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On Primal-Dual Stability in Convex Optimization

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Conditions are given ensuring the stability of the classical duality scheme of Ekeland-Temam-Rockafellar with respect to the slice topology (a generalization of the Mosco topology to the non reflexive setting). Both primal and dual aspect are tackled allowing convergence of primal solutions along with the Lagrange multipliers. An example of application is given to optimal control of distributed parameter systems.

1. Introduction and notations

There has been recently a renewed interest in convex duality, mainly due to its connections with fully nonlinear optimal control problems, revealed by R. Vinter in [34]. On the other hand several notions of variational convergence for convex functions have been introduced and thoroughly studied by many authors ([1], [2], [3], [5], [6], [7], [9], [11], [12], [16], [17], [18], [19], [25], [26], [27], [31], [32], [33]). These variational notions of convergence lead to convergence of exact and approximate solutions of the associated minimization problems. It is quite natural to investigate the behaviour of the primal and dual functionals attached to a convex perturbation function when this perturbation function moves along some topology. This was firstly done in [13] in the finite dimensional setting. The natural topologies to be considered in this problems are those making continuous the conjugacy operation. In this work we deal with general normed spaces and we focus our attention on the slice topology introduced in [11], [12], [31], [32], [33] (see also [16], [17], [7], which specializes to the Mosco topology ([25], [26]) in the nonreflexive setting. In section 2, we study the primal stability of the duality scheme. To this end we prove a result on the slice convergence of the sum of two convex functions which might be useful in other fields of convex analysis. In section 3 we treat the case of the stability both of the primal and the dual functionals. Our approach differs from [10] where primal- dual stability was obtained by means of conditions bearing on the convex-concave Lagrangians. In our framework the stability is studied in terms of perturbations of the convex function which generates the Lagrangian. The assumptions used to prove our primal-dual stability result, namely, the equi-continuity at 0 of the value functions and the equi-coerciveness of primal functionals, are quite natural. Our result differs also from the one given in [15] and also from [30] which rely, in the finite dimensional setting, on a stronger notion of convergence for convex functions. An application is given in section 4 to a singular

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perturbation problem in optimal control of distributed parameter systems.

Let us give some definitions. Let $(X, \|.\|)$ be a normed vector space and let $f : X \longrightarrow \overline{\mathbb{R}}$ be a function. We denote by

$$\operatorname{epi} f = \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$$

its *epigraph* and by

$$\operatorname{epi}' f = \{(x,t) \in X \times \mathbb{R} : f(x) < t\}$$

its *strict epigraph*. We also denote by

$$\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$$

its *effective domain* and by

$$[f \leq c] = \{x \in X : f(x) \leq c\}$$

its sublevel sets. We say that f is proper whenever $\operatorname{dom} f \neq \emptyset$ and $-\infty \notin f(X)$. We denote by $\overline{f}$ the *lower semi continuous regularization* of f characterised by

$$\operatorname{epi} \overline{f} = \overline{\operatorname{epi} f}$$

where $\overline{\operatorname{epi} f}$ stands for the closure of $\operatorname{epi} f$. We denote by $\operatorname{Conv}(X)$ the set of proper convex functions defined on X with values in $\mathbb{R} \cup \{+\infty\}$ and by $\Gamma_0(X)$ the set of those $f \in \operatorname{Conv}(X)$ which are lower semicontinuous on X. Given $f \in \operatorname{Conv}(X)$ (resp. $g \in \operatorname{Conv}(X^*)$), its conjugate is defined on X^* (resp. on X) by

$$f^*(y) = \sup\{\langle x, y\rangle - f(x) : x \in X\} \text{ for all } y \in X^*,$$

and

$$g^*(x) = \sup\{\langle x, y\rangle - g(y) : y \in X^*\} \text{ for all } x \in X.$$

The *indicator function* of a subset $A \subset X$ is the function i_A defined by $i_A(x) = 0$ if $x \in A$ and $i_A(x) = +\infty$ if $x \notin$A. We denote by $\sigma_C = i_C^*$ the *support function* of a convex subset C of X, and by $C^\circ \subset X^*$ its *polar set* defined by

$$C^\circ = \{y \in X^* : \sigma_C(y) \leq 1\}.$$

The *gauge function* of a subset $C \subset X$ is the function $j : X \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined for all $x \in X$ by

$$j(x) = \inf\{t > 0 : x \in tC\}$$

with the convention $\inf \emptyset = +\infty$. It is known (see [24], p. 34) that

$$\sigma_C = j_{C^\circ}. \tag{1}$$

whenever $C \subset X$ is convex and contains 0.

