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# Generalized Monotone Nonsmooth Maps

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Recent characterizations of various types of differentiable generalized monotone maps by Karamardian–Schaible–Crouzeix and their strengthened versions by Crouzeix–Ferland are extended to the nonsmooth case. For nondifferentiable locally Lipschitz maps necessary and/or sufficient conditions for quasimonotonicity, pseudomonotonicity and strict/ strong pseudomonotonicity are derived. To accomplish this, the generalized Jacobian in the sense of Clarke is employed.

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## 1. Introduction

Generalizations of convex functions have been used in a variety of fields such as economics, business administration, engineering, statistics, applied sciences and numerical mathematics. In 1949 de Finetti introduced one of the primary types, later called quasiconvex function, which is characterized by convex level sets. Since then numerous types of generalized convex functions have been proposed according to needs arising in particular applications. In 1988 a first monograph on the theory and applications of generalized convexity was published [1].

A characterization of pseudoconvex functions in terms of the gradient without using function values was derived by Karamardian in 1976 [13]. Meanwhile similar characterizations of nondifferentiable generalized convex functions appeared, see e.g. [7], [20], [21], [22]. Karamardian used his characterization of pseudoconvex functions to introduce pseudomonotonicity of maps which are not necessarily the gradient of a function. For nonlinear complementarity problems involving a pseudomonotone map he established the existence of a solution [13].

Since the article by Karamardian and Schaible in 1990 [14], several papers appeared that introduce various types of generalized monotone maps, e.g. [2], [3], [8], [17], [18],

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[24], [28]. Usually these are defined in such a way that in case of a gradient map they characterize some type of generalized convexity of the underlying function. The majority of the articles in this emerging area of research deals with conceptual matters. However uses of generalized monotonicity of operators in establishing the existence of solutions of complementarity problems and variational inequality problems are appearing as well in the recent literature;e.g. [2], [5], [9], [10], [11], [25], [26], [27].

The present paper attempts to broaden the conceptual basis of generalized monotonicity. The starting point are necessary and sufficient conditions for differentiable generalized monotone maps in terms of the Jacobian derived in [15]. These characterizations are extended to nondifferentiable locally Lipschitz maps using the generalized Jacobian in the sense of Clarke [4]. For other characterizations of nondifferentiable generalized monotone maps see [2], [16], [17], [18], [19].

Very recently the characterizations in [15] for differentiable quasimonotone and pseudomonotone maps have been extended by Crouzeix and Ferland [6] by significantly weakening the sufficient conditions. We will show in the present paper that the same extensions can be carried out in the nondifferentiable case. The proofs combine techniques in [6] with those by Luc [23].

The paper is organized as follows. The remainder of this section presents definitions and notation used in this paper. Section 2 deals with nondifferentiable monotone maps which will be characterized in terms of the generalized Jacobian. In Section 3 various necessary and/or sufficient conditions for quasimonotone maps are derived with help of the generalized Jacobian. In Section 4 analoguous results for pseudomonotone maps are obtained. Finally Section 5 presents similar results for strictly and strongly pseudomonotone maps.

Throughout this paper $S$ denotes a nonempty convex open subset of the $n$-dimensional Euclidean space $R^n$ and $F$ denotes a locally Lipschitz map from $S$ to $R^n$. The generalized Jacobian of $F$ at $x \in S$ in the sense of Clarke [4] is the set of matrices

$$\partial F(x) := \operatorname{conv}\{\lim DF(x_i) : x_i \to x, F \text{ is differentiable at } x_i\} ,$$

where "conv" denotes the convex hull and $DF(x)$ is the usual Jacobian of $F$ at $x$. Elements of $\partial F(x)$ are called subgradients.

Furthermore, let $g$ be a real valued differentiable function on $S$. If the gradient map $Dg$ from $S$ to $R^n$ is locally Lipschitz, i.e. $g$ is a $C^{1,1}$ function in the terminology of [12], the generalized Jacobian of this map at $x \in S$ is called generalized Hessian of $g$ and denoted by $\partial^2 g(x)$.

