta^G$ on $CL(X)$. □

Observe that actually it is not necessary to suppose the separability of X (Proposition 3.1).

Remark 3.7. If (X,d) is a separable complete metric space then the Wijsman topology induced by the metric d on $CL(X)$ is Polish ([2]). So it is of interest to know conditions on Δ under which τ_Δ^G is different from the Wijsman topology. Results of this type can be found in [7].

Remark 3.8. By using Theorem 3.3 we can find another proof of Polishness of the slice topology. Let X be a Banach space with strongly separable dual. Let E be a countable dense set in X and E^* a countable dense set in X^*. Put

$$\mathcal{B} = \{B(e,q) : \ e \in E, \ q \in Q\}$$

where Q are positive rationals. The family $\mathcal{S}$ of slices of balls from $\mathcal{B}$ generated by elements from E^* and rationals is countable. Put $\Delta = \overline{\mathcal{S}}$, where the bar means the closure in the Hausdorff metric induced by the norm of X. By Theorem 3.3 we have that $(CL(X), \tau_\Delta^G)$ is Polish. The family $C(X)$ of the nonempty closed convex sets is closed in $(CL(X), \tau_\Delta^G)$, so $(C(X), \tau_\Delta^G)$ is Polish and τ_Δ^G on $C(X)$ is just the slice topology.

4. Weak topologies generated by gap and excess functionals

Let (X,d) be a metric space and Δ a nonempty subfamily of $CL(X)$. In this section we will study the weak topology τ_Δ^{GE} on $CL(X)$. If $\Delta = K(X)$ we have the finite Hausdorff topology τ_{fH}^G ([2]).

We start with the following Proposition:

Proposition 4.1. *Let (X,d) be a separable metric space and Δ a nonempty subfamily of $CL(X)$ which contains the singletons. The following are equivalent:*

(1) $(CL(X),\tau_{\Delta}^{GE})$ *is metrizable;*
(2) $(CL(X),\tau_{\Delta}^{GE})$ *is second countable;*
(3) *There is a countable subfamily $\Sigma \subset \Delta$ such that $\tau_{\Sigma}^{GE} = \tau_{\Delta}^{GE}$ on $CL(X)$.*

Proof. (1) $\Rightarrow$ (2) It is sufficient to prove that $(CL(X),\tau_{\Delta}^{GE})$ is separable. Let E be a countable dense set in X. It is easy to verify that finite subsets of E form a dense set in $(CL(X),\tau_{\Delta}^{GE})$.
(2) $\Rightarrow$ (3) This is clear.
(3) $\Rightarrow$ (1) $(CL(X),\tau_{\Delta}^{GE})$ is completely regular (as a weak topology), has a countable base and it is Hausdorff. So $(CL(X),\tau_{\Delta}^{GE})$ is metrizable. □

Now we will formulate the main result of this part:

Theorem 4.2. *Let (X,d) be a separable complete metric space and Δ a subfamily of $CLB(X)$ which contains the singletons. If $(CL(X),\tau_{\Delta}^{GE})$ is metrizable, then $(CL(X),\tau_{\Delta}^{GE})$ is Polish.*

We will deduce the above Theorem from the following one:

Theorem 4.3. *Let (X,d) be a complete metric space and Δ a subfamily of $CLB(X)$ separable with respect to the induced Hausdorff metric H and containing the singletons. Then $(CL(X),\tau_{\Delta}^{GE})$ is a Polish space.*

The proof of theorem 4.3 is based on the following remarks and lemmas.
Suppose $\Delta \subset CL(X)$ contains the singletons. A net $\{A_\sigma : \sigma \in \Omega\}$ in $CL(X)$ τ_{Δ}^{GE}-converges to $A \in CL(X)$ if and only if for every $B \in \Delta$ the net $\{D(A_\sigma,B) : \sigma \in \Omega\}$ converges to $D(A,B)$ and the net $\{e(A_\sigma,B) : \sigma \in \Omega\}$ converges to $e(A,B)$.
Let u be the usual metric on $[0,\infty)$. By $[0,\infty]$ we mean the one-point compactification of $[0,\infty)$. So $[0,\infty]$ can be equipped with a separable complete metric f. Put $Y = [0,\infty) \times [0,\infty]$ and consider the box metric ϱ on Y generated by u and f.
Let Δ be equipped with the induced Hausdorff metric H. Under the identification

$$A \leftrightarrow (D(A,.),e(A,.))$$

(D(A,.) and e(A,.) are defined on Δ) we can consider $(CL(X),\tau_{\Delta}^{GE})$ as a topological subset of $C_p(\Delta,Y)$.