Let us now review the definitions we shall need about topologies on hyperspaces. We denote by $\mathcal{F}(X)$ the set of nonempty closed subsets of X and by $\mathcal{C}(X)$ the set of nonempty closed convex subsets of X. The *lower Vietoris topology* τ_V^- is the topology on $\mathcal{F}(X)$ whose a subbase is constituted by the sets

$$V^- = \{C \in \mathcal{F}(X) : C \cap V \neq \emptyset\}$$

where V ranges over the family of norm open subsets of X. A sequence $(C_n) \subset \mathcal{F}(X)$ converges to $C \in \mathcal{F}(X)$ with respect to τ_V^- if and only if

$$C \subset \liminf_{n\to\infty} C_n$$

where, given a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of X, one denotes by $\liminf_{n\to\infty} C_n$ the set of $x \in X$ such that there exists a sequence (x_n) which converges to x with $x_n \in C_n$ eventually.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\overline{\mathbb{R}}^X$, we introduce its *epigraphical upper limit* $f = e\text{-}\limsup_{n\to\infty} f_n$ (see [1]) defined by

$$\operatorname{epi} f = \liminf_{n\to\infty}(\operatorname{epi} f_n). \tag{2}$$

In [16] and [17] (see also [6], [9], [11]) was introduced the *Joly topology* τ_J in $\Gamma_0(X)$ as the weakest topology making continuous the mappings

$$E : \Gamma_0(X) \longrightarrow (\mathcal{C}(X \times \mathbb{R}), \tau_V^-)$$

and

$$E^* : \Gamma_0(X) \longrightarrow (\mathcal{C}^*(X^* \times \mathbb{R}), \tau_V^-),$$

where $\mathcal{C}^*(X^* \times \mathbb{R})$ is the set of w^*-closed convex subsets of $X^* \times \mathbb{R}$, defined for all $f \in \Gamma_0(X)$ by

$$E(f) = \operatorname{epi} f \text{ and } E^*(f) = (E \circ \mathcal{L})(f) = \operatorname{epi} f^*.$$

Analogously, the *dual Joly topology* τ_J^* is defined on $\Gamma_0(X^*)$ by exchanging the roles played by X and X^* in the definition of τ_J. From the very definitions of these topologies, it follows that the conjugacy operation is continuous from $(\Gamma_0(X), \tau_J)$ into $(\Gamma_0(X^*), \tau_J^*)$ and from $(\Gamma_0(X^*), \tau_J^*)$ into $(\Gamma_0(X), \tau_J)$.

A sequence $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to τ_J in $\Gamma_0(X)$ if and only if

$$\operatorname{epi} f \subset \liminf_{n\to\infty}(\operatorname{epi} f_n) \text{ and } \operatorname{epi} f^* \subset \liminf_{n\to\infty}(\operatorname{epi} f_n^*), \tag{3}$$

which is equivalent using (2) to

$$e\text{-}\limsup_{n\to\infty} f_n \le f \text{ and } e\text{-}\limsup_{n\to\infty} f_n^* \le f^*.$$

If the case when X is a general normed space, this topology has been characterized only recently (see [11], [31]). Following G. Beer in [11] one defines the topology τ_S^+ on $\mathcal{C}(X)$ as the topology whose a subbase is constituted by the family of sets

$$(B^c)^{++} = \{C \in \mathcal{C}(X) : D(B, C) > 0\}$$

where B ranges over the family of closed bounded convex sets of X and

$$D(B,C) = \inf\{\|b-c\| : (b,c) \in B \times C\}.$$

The *slice topology* τ_S is defined by

$$\tau_S = \tau_S^- \vee \tau_S^+ \tag{4}$$

where $\tau_S^- = \tau_V^-$ with a similar definition for τ_S^* where B is w^*-compact convex. A net $(C_i)_{i\in I}$ converges in $(\mathcal{C}(X), \tau_S)$ if and only if for all $B \subset X$ closed bounded convex we have

$$D(B,C) \le \liminf_I D(B,C_i) \text{ and } \limsup_I D(B,C_i) \le D(B,C). \tag{5}$$