We shall often refer to the following mean value theorem in [4]: for every pair of points $a, b \in S$ there exists a matrix $A \in conv\{\partial F(x) : x \in [a, b]\}$ such that $F(b) - F(a) = A(b - a)$. Moreover, if $f$ is a locally Lipschitz function from $S$ to $R$, then there exist a point $x \in (a, b)$ and a vector $A \in \partial f(x)$ such that $f(b) - f(a) = A(b - a)$.

In the sequel we will make use of the one-dimensional restriction of $F$, namely $\varphi(t) = \langle F(x + tu), u\rangle$ defined on the set $\{t \in R : x + tu \in S\}$ for a fixed $x \in S$ and $u \in R^n$.

Finally we recall the following definitions from [8], [14]:

$F$ is *monotone* if for all $x, y \in S, \langle F(x), y - x\rangle + \langle F(y), x - y\rangle \le 0$;

$F$ is *strictly monotone* if the above inequality is strict for all distinct pairs $x, y \in S$;

$F$ is *quasimonotone* if for all $x, y \in S, \langle F(x), y - x\rangle > 0$ implies $\langle F(y), y - x\rangle \ge 0$;

$F$ is *pseudomonotone* if for all $x, y \in S, \langle F(x), y - x\rangle > 0$ implies $\langle F(y), y - x\rangle > 0$;

or equivalently $\langle F(x), y - x\rangle \ge 0$ implies $\langle F(y), y - x\rangle \ge 0$;

$F$ is *strictly pseudomonotone* if for all $x, y \in S, x \neq y, \langle F(x), y-x\rangle \geq 0$ implies $\langle F(y), y-x\rangle > 0$;
$F$ is *strongly pseudomonotone* if for all $x \in S, u \in R^n, \|u\| = 1, \langle F(x), u\rangle = 0$ implies the existence of positive numbers $\varepsilon$ and $\beta$ such that $\langle F(x+tu), u\rangle \geq \beta t$ for all $t \in [0, \varepsilon]$.
The following implications hold: monotone $\Rightarrow$ pseudomonotone $\Rightarrow$ quasimonotone ; strictly monotone $\Rightarrow$ strictly pseudomonotone ; strongly pseudomonotone $\Rightarrow$ strictly pseudomonotone.

## 2. Monotone Maps

Let us first characterize monotone maps in terms of the generalized Jacobian.

**Proposition 2.1.** *The map $F$ is monotone on $S$ if and only if for every $x \in S$ the subgradients $A \in \partial F(x)$ are positive semidefinite.*

**Proof.** For the "if" part, let $x, y \in S$. By the mean value theorem,

$$F(x) - F(y) \in \operatorname{conv}\{\partial F(z)(x-y) : z \in [x, y]\} .$$

Hence

$$\langle F(x) - F(y), x-y\rangle \in \operatorname{conv}\{\langle x-y, A(x-y)\rangle : A \in \bigcup_{z\in[x,y]} \partial F(z)\} .$$

By the hypothesis every element of the above convex hull is nonnegative, consequently

$$\langle F(x) - F(y) , x-y\rangle \geq 0$$

which implies that $F$ is monotone on $S$.
Conversely, suppose that $F$ is monotone. By the definition of the generalized Jacobian, it suffices to show that the usual Jacobian $DF(x)$ is positive semidefinite whenever it exists. In fact, let $x \in S$ where $DF(x)$ exists. If $DF(x)$ is not positive semidefinite, then there can be found $u \in R^n$ such that $\langle u, DF(x)u\rangle < 0$.
By the definition of the Jacobian,

$$DF(x)u = \lim_{t\to 0} \frac{F(x+tu) - F(x)}{t} .$$

Hence for $t > 0$ sufficiently small,

$$\langle u, F(x+tu) - F(x)\rangle < 0 .$$

Setting $y = x + tu$ for a fixed small $t > 0$ , we obtain

$$\langle y-x, F(y) - F(x)\rangle < 0$$

which contradicts the monotonicity of $F$. $\square$

We now add a sufficient condition for strict monotonicity.