Lemma 4.4. *Let (X,d) be a metric space and Δ a subfamily of $CL(X)$ containing the singletons. Let Δ be equipped with H and Y with ϱ. Then $CL(X) \subset C_p(\Delta,Y)$ is equicontinuous.*

Proof. Let $B \in \Delta$ and $\varepsilon > 0$. There is $\delta > 0$, $\delta < \varepsilon$ such that whenever $u(x,y) < \delta$ $x,y \in [0,\infty)$ then $f(x,y) < \varepsilon$. By proposition 3.2.5. in [2] we can deduce that if $C \in \Delta$ is such that $H(B,C) < \delta$, then

$$\varrho((D(A,B),e(A,B)),(D(A,C),e(A,C)) < \varepsilon$$

for every $A \in CL(X)$. □

Lemma 4.5. *Let (X,d) be a metric space, Δ a subfamily of $CL(X)$ separable with respect to the induced Hausdorff metric and containing the singletons and let Y be equipped with the metric ϱ. Then $\overline{CL(X)}$ in $C_p(\Delta,Y)$ is Polish, where $\overline{CL(X)}$ is the closure of $CL(X)$ in $C_p(\Delta,Y)$.*

Proof. $\overline{CL(X)}$ in $C_p(\Delta,Y)$ is equicontinuous, since by Lemma 4.4 $CL(X)$ is equicontinuous. By Lemma 2.5.2 in [2] $\overline{CL(X)}$ is second countable. On $\overline{CL(X)}$ the topology of pointwise convergence and the topology of uniform convergence on compact sets coincide. By Lemma 2.1 the uniformity of uniform convergence on compact sets on $\overline{CL(X)}$ is metrizable and it is complete, since Y is complete. □

In what follows we will suppose that Δ is a subset of $CLB(X)$. So if $A \in CLB(X)$ then the function $e(A,.)$ defined on Δ has values in $[0,\infty)$ and if $A \in CL(X)$ is unbounded then $e(A,.)$ defined on Δ is identically equal to ∞. The following lemma describes the behaviour of the elements of $\overline{CL(X)}$.

Lemma 4.6. *Let (X,d) be a metric space and Δ a subfamily of $CLB(X)$ containing the singletons. Let $F=(F_1,F_2) \in \overline{CL(X)}$ in $C_p(\Delta,Y)$. If there is $B \in \Delta$ with $F_2(B)=\infty$ then for every $K \in \Delta$ $F_2(K)=\infty$.*

Proof. Of course if $F \in CL(X)$ we are done. Otherwise suppose there is $K \in \Delta$ such that $F_2(K)<\infty$. Let $\varepsilon>0$. $F \in \overline{CL(X)}$, so there is $A \in CL(X)$ such that

$$e(A,K) \in (F_2(K)-\varepsilon, F_2(K)+\varepsilon) \quad \text{and} \quad e(A,B) > F_2(K)+\varepsilon+H(K,B).$$

Since $e(A,K)<\infty$ also $e(A,B)<\infty$ and by Proposition 3.2.5 in [2] we have

$$e(A,B) \le e(A,K)+H(K,B).$$

So $e(A,B) < F_2(K)+\varepsilon+H(K,B)$, a contradiction. □

Lemma 4.7. *Let (X,d) be a metric space and Δ a subfamily of $CLB(X)$ containing the singletons. Let $F=(F_1,F_2) \in \overline{CL(X)}$ in $C_p(\Delta,Y)$ be such that $F_2(B)<\infty$ for every $B \in \Delta$. Then for every $\varepsilon>0$ the set $H=\{x \in X: \ F_1(\{x\})<\varepsilon\}$ is bounded.*