In [11] it has been proved that

$$\tau_J = \tau_S \text{ and } \tau_J^* = \tau_S^*. \tag{6}$$

The convergence with respect to the slice topology generalizes to non reflexive spaces the convergence in the sense of Mosco (see [25], [26], [1]). The *Mosco-Beer topology* τ_M is the topology defined on $\mathcal{C}(X)$ by

$$\tau_M = \tau_V^- \vee \tau_M^+$$

where τ_M^+ is the topology on $\mathcal{C}(X)$ whose a basis is constituted by the sets

$$K^{c+} = \{C \in \mathcal{C}(X) : K \cap C = \emptyset\}$$

with K weakly compact. The *dual Mosco-Beer topology* τ_M^* is defined on the set $\mathcal{C}^*(X^*)$ of w^*-closed convex subsets of X^* by

$$\tau_M^* = \tau_V^- \vee \tau_M^{*+}$$

where τ_M^{*+} is the topology on $\mathcal{C}^*(X^*)$ whose a basis is constituted by the sets

$$K^{c+} = \{C \in \mathcal{C}^*(X^*) : K \cap C = \emptyset\}$$

with K w^*-compact.

For the convenience of the reader we give the following well known lemma.

Lemma 1.1. *Let X be a normed vector space and let $(f_n)_{n\in\mathbb{N}}$, $f \in \overline{\mathbb{R}}^X$. Then $e\text{-}\limsup_{n\to\infty} f_n \le f$ if and only if*

$$\textit{for all } x \in X \textit{ there exists } (x_n) \longrightarrow x \textit{ such that } f(x) \ge \limsup_{n\to\infty} f_n(x_n).$$

Proof. Suppose that $f \ge e\text{-}\limsup_{n\to\infty} f_n$ and let $x \in X$. We can suppose that $f(x) < +\infty$. Then given $\lambda \in \mathbb{R}$ with $\lambda > f(x)$, there exists a sequence $(x_n, \lambda_n) \in \operatorname{epi} f_n$ such that (x_n, λ_n) converges to (x, λ). Thus we have

$$\limsup_{n\to\infty} f_n(x_n) \le \limsup_{n\to\infty} \lambda_n = \lambda.$$

Letting λ decrease to $f(x)$ and using a diagonalization procedure, one gets a sequence $(x_n)_{n\in\mathbb{N}}$ which converges to x and which satisfies $f(x) \geq \limsup_{n\to\infty} f_n(x_n)$. Conversely, let $(x,\lambda) \in \operatorname{epi} f$ and let (x_n) be a sequence converging to x such that

$$t := \limsup_{n\to\infty} f_n(x_n) \leq f(x).$$

If $t = -\infty$ then $f_n(x_n) < \lambda$ eventually, thus $(x_n, \lambda) \in \operatorname{epi} f_n$ eventually and the sequence $((x_n, \lambda))$ converges to (x, λ). If $t \neq -\infty$, let us set

$$\lambda_n = \lambda - t + \max(t, f_n(x_n)),$$

we get $(x_n, \lambda_n) \in \operatorname{epi} f_n$ and the sequence $((x_n, \lambda_n))$ converges to (x, λ), hence the result. $\square$

We shall use the following characterization of the convergence of sequence in $\Gamma_0(X)$ with respect to the slice topology (see [3]).

Proposition 1.2. *Let X be a normed vector space and let $(f_n)_{n\in\mathbb{N}}$, $f \in \Gamma_0(X)$. Then $f = \tau_S - \lim_{n\to\infty} f_n$ if and only if*

$$\begin{cases} \text{for all } x \in X, & \text{there exists } (x_n) \longrightarrow x : \quad f(x) \geq \limsup\limits_{n\to\infty} f_n(x_n) \\ \text{for all } y \in X^*, & \text{there exists } (y_n) \longrightarrow y : \quad f^*(y) \geq \limsup\limits_{n\to\infty} f_n^*(y_n). \end{cases}$$

Proof. This is immediate from Lemma 1.1, from (3) and from the fact that $\tau_J = \tau_S$. $\square$