**Proposition 2.2.** *The map $F$ is strictly monotone on $S$ if for every $x \in S$, the subgradients $A \in \partial F(x)$ are positive definite.*

**Proof.** The argument of the previous proof can be used, assuming $x, y \in S$, $x \neq y$, and noting that every element of the set

$$\operatorname{conv}\{\langle x-y, A(x-y)\rangle : A \in \bigcup_{z\in[x,y]} \partial F(z)\}$$

is strictly positive. □

It should be noticed that the above condition for strict monotonicity is only sufficient but not necessary, as in the differentiable case.

## 3. Quasimonotone Maps

In preparation of the first result on nonmonotone maps, we recall the following notation from [23]. For every $x \in S$, $u \in R^n$ let

$$D_+F(x;u) = \sup\{\langle u, Au\rangle : A \in \partial F(x)\}\ ,$$

$$D_-F(x;u) = \inf\{\langle u, Au\rangle : A \in \partial F(x)\}\ .$$

In the above expressions, sup and inf can be replaced by max and min respectively because $F$ is locally Lipschitz and the set $\partial F(x)$ is nonempty compact.

**Proposition 3.1.** *The map $F$ is quasimonotone on $S$ if and only if the following conditions hold for every $x \in S$, $u \in R^n$:*

(i) $\langle F(x), u\rangle = 0$ *implies* $D_+F(x;u) \geq 0$ ;

(ii) $\langle F(x), u\rangle = 0$ , $0 \in \{\langle u, Au\rangle : A \in \partial F(x)\}$ *and* $\langle F(x+\bar{t}u), u\rangle > 0$ *for some* $\bar{t} < 0$ *imply the existence of* $\tilde{t} > 0$ *such that*

$$\langle F(x+tu), u\rangle \geq 0 \quad \textit{for all} \quad t \in [0, \tilde{t}\,]\ .$$

**Proof.** We begin with the "only if" part. If (i) does not hold, one can find $x \in S$ and $u \in R^n$ such that $\langle F(x), u\rangle = 0$ and $D_+F(x;u) < 0$.
Consider the function

$$\varphi(t) = \langle F(x+tu), u\rangle$$

as defined in Section 1. This function is quasimonotone and $\varphi(0) = 0$. Using the mean value theorem, for every $t$ sufficiently small, there can be found $t_0$ between 0 and $t$ , $\alpha \in \partial\varphi(t_0)$ such that

$$\varphi(t) - \varphi(0) = \alpha t\ . \tag{3.1}$$

By the calculus rule of the generalized Jacobian [4],

$$\partial\varphi(t_0) \subseteq \{\langle u, Au\rangle : A \in \partial F(x+t_0u)\}\ .$$

By the upper semicontinuity of the function $D_+F(\cdot, u)$ (Lemma 2.2 of [23]),

$$D_+F(x+tu;u) < 0 \quad \text{for } t \text{ small enough}\ .$$

This and (3.1) show that $\varphi(t) > 0$ if $t < 0$ and $\varphi(t) < 0$ if $t > 0$, which contradict the quasimonotonicity of $\varphi$.