Proof. Let $\varepsilon>0$ and suppose that

$$H=\{x \in X: \ F_1(\{x\})<\varepsilon\}$$

is unbounded. Let $x_0 \in X$ be a fixed point of X. Let $M \in Z^+$ be such that $F_2(\{x_0\})<M$ and let $K \in Z^+$ be such that $KM > M+\varepsilon$. There is $a \in H$ such that $F_1(\{a\})<\varepsilon$ and $a \notin B(x_0, KM+2\varepsilon)$. $F \in \overline{CL(X)}$, so there is $B \in CL(X)$ such that

$$|D(B,\{a\})-F_1(\{a\})|<\varepsilon \text{ and also } |e(B,\{x_0\})-F_2(\{x_0\})|<\varepsilon.$$

Thus

$$e(B,\{x_0\}) < F_2(\{x_0\})+\varepsilon \text{ and } D(B,\{a\})<2\varepsilon.$$

Let $b \in B$ be such that $d(b, a) < 2\varepsilon$. We claim that $d(b, x_0) > KM$. Suppose $d(b, x_0) \leq KM$. Then

$$d(x_0, a) \leq d(b, x_0) + d(b, a) \leq KM + 2\varepsilon,$$

a contradiction. Thus $d(b, x_0) > KM$ and so $e(B, \{x_0\}) > KM$, a contradiction. $\square$

Lemma 4.8. *Let (X, d) be a metric space and Δ a subfamily of $CLB(X)$ containing the singletons. Let $F = (F_1, F_2) \in \overline{CL(X)}$ in $C_p(\Delta, Y)$. Let $A = \{x \in X : F_1(\{x\}) = 0\}$.*

(1) *If for every $B \in \Delta$ $F_2(B) < \infty$, then $(F_1, F_2) = (D(A, .), e(A, .)) \Leftrightarrow A \neq \emptyset$, A is bounded and for each $K \in \Delta$, $D(A, K) \leq F_1(K)$ and $e(A, K) \geq F_2(K)$.*
(2) *If for every $B \in \Delta$ $F_2(B) = \infty$ then $(F_1, F_2) = (D(A, .), e(A, .)) \Leftrightarrow A \neq \emptyset$, A is unbounded and, for each $K \in \Delta$, $D(A, K) \leq F_1(K)$.*

Proof. 1) $\Rightarrow$ This implication is obvious.
$\Leftarrow$ Suppose $A \neq \emptyset$, A bounded and for each $K \in \Delta$

$$D(A, K) \leq F_1(K) \text{ and } e(A, K) \geq F_2(K).$$

We have to prove also the opposite inequalities. Take $K \in \Delta$ and $\varepsilon > 0$. There is $a \in A$ such that

$$d(a, K) > e(A, K) - \frac{\varepsilon}{3}.$$

$F \in \overline{CL(X)}$, so there is $B \in CL(X)$ such that

$$|F_2(K) - e(B, K)| < \frac{\varepsilon}{3} \text{ and also } |F_1(\{a\}) - D(B, \{a\})| < \frac{\varepsilon}{3}.$$

For every $C \in CL(X)$ $d(a, K) \leq e(C, K) + D(C, a)$. (Take $\eta > 0$. There is $c \in C$ such that $d(a, C) < D(C, a) + \eta$. Then $d(a, K) \leq d(c, K) + d(a, c) \leq e(C, K) + D(C, a) + \eta$.) We have

$$\begin{aligned} e(A, K) &< d(a, K) + \frac{\varepsilon}{3} < e(B, K) + D(B, a) + \frac{\varepsilon}{3} \\ &< F_2(K) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = F_2(K) + \varepsilon. \end{aligned}$$

To prove $F_1(K) \leq D(A, K)$ we use the same idea as in the proof of Lemma 3.5.
2) $\Rightarrow$ This implication is obvious.
$\Leftarrow$ Suppose $A \neq \emptyset$, A is unbounded and for each $K \in \Delta$ $D(A, K) \leq F_1(K)$. With the same proof of the Lemma 3.5 we obtain $F_1(K) = D(A, K)$ for every $K \in \Delta$. The unboundedness of A implies that $e(A, K) = \infty$ for every $K \in \Delta$. So $F_2(K) = \infty = e(A, K)$ for every $K \in \Delta$. $\square$