2. Primal stability

Let us begin by a quick review on the main features in convex duality (see [14], [20] and [29]). Let X, U be normed vector spaces whose topological dual spaces are denoted by $V = X^*$ and $Y = U^*$. Given $F \in \Gamma_0(X \times U)$, the primal problem associated to the perturbation function F consists in

$$\inf_{x\in X} F(x, 0) = \inf_{x\in X} f(x).$$

We shall always assume that the function $f(x) = F(x, 0)$ is proper. We set for all $u \in U$, $\varphi(u) = \inf_{x\in X} F(x, u)$. The value function φ is convex but not necessarily lower semicontinuous. Observe that $\varphi^*(y) = F^*(0, y)$. The associated dual problem consists in

$$\inf_{y\in Y} F^*(0, y).$$

It is also useful to introduce for all $v \in V$ the value function $\psi(v) = \inf_{y\in Y} F^*(v, y)$. One has for all $x \in X$, $\psi^*(x) = F(x, 0)$. The convex-concave Lagrangian associated to the perturbation function is the function $L : X \times Y \longrightarrow \overline{\mathbb{R}}$ defined for all $(x, y) \in X \times Y$ by

$$L(x, y) = \inf_{w\in U} (F(x, w) - \langle w, y \rangle).$$

It is important to observe that

$$F^*(0,y) = \sup_{x\in X}(-L(x,y)). \tag{7}$$

A pair $(\bar{x},\bar{y}) \in X \times Y$ solves

$$F(\bar{x},0) = \inf_{x\in X} F(x,0) \text{ and } F^*(0,\bar{y}) = \inf_{y\in Y} F^*(0,y) \text{ with } F(\bar{x},0,) = -F^*(0,\bar{y})$$

if and only if $(\bar{x},\bar{y})$ is a saddle-point of the Lagrangian that is

$$L(\bar{x},y) \leq L(\bar{x},\bar{y}) \leq L(x,\bar{y}) \text{ for all } (x,y) \in X \times Y.$$

Moreover, assuming that $L(\bar{x},\bar{y})$ is finite, this is equivalent to

$$L'_x((\bar{x},\bar{y}); x-\bar{x}) \geq 0 \text{ and } L'_y((\bar{x},\bar{y}); y-\bar{y}) \leq 0 \tag{8}$$

for all $(x,y) \in X \times Y$ where $L(\bar{x},y)$ and $L(x,\bar{y})$ are finite with

$$L'_x((\bar{x},\bar{y}); x-\bar{x}) = \lim_{t\downarrow 0} \frac{L(\bar{x}+t(x-\bar{x}),\bar{y}) - L(\bar{x},\bar{y})}{t}$$

and

$$L'_y((\bar{x},\bar{y}); y-\bar{y}) = \lim_{t\downarrow 0} \frac{L(\bar{x},\bar{y}+t(y-\bar{y})) - L(\bar{x},\bar{y})}{t}.$$

We shall use the following well-known result.

Lemma 2.1. *Let X, U be normed vector spaces, let $F_n : X \times U \longrightarrow \mathbb{R} \cup \{+\infty\}$ and let $F : X \times U \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that* $\operatorname{epi} F \subset \liminf_{n\to\infty}(\operatorname{epi} F_n)$. *Then*

$$\operatorname{epi}\varphi \subset \liminf_{n\to\infty}(\operatorname{epi}\varphi_n).$$

Proof. One has $\operatorname{epi}'\varphi = \operatorname{proj}_{U\times\mathbb{R}}(\operatorname{epi}' F)$. Let $(u,\lambda) \in \operatorname{epi}'\varphi$, there exists $x \in X$ such that $(x,u,\lambda) \in \operatorname{epi}' F$. Thus there exists a sequence (x_n,u_n,λ_n) such that $(x_n,u_n,\lambda_n) \in \operatorname{epi} F_n$ and $((x_n,u_n,\lambda_n))$ converges to (x,u,λ). As $(u_n\lambda_n) \in \operatorname{epi}\varphi_n$ and $((u_n,\lambda_n))$ converges to (u,λ) we derive that $\operatorname{epi}'\varphi \subset \liminf_{n\to\infty}(\operatorname{epi}\varphi_n)$, thus

$$\operatorname{epi}\varphi \subset \overline{\operatorname{epi}'\varphi} \subset \liminf_{n\to\infty}(\operatorname{epi}\varphi_n).$$

□

The following lemma in the spirit of [5] plays a prominent role in the sequel.