If (ii) does not hold, there exists $t_0 > 0$ such that $\langle F(x), u) \rangle = 0$ , $\langle F(x + \bar{t}u), u\rangle > 0$ for some $\bar{t} < 0$ and $\langle F(x + t_0 u), u\rangle < 0$. Define $x_0 = x + \bar{t}u$ and $y_0 = x + t_0 u$. We have

$$\langle F(y_0), x_0 - y_0\rangle = \langle F(x + t_0 u), (\bar{t} - t_0)u\rangle > 0 \ ,$$

$$\langle F(x_0), y_0 - x_0\rangle = \langle F(x + \bar{t}u), (t_0 - \bar{t})u\rangle > 0 \ .$$

These inequalities contradict the quasimonotonicity of $F$.
Conversely, suppose that $F$ is not quasimonotone on $S$. Then there exist $x, y \in S$ such that

$$\langle F(x), y - x\rangle > 0 \quad \text{and} \quad \langle F(y), x - y\rangle > 0 \ .$$

Let $u = y - x$ and consider the function $\varphi(t) = \langle F(x + tu), u\rangle$ . We have $\varphi(0) > 0$ , $\varphi(1) < 0$. Since $\varphi$ is continuous, there exists $t_0 \in (0, 1)$ such that

$$\varphi(t_0) = 0 \quad \text{and} \quad \varphi(t) < 0 \quad \text{for all } t,\ 1 \geq t > t_0 \ . \tag{3.2}$$

Define $x_0 = x + t_0 u$. Then $\varphi(t_0) = \langle F(x_0), u\rangle = 0$ and $D_- F(x_0; u) \leq 0$. Consequently $0 \in \{\langle u, Au\rangle : A \in \partial F(x_0)\}$ because of (i). It follows from (ii) that one can find a positive $\tilde{t}$ such that

$$\varphi(t_0 + t) = \langle F(x_0 + tu), u\rangle \geq 0 \quad \text{for all } t \in [0, \tilde{t}\,] \ .$$

This contradicts (3.2) and the proof is complete. □

The above proof reveals that conditions (i) and (ii) are equivalent to the following one: $\langle F(x), u\rangle = 0$, $D_- F(x; u) \leq 0$ and $\langle F(x + \bar{t}u), u\rangle > 0$ for some $\bar{t} < 0$ imply the existence of $\tilde{t} > 0$ such that $\langle F(x + tu), u\rangle \geq 0$ for all $t \in [0, \tilde{t}\,]$
In light of the recent characterization of quasimonotone maps by Crouzeix and Ferland [6], we shall present a nonsmooth version of Theorem 2.3 of [6]. The following lemma will be needed in the sequel (see also the proof of Theorem 2.2 of [6] for the smooth case).

**Lemma 3.2.** *Suppose that for each $x \in S$, $u \in R^n$, $\langle F(x), u\rangle = 0$ implies $D_- F(x, u) \geq 0$. Then for every $x \in S$, $u \in R^n$ such that $F(x) \neq 0$ and $\langle F(x), u\rangle = 0$ there exists a positive $\tilde{t}$ such that $\langle F(x + tu), u\rangle \geq 0$ for all $t \in [0, \tilde{t}]$*

**Proof.** Suppose to the contrary that for some $x_0 \in S$ with $F(x_0) \neq 0$ and $\langle F(x_0), u\rangle = 0$, for every positive $\bar{t}$ sufficiently small there exists $t_1 \in [0, \bar{t}\,]$ such that $\langle F(x_0 + t_1 u), u\rangle < 0$. It follows from the continuity of $F$ that one can find a small positive $\tilde{t}$ and $t_0 \in [0, \tilde{t}\,]$ such that

$$\begin{aligned} \langle F(x_0 + t_0 u), u\rangle &= 0 \\ \langle F(x_0 + tu), u\rangle &< 0 \end{aligned} \tag{3.3}$$

for every $t \in [t_0, \tilde{t}\,]$, and $F(x_0 + t_0 u) \neq 0$.
Without loss of generality, we may assume that $x_0 + t_0 u = 0$; $F(0) = (0, \cdots, 0, 1)$ and

$$\langle F(tu), u\rangle < 0 \quad \text{for all} \quad t \in (0, \tilde{t}]. \tag{3.4}$$