Proof of Theorem 4.3. Now we prove that $CL(X)$ is a G_δ-set in $\overline{CL(X)}$.
Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X and $\{B_1, B_2, ..., B_n, ...\}$ a countable

dense set in Δ. For every $n,k \in Z^+$ put

$$\Omega^1(n,k) = \{F \in \overline{CL(X)}: \ \{x \in X: \ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and}$$
$$d(x_k, \{x \in X: \ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(\{x_k\}) + \frac{1}{2^n}\}$$
$$\cap \ \{F \in \overline{CL(X)}: \ F_2(B_k) \in R\}$$
$$\cap \ \{F \in \overline{CL(X)}: \ D(B_k, \{x \in X: \ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(B_k) + \frac{1}{2^n}\}$$
$$\cap \ \{F \in \overline{CL(X)}: \ e(\{x \in X: \ F_1(\{x\} < \frac{1}{2^{n+1}}\}, B_k) > F_2(B_k) - \frac{1}{2^n}\}.$$

It is easy to verify that if $A \in CL(X)$ is bounded, then

$$(D(A,.), e(A,.)) \in \bigcap_{n,k} \Omega^1(n,k).$$

For every $n,k \in Z^+$ $\Omega^1(n,k)$ is open in $\overline{CL(X)} \subset C_p(\Delta, Y)$. We prove only that the set

$$L = \{F \in \overline{CL(X)}: \ \{x \in X: \ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset\}$$
$$\cap \{F \in \overline{CL(X)}: \ F_2(B_k) \in R\}$$
$$\cap \{F \in \overline{CL(X)}: \ e(\{x \in X: \ F_1(\{x\}) < \frac{1}{2^{n+1}}\}, B_k)$$
$$> F_2(B_k) - \frac{1}{2^n}\}$$

is open in $\overline{CL(X)}$.
Let $G \in L$. By Lemma 4.7 the set $\{x \in X: \ G_1(\{x\}) < \frac{1}{2^{n+1}}\}$ is bounded. Let $\varepsilon > 0$ be such that

$$e(\{x \in X: \ G_1(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) - \varepsilon > G_2(B_k) - \frac{1}{2^n} + \varepsilon.$$

Let $x_0 \in X$ be such that

$$G_1(\{x_0\}) < \frac{1}{2^{n+1}} \text{and} d(x_0, B_k) > e(\{x \in X: \ G_1(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) - \varepsilon.$$

Then the set

$$\{F \in \overline{CL(X)}: \ F_1(\{x_0\}) < \frac{1}{2^{n+1}}\}$$
$$\cap \{F \in \overline{CL(X)}: \ F_2(B_k) \in R\}$$
$$\cap \{F \in \overline{CL(X)}: \ F_2(B_k) < G_2(B_k) + \varepsilon\}$$

is an open neighbourhood of G which is contained in L.
Thus $\bigcap_{n,k} \Omega^1(n,k)$ is a G_δ-set. Now we show that $\bigcap_{n,k} \Omega^1(n,k) = CLB(X)$. Let

$$G \in \bigcap_{n,k} \Omega^1(n,k).$$

So

$$G \in \bigcap_{n,k} \{F \in \overline{CL(X)} : \; \{x \in X : \; F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset\}$$
$$\cap \{F \in \overline{CL(X)} : \; d(x_k, \{x \in X : \; F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1\{x_k\} + \frac{1}{2^n}\}.$$

Following the proof of Theorem 2.5.4 in [2] we can see that $A = \{x \in X : \; G_1(\{x\}) = 0\} \neq \emptyset$ and $G_1(\{x\}) = d(x, A)$ for every $x \in X$. By Lemma 4.7 A is bounded. By Lemma 4.8 1) it is sufficient to prove that $D(A, K) \leq G_1(K)$ and $e(A, K) \geq G_2(K)$ for every $K \in \Delta$. But this is routine.
Now let x_0 be a fixed point from X. Put

$$\Omega^2(n,k) = \{F \in \overline{CL(X)} : \; \{x \in X : \; F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and } F_2(B_k) > n\}$$
$$\cap \{F \in \overline{CL(X)} : \; d(x_k, \{x \in X : \; F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(\{x_k\}) + \frac{1}{2^n}\}$$
$$\cap \{F \in \overline{CL(X)} : D(B_k, \{x \in X : \; F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(B_k) + \frac{1}{2^n}\}$$
$$\cap \{F \in \overline{CL(X)} : \{x \in X : \; F_1(\{x\}) < \frac{1}{2^{n+1}}\} \cap B(x_0, n)^c \neq \emptyset\}.$$