Lemma 2.2. *Let X be a normed vector space and $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}} \in \operatorname{Conv}(X)$. Assume that for some real numbers $s > 0$, $r \geq 0$, $c \in \mathbb{R}$*

$$sB_X \subset \bigcap_{n\in\mathbb{N}} \Big([f_n \leq c] \cap rB_X - [g_n \leq c] \cap rB_X\Big). \tag{9}$$

Then if f, $g \in \mathrm{Conv}(X)$ are such that e- $\limsup_{n\to\infty} f_n \le f$ and e-$\limsup_{n\to\infty} g_n \le g$, one has

$$e\text{-}\limsup_{n\to\infty}(f_n + g_n) \le f + g.$$

Proof. Let $x \in \mathrm{dom}(f+g)$. There exist sequences (x_{1n}) and (x_{2n}) converging to x such that $\limsup_{n\to\infty} f_n(x_{1n}) \le f(x)$ and $\limsup_{n\to\infty} g_n(x_{2n}) \le g(x)$. Let us set $t_n = \|x_{1n} - x\| + \|x_{2n} - x\|$. From (9) there exist sequences (z_{1n}) and (z_{2n}) with $z_{1n} \in [f_n \le c] \cap rB_X$, $z_{2n} \in [g_n \le c] \cap rB_X$ and

$$s(x_{2n} - x_{1n}) = t_n(z_{1n} - z_{2n}).$$

Let us set $x_n = (t_n + s)^{-1}(t_n z_{1n} + s x_{1n}) = (t_n + s)^{-1}(t_n z_{2n} + s x_{2n})$. One has

$$\begin{aligned} \|x_n - x\| &\le (t_n + s)^{-1}(t_n\|z_{1n} - x\| + s\|x_{1n} - x\|) \\ &\le s^{-1}t_n(r + \|x\|) + \|x_{1n} - x\|, \end{aligned}$$

thus (x_n) converges to x. Using the fact that

$$\begin{aligned} f_n(x_n) &\le (t_n + s)^{-1}t_n f_n(z_{1n}) + (t_n + s)^{-1}s f_n(x_{1n}) \\ &\le s^{-1}t_n c + f_n(x_{1n}) \end{aligned}$$

and

$$\begin{aligned} g_n(x_n) &\le (t_n + s)^{-1}t_n g_n(z_{2n}) + (t_n + s)^{-1}s g_n(x_{2n}) \\ &\le s^{-1}t_n c + g_n(x_{2n}), \end{aligned}$$

we get

$$\begin{aligned} \limsup_{n\to\infty}(f_n + g_n)(x_n) &\le \limsup_{n\to\infty} f_n(x_{1n}) + \limsup_{n\to\infty} g_n(x_{2n}) \\ &\le (f+g)(x) \end{aligned}$$

which ends the proof of the lemma. □

Applying the preceding lemma to the indicator functions of C_n and C, we get the following corollary.

Corollary 2.3. *Let X be a normed vector space, let $(C_n)_{n\in\mathbb{N}}$, C, $(D_n)_{n\in\mathbb{N}}$, D be convex subsets such that*

$$C \subset \liminf_{n\to\infty} C_n \text{ and } D \subset \liminf_{n\to\infty} D_n.$$

Assume that there exists $s > 0$, $r \ge 0$, $c \in \mathbb{R}$ such that

$$sB_X \subset \bigcap_{n\in\mathbb{N}} \Big(C_n \cap rB_X - D_n \cap rB_X\Big). \tag{10}$$

Then

$$C \cap D \subset \liminf_{n\to\infty}(C_n \cap D_n).$$

We shall also need the following results of [4].

Theorem 2.4. *Let $(X, \|.\|)$ be a normed vector space and let $f,\ g \in \mathrm{Conv}(X)$. Assume that for some real numbers $s > 0$, $r \geq 0$, $c \in \mathbb{R}$*

$$sB_X \subset [f \leq c] \cap rB_X - [g \leq c] \cap rB_X.$$

Then for all $y \in X^*$ one has

$$(f+g)^*(y) = \min_{y' \in X^*} f^*(y') + g^*(y - y').$$

Lemma 2.5. *Let X, U be normed vector spaces, let $(F_n)_{n\in\mathbb{N}}$, $F \in \Gamma_0(X \times U)$ be such that $F = \tau_S\text{-} \lim_{n\to\infty} F_n$. Assume that there exist $s > 0$, $r \geq 0$, $c \in \mathbb{R}$ such that*

$$sB_{X\times U} \subset \bigcap_{n\in\mathbb{N}} \Big(X \times \{0\} \cap rB_{X\times U} - [F_n \leq c] \cap rB_{X\times U} \Big). \tag{11}$$