We shall write $x = (y, z) \in R^{n-1} \times R$. Let $H(x)$ be the last coordinate of $F(x)$ and $G(x)$ the first $(n - 1)$ coordinates, i.e. $F(x) = (G(x), H(x)) \in R^{n-1} \times R$. $G$ and $H$ are

then locally Lipschitz functions of $(y, z) \in R^{n-1} \times R$. We have $G(0, 0) = 0 \in R^{n-1}$ and $H(0, 0) = 1$.
Observe also that $u = (v_0, 0)$ with some $v_0 \in R^{n-1}$ because $\langle F(0), u\rangle = 0$.
Following [6] let us consider the following differential equation:

$$H(tv_0, g(t))g'(t) + \langle G(tv_0, g(t)), v_0\rangle = 0, \quad g(0) = 0. \tag{3.5}$$

Since $G$ and $H$ are locally Lipschitz and $H(0, 0) = 1$ , the system (3.5) has a differentiable solution $g(y)$ with locally Lipschitz gradient, i.e. $g$ is a $C^{1,1}$ function on a sufficiently small interval $(-\epsilon, \epsilon) \subseteq (-\tilde{t}, \tilde{t})$ on which

$$H(tv_0, g(t)) > 0,\ t \in (-\epsilon, \epsilon). \tag{3.6}$$

We now prove that $g$ is concave. In fact, because of (3.5), for every $t \in (-\epsilon, \epsilon)$ one has

$$\langle F(tv_0, g(t)), (v_0, g'(t))\rangle = \langle G(tv_0, g(t)), v_0\rangle + H(tv_0, g(t)) \,.\, g'(t) = 0 \,.$$

Hence

$$D_{-}F((tv_0, g(t))\,;\ (v_0, g'(t))) \geq 0, \tag{3.7}$$

according to the assumption of the lemma. By a technique of the proof of Proposition 4.1 in [23], we have

$$\partial^2 g(t) \subseteq -\frac{1}{H(tv_0, g(t))}\,\{v_0 A_{yy} v_0 + v_0 A_{yz} g'(t) + [A_{zy}^T v_0 + A_{zz} g'(t)]g'(t) :$$

$$\begin{pmatrix} A_{yy} & A_{yz} \\ A_{zy} & A_{zz} \end{pmatrix} \in \partial F(tv_0, g(t))\} \,.$$

Consequently, every element of $\partial^2 g(t)$ is nonpositive because of (3.7). Thus $g$ is concave on $(-\epsilon, \epsilon)$.
Observe that $g(t) \leq 0$ and $g'(t) \leq 0$ for $t \in [0, \epsilon)$ because $g$ is concave and $g(0) = g'(0) = 0$.
Actually we have the strict inequalities

$$g(t) < 0 \text{ and } g'(t) < 0 \quad \text{for all} \quad t \in (0, \epsilon). \tag{3.8}$$

In fact, if for some $t \in (0, \epsilon)$ one had $g'(t) = 0$, then using (3.5) one should obtain $\langle F(tu), u\rangle = 0$, a contradiction with (3.4). The first inequality in (3.8) follows from the second one and the initial condition $g(0) = 0$.
Pick $\delta \in (0, \epsilon)$. On one hand $\langle F(\delta v_0, 0), u\rangle < 0$ according to (3.4). On the other hand $\langle F(\delta v_0, g(\delta)), u\rangle > 0$ because of (3.5). Hence by continuity of $F$ there exists a negative $\lambda > g(\delta)$ such that

$$\langle F(\delta v_0, \lambda g(\delta)), u\rangle = 0. \tag{3.9}$$

Let us further consider the same differential equation as (3.5) but with another initial condition:

$$H(tv_0, h(t))h'(t) + \langle G(tv_0, h(t)), v_0\rangle = 0\,, \quad h(\delta) = \lambda. \tag{3.10}$$

As before the solution of this equation is concave and exists on an interval $[\tau, \delta]$ (where $\tau \in [0, \delta]$) on which the relation $h(t) \geq g(t)$ holds (using (3.6)).