It is easy to verify that $\Omega^2(n,k)$ is open in $C_p(\Delta, Y)$ for every $n, k \in Z^+$. If $A \in CL(X)$ is unbounded, then $A \in \bigcap_{n,k} \Omega^2(n,k)$.
Now suppose that $(F_1, F_2) \in \bigcap_{n,k} \Omega^2(n,k)$. Again from the proof of Theorem 2.5.4 in [2]

$$A = \{x \in X : \; F_1(\{x\}) = 0\} \neq \emptyset.$$

We prove that A is unbounded. Let $n \in Z^+$. We show that $B(x_0, n)^c \cap A \neq \emptyset$. As

$$\{x \in X : \; F_1(\{x\}) < \frac{1}{2^{2n+1}}\} \cap B(x_0, 2n)^c \neq \emptyset,$$

there is $z \in X$ such that

$$F_1(\{z\}) < \frac{1}{2^{2n+1}} \quad \text{and} \quad d(z, x_0) > 2n.$$

So

$$F_1(\{z\}) = d(z, A) < \frac{1}{2^{2n+1}},$$

and there is $a \in A$ such that $d(z, a) < \frac{1}{2^{2n+1}}$. We claim that $d(a, x_0) > 2n - \frac{1}{2^{2n+1}}$. Suppose not. Then

$$d(z, x_0) \leq d(a, x_0) + d(z, a) \leq 2n - \frac{1}{2^{2n+1}} + \frac{1}{2^{2n+1}} = 2n,$$

a contradiction. From the above results we have

$$CL(X) = \bigcap_{n,k} \Omega^1(n,k) \cup \bigcap_{n,k} \Omega^2(n,k).$$

So $CL(X)$ is a G_δ-set in $\overline{CL(X)}$, i.e. $(CL(X), \tau_\Delta^{GE})$ is a Polish space. □

Now let (X,d) be a separable completely metrizable space. We are going to show that if Δ is a subfamily of $CLB(X)$ containing the singletons and separable with respect to the induced Hausdorff metric, then τ_Δ^G and τ_Δ^{GE} are Polish. As a corollary we obtain that the finite Hausdorff topology for a separable completely metrizable space is Polish.
We prove only the case of τ_Δ^G; the proof for τ_Δ^{GE} is the same.
Let $(\tilde{X}, \tilde{d})$ be the completion of (X,d). Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X and $\{B_1, B_2, ..., B_n, ...\}$ a countable dense set in (Δ, H).
Put

$$\Delta_{\tilde{X}} = \{CL_{\tilde{X}}B_1, CL_{\tilde{X}}B_2, ..., CL_{\tilde{X}}B_n, ...\} \cup \{x : \ x \in \tilde{X}\}.$$

Define the map

$$L : \ (CL(X), \tau_\Delta^G) \to (CL(\tilde{X}), \tau_{\Delta_{\tilde{X}}}^G)$$

as follows: $L(A) = CL_{\tilde{X}}A$. It is easy to verify that L is a topological embedding. By Theorem 3.3 $(CL(\tilde{X}), \tau_{\Delta_{\tilde{X}}}^G)$ is a Polish space. By [6] $CL(X)$ is a G_δ-subset of $CL(\tilde{X})$ equipped with the Wijsman topology induced by $\tilde{d}$, so $CL(X)$ is a G_δ-subset of $CL(\tilde{X})$ equipped with $\tau_{\Delta_{\tilde{X}}}^G$, since the Wijsman topology on $CL(\tilde{X})$ induced by $\tilde{d}$ is weaker than $\tau_{\Delta_{\tilde{X}}}^G$.

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