Then

$$F(.,0) = \tau_S\text{-} \lim_{n\to\infty} F_n(.,0).$$

Proof. From assumption (11) and Lemma 2.1, we get

$$\mathrm{epi}\,(F + i_{X\times\{0\}}) \subset \liminf_{n\to\infty} \mathrm{epi}\,(F_n + i_{X\times\{0\}})$$

which readily turns to

$$\mathrm{epi}\,(F(.,0)) \subset \liminf_{n\to\infty} \Big(\mathrm{epi}\,(F_n(.,0)) \Big). \tag{12}$$

On the other hand, we derive from Theorem 2.4 that, for all $v \in V$

$$(F_n + i_{X\times\{0\}})^*(v,0) = \min_{(v',y')\in V\times Y} F_n^*(v - v', -y') + i_{\{0\}\times Y}(v',y') = \psi_n(v).$$

It follows that $\psi_n \in \Gamma_0(V)$. As $\psi_n^* = f_n = F_n(.,0)$, we get $(F_n(.,0))^* = \psi_n$. Relying on the slice convergence of the sequence (F_n) to F and using Lemma 2.1, we derive that

$$\mathrm{epi}\,(F(.,0))^* \subset \liminf_{n\to\infty} \mathrm{epi}((F_n(.,0))^* \tag{13}$$

which combined with (12) yields the announced result. □

One can apply Lemma 2.5 to the particular case of Fenchel duality. Let us consider a primal problem of the type

$$\inf_{x\in X} \Big(f(x) + g(A(x)) \Big)$$

where f and g belong respectively to $\Gamma_0(X)$ and $\Gamma_0(U)$ and where $A : X \longrightarrow U$ is a linear continuous operator. The associated perturbation function is

$$F(x,u) = f(x) + g(A(x) + u).$$

Its conjugate is given by

$$F^*(v, y) = f^*(-A^*(v) + y) + g^*(v).$$

Theorem 2.6. *Let X, U be normed vector spaces, let $A \in L(X, U)$ and let $(f_n)_{n\in\mathbb{N}}$, $f \in \Gamma_0(X)$, $(g_n)_{n\in\mathbb{N}}$, $g \in \Gamma_0(U)$ be such that*

$$\tau_S\text{-}\lim_{n\to\infty} f_n = f \ \text{ and } \tau_S\text{-}\lim_{n\to\infty} g_n = g.$$

Assume there exist $s > 0$, $r \geq 0$, $c \in \mathbb{R}$ such that

$$sB_U \subset \bigcap_{n\in\mathbb{N}} \Big([g_n \leq c] \cap rB_U - A([f_n \leq c] \cap rB_X)\Big). \tag{14}$$

Then

$$\tau_S\text{-}\lim_{n\to\infty} (f_n + (g_n \circ A)) = f + (g \circ A).$$

Proof. Let us set

$$F(x, u) = f(x) + g(A(x) + u) \text{ and } F_n(x, u) = f_n(x) + g_n(A(x) + u).$$

We claim that

$$\tau_S\text{-}\lim_{n\to\infty} F_n = F.$$

Indeed, let $(x, u) \in X \times U$. From the slice convergence of (f_n) and (g_n) to f and g there exist sequences (x_n) and (y_n) converging respectively to x and $A(x) + u$ such that

$$\limsup_{n\to\infty} f_n(x_n) \leq f(x) \text{ and } \limsup_{n\to\infty} g_n(y_n) \leq g(A(x) + u).$$

Setting $u_n = y_n - Ax_n$, we observe that u_n converges to u and we get

$$\begin{aligned}
\limsup\nolimits_{n\to\infty} F_n(x_n, u_n) &\leq \limsup\nolimits_{n\to\infty} f_n(x_n) + \limsup\nolimits_{n\to\infty} g_n(A(x_n) + u_n) \\
&\leq f(x) + g(Ax + u) \\
&= F(x, u).
\end{aligned}$$