Observe that $h'(\delta) = 0$ according to (3.9). Moreover

$$h(t) \le \lambda \quad \text{and} \quad h'(t) \ge 0 \quad \text{for all} \quad t \in [\tau, \delta]. \tag{3.11}$$

This and the fact that $g(0) = 0 > \lambda$ imply the existence of a negative $\lambda_0 \in [\tau, \delta]$ such that $h(\lambda_0) = g(\lambda_0)$. For this $\lambda_0$ one has $\langle G(\lambda_0 v_0, g(\lambda_0)), v_0\rangle > 0$ according to (3.5), (3.8), and $\langle G(\lambda_0 v_0, h(\lambda_0)), v_0\rangle \le 0$ according to (3.10) and (3.11). The contradiction completes the proof. □

With help of the above lemma we are able to derive the following result.

**Proposition 3.3.** *Suppose that for every $x \in S$ , $u \in R^n$, $u \neq 0$ the following conditions hold*

(i) $\langle F(x), u\rangle = 0$ *implies* $D_-F(x; u) \ge 0$ ;

(ii) $F(x) = 0$ , $D_-F(x; u) = 0$ *and* $\langle F(x + \bar{t}u), u\rangle > 0$ *for some* $\bar{t} < 0$ *imply the existence of* $\tilde{t} > 0$ *such that*

$$\langle F(x + tu), u\rangle \ge 0 \quad \textit{for all} \quad t \in [0, \tilde{t}\,] .$$

*Then $F$ is quasimonotone on $S$.*

**Proof.** Let us show that conditions (i) and (ii) of Proposition 3.1 are satisfied. Obviously (i) holds because $D_+F(x; u) \ge D_-F(x; u)$ , and (ii) holds if $F(x) = 0$, in view of the second condition of Proposition 3.3. For the remaining case $F(x) \neq 0$ , we invoke Lemma 3.2 to obtain the existence of $\tilde{t}$ in (ii) of Proposition 3.1. This completes the proof. □

Note that the sufficient condition of Proposition 3.3 is not necessary as can be seen from an example in [23].

## 4. Pseudomonotone Maps

In this section we shall derive characterizations of pseudomonotone maps similar to the ones in the quasimonotone case.

**Proposition 4.1.** *The map $F$ is pseudomonotone on $S$ if and only if the following conditions hold for every $x \in S$, $u \in R^n$:*

(i) $\langle F(x), u\rangle = 0$ *implies* $D_+F(x; u) \ge 0$ ;

(ii) $\langle F(x), u\rangle = 0$ *and* $0 \in \{\langle u, Au\rangle : A \in \partial F(x)\}$ *imply the existence of* $\tilde{t} > 0$ *such that*

$$\langle F(x + tu), u\rangle \ge 0 \quad \textit{for all} \quad t \in [0, \tilde{t}\,] .$$

**Proof.** Let us begin with the "only if" case. Since pseudomonotonicity implies quasimonotonicity, (i) follows from Proposition 3.1.

For (ii), if this is not true, there exists $\hat{t} > 0$ sufficiently small such that $\langle F(x+\hat{t}u), u\rangle < 0$. Let $y = x + \hat{t}u$. Since $\langle F(x), u\rangle = 0$, one has

$$\langle F(x), y - x\rangle = \langle F(x), \hat{t}u\rangle = 0 . \tag{4.1}$$

On the other hand

$$\langle F(y), x - y\rangle = \langle F(x + \hat{t}u), -\hat{t}u\rangle > 0 .$$

However pseudomonotonicity of $F$ implies $\langle F(x), y-x\rangle < 0$ , which contradicts (4.1) . Conversely, if $F$ is not pseudomonotone, there exists $x, y \in S$ such that $\langle F(x), y-x\rangle \geq 0$ and $\langle F(y), x-y\rangle > 0$. Let $u = y - x$ and consider the function $\varphi(t) = \langle F(x+tu), u\rangle$. We have $\varphi(0) \geq 0$ and $\varphi(1) < 0$. By the continuity of $\varphi$, there exists $t_0 \in [0,1]$ such that

$$\varphi(t_0) = 0 \quad \text{and} \quad \varphi(t) < 0 \tag{4.2}$$

for all $t$ with $t_0 < t \leq 1$. It is evident that $D_-F(x+t_0u; u) \leq 0$. Hence $0 \in \{\langle u, Au\rangle : A \in \partial F(x+t_0u)\}$ in view of (i). It follows from (ii) that $\langle F(x+tu), u\rangle \geq 0$ for all $t \geq t_0$, sufficiently close to $t_0$. This contradicts (4.2) and the proof is complete. □

Similar to the quasimonotone case, conditions (i) and (ii) of the preceding proposition are equivalent to the following one: $\langle F(x), u\rangle = 0$ and $D_-F(x; u) \leq 0$ imply the existence of $\tilde{t} > 0$ such that $\langle F(x+tu), u\rangle \geq 0$ for all $t \in [0, \tilde{t}\,]$ .
Now we present sufficient conditions for pseudomonotonicity.

**Proposition 4.2.** *The map $F$ is pseudomonotone on $S$ if the following conditions hold for every $x \in S$, $u \in R^n$:*
(i) $\langle F(x), u\rangle = 0$ *implies* $D_+F(x; u) \geq 0$ ;
(ii) $\langle F(x), u\rangle = 0$ *and* $0 \in \{\langle u, Au\rangle : A \in \partial F(x)\}$ *imply the existence of* $\tilde{t} > 0$ *such that*

$$D_-F(x+tu; u) \geq 0 \quad \textit{for all} \quad t \in [0, \tilde{t}\,] \,.$$

**Proof.** By Proposition 4.1, it suffices to show that $\langle F(x+tu), u\rangle \geq 0$ for all $t \in [0, \tilde{t}]$. In fact, if this is not the case, i.e. $\langle F(x+\bar{t}u), u\rangle < 0$ for some $\bar{t} \in (0, \tilde{t}\,]$, one has $\varphi(0) = 0$ and $\varphi(\bar{t}) < 0$ where as before $\varphi(t) = \langle F(x+tu), u\rangle$. By the mean value theorem, there can be found $t_0 \in (0, \bar{t})$ and $\alpha \in \partial\varphi(t_0)$ such that $t_0\alpha = \varphi(t) - \varphi(0) < 0$. Consequently, $D_-F(x+t_0u; u) < 0$ because $\alpha \in [D_-F(x+t_0u; u)\ ,\ D_+F(x+t_0u; u)]$. This contradicts the assumption of the proposition. □

In addition, we have a nonsmooth version of Theorem 2.3, (ii) [6] for pseudomonotonicity.

**Proposition 4.3.** *Suppose that for every $x \in S$, $u \in R^n$ the following conditions hold:*
(i) $\langle F(x), u\rangle = 0$ *implies* $D_-F(x; u) \geq 0$ ;
(ii) $F(x) = 0$ *and* $D_-F(x; u) = 0$ *imply the existence of* $\tilde{t} > 0$ *such that*

$$\langle F(x+tu), u\rangle \geq 0 \textit{ for all } t \in [0, \tilde{t}\,] \,.$$

*Then $F$ is pseudomonotone on $S$.*

**Proof.** We can follow the same arguments as in the proof of Proposition 3.3 by using Lemma 3.2 and Proposition 4.1. □

As in the quasimonotone case, the condition of Proposition 4.3 is not necessary for pseudomonotonicity as seen by the same example in [23]. An immediate consequence of Proposition 4.3 is the following result.

**Corollary 4.4.** *Suppose that for every $x \in S$ and $u \in R^n$ the following conditions hold:*
(i) $\langle F(x,), u\rangle = 0$ *implies* $D_-F(x; u) \geq 0$ ;
(ii) $F(x) = 0$ *implies* $D_-F(x; u) > 0$.
*Then $F$ is pseudomonotone on $S$.*

## 5. Strictly and Strongly Pseudomonotone Maps

In this last section we give sufficient conditions and necessary conditions for special types of pseudomonotone maps.

**Proposition 5.1.** *The map $F$ is strictly pseudomonotone on $S$ if the following condition holds: for every $x \in S$, $u \in R^n$ , $\langle F(x), u\rangle = 0$ implies $D_-F(x; u) > 0$.*

**Proof.** It is clear that the condition above implies conditions (i) and (ii) of Proposition 4.2. Hence $F$ is pseudomonotone. If it is not strictly pseudomonotone, then there can be found $x, y \in S$ such that $\langle F(x), y - x\rangle \geq 0$ and $\langle F(y), x - y\rangle \geq 0$.
By pseudomonotonicity, it follows that

$$\langle F(x), y - x\rangle = 0 \quad \text{and} \quad \langle F(y), y - x\rangle = 0 . \tag{5.1}$$

The condition of the proposition implies that $D_-F(x; y - x) > 0$. Hence for $t > 0$ sufficiently close to $t = 0$, one has $\langle F(x + t(y - x)), y - x\rangle > 0$ . The latter inequality in turn implies that $\langle F(y), y - (x + t(y - x))\rangle > 0$ by pseudomonotonicity. Hence $\langle F(y), y - x\rangle > 0$, which contradicts (5.1). □

**Proposition 5.2.** *If the map $S$ is strongly pseudomonotone on $S$, then for every $x \in S$, there exists $\beta > 0$ such that $\langle F(x), u\rangle = 0$ and $||u|| = 1$ imply $D_+F(x; u) \geq \beta$.
Conversely, if for every $x \in S$ there exists $\beta > 0$ such that $\langle F(x), u\rangle = 0$ and $||u|| = 1$ imply $D_-F(x; u) \geq \beta$, then $F$ is strongly pseudomonotone.*

**Proof.** To prove the first part of the proposition, it suffices to show that $\langle F(x+tu), u\rangle \geq \beta t$ for all $t \in [0, \varepsilon]$ implies $D_+F(x; u) \geq \beta$. For this purpose, let us consider the function $\varphi(t) = \langle F(x + tu), u\rangle$. One has $\varphi(0) = 0$ and $\varphi(t) \geq \beta t$ for every $t \in [0, \varepsilon]$. Using the mean value theorem, for each $t \in (0, \varepsilon)$ one can find $t_0 \in (0, t)$ and $\alpha \in \partial\varphi(t_0)$ such that $\alpha t = \varphi(t) - \varphi(0) \geq \beta t$. Consequently $\alpha \geq \beta$ and $D_+F(x; u) \geq \beta$.
For the second part, suppose to the contrary that $F$ is not strongly pseudomonotone. There exist $x \in S$ and $u \in R^n$ with $\langle F(x), u\rangle = 0$ and $||u|| = 1$ such that for each positive $\varepsilon$ and $\beta$ there can be found $t_{\varepsilon\beta} \in [0, \varepsilon]$ satisfying the inequality

$$\langle F(x + t_{\varepsilon\beta}u), u\rangle < \beta t_{\varepsilon\beta} .$$

Applying the mean value theorem to the function $\varphi$, we can find $t_0 \in (0, t_{\varepsilon\beta})$ and $\alpha \in \partial\varphi(t_0)$ such that

$$t_{\varepsilon\beta}\alpha = \varphi(t_{\varepsilon\beta}) - \varphi(0) < \beta t_{\varepsilon\beta} .$$

Since $\beta$ is arbitrary small, the latter inequality shows that $D_-F(x; u) \leq 0$. This contradicts the assumption. □

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**Leere Seite**
**206**