It follows that $e\text{-}\limsup_{n\to\infty} F_n \leq F$. On the other hand as

$$F^*(v, y) = f^*(-A^*(y) + v) + g^*(y) \text{ and } F_n^*(v, y) = f_n^*(-A^*(y) + v) + g_n^*(y).$$

we obtain by the same reasoning as above that

$$e\text{-}\limsup_{n\to\infty} F_n^* \leq F^*.$$

Now let $u \in sB_U$ and $\xi \in sB_X$. For all $n \in \mathbb{N}$, there exist $u_n \in [g_n \leq c] \cap rB_U$ and $x_n \in [f_n \leq c] \cap rB_X$ with $u = u_n - A(x_n)$. Thus we get $(\xi, u) = (x_n, u) - (x_n - \xi, 0)$ with

$$(x_n, u) \in [F_n \leq 2c] \cap (r + s)B_{X\times U} \text{ and } (x_n - \xi, 0) \in (X \times \{0\}) \cap (r + s)B_{X\times U}$$

yielding

$$sB_{X\times U} \subset \bigcap_{n\in\mathbb{N}} \Big([F_n \le 2c] \cap (r+s)B_{X\times U} - (X\times\{0\}) \cap (r+s)B_{X\times U}\Big).$$

Hence we can apply Lemma 2.5 which ends the proof of the theorem. □

Corollary 2.7. *Let X be a normed vector space, let $(f_n)_{n\in\mathbb{N}}$, f and let $(g_n)_{n\in\mathbb{N}}$, $g \in \Gamma_0(X)$ be such that*

$$\tau_S\text{-}\lim_{n\to\infty} f_n = f \text{ and } \tau_S\text{-}\lim_{n\to\infty} g_n = g. \tag{15}$$

Assume that for some real numbers $s>0$, $r\ge 0$, $c\in\mathbb{R}$

$$sB_X \subset \bigcap_{n\in\mathbb{N}} \Big([f_n \le c] \cap rB_X - [g_n \le c] \cap rB_X\Big). \tag{16}$$

Then

$$\tau_S\text{-}\lim_{n\to\infty} (f_n+g_n) = f+g.$$

Proof. Apply Theorem 2.6 with $U=X$ and $A=I_X$. □

One readily deduces from the above theorem a result on the continuity of the intersection of convex sets with respect to the slice topology.

Corollary 2.8. *Let X be a normed vector space, let $(C_n)_{n\in\mathbb{N}}$, $(D_n)_{n\in\mathbb{N}}$, C, $D\in\mathcal{C}(X)$ such that*

$$\tau_S\text{-}\lim_{n\to\infty} C_n = C \text{ and } \tau_S\text{-}\lim_{n\to\infty} D_n = D.$$

Assume that for some real numbers $s>0$, $c\in\mathbb{R}$

$$sB_X \subset \bigcap_{n\in\mathbb{N}} \Big(C_n \cap rB_X - D_n \cap rB_X\Big). \tag{17}$$

Then

$$\tau_S\text{-}\lim_{n\to\infty} (C_n \cap D_n) = C\cap D.$$

Remark 2.9. Corollary 2.7 strictly extends Theorem 3.3 of [18] in which the continuity of the sum was obtained under the following assumption:

$$\text{there exists } s>0,\ \lambda\in\mathbb{R},\ x_0\in \operatorname{dom} g \text{ such that } \sup_{(n,x)\in\mathbb{N}\times B(x_0,s)} f_n(x) \le \lambda. \tag{18}$$

Assume that this assumption is satisfied. From $g=\tau_S\text{-}\lim_{n\to\infty} g_n$ and Proposition 1.2 it follows the existence of $(u_n)\to 0$ such that

$$g(x_0) \ge \limsup_{n\to\infty} g_n(x_0+u_n).$$

Thus there exists a constant $\mu \in \mathbb{R}$ such that for all n sufficiently large

$$g_n(x_0 + u_n) \leq \mu \text{ and } u_n \in \frac{s}{2} B_X.$$

Let $u \in \frac{s}{2} B_X$. Let us set $v_n = u + u_n$. One has $v_n \in sB_X$, $u = (x_0 + v_n) - (x_0 + u_n)$ and $f_n(x_{0n} + v_n) \leq \lambda$, $g_n(x_{0n} + u_n) \leq \mu$. Thus there exists $r \geq 0$ and $c \in \mathbb{R}$ such that

$$sB_X \subset \bigcap_{n \in \mathbb{N}} \Big([f_n \leq c] \cap rB_X - [g_n \leq c] \cap rB_X \Big)$$

which is assumption (16) of Corollary 2.7. On the other hand, given a closed hyperplane $H \subset X$ and a vector $u \notin H$ and setting $f_n \equiv i_H$, $g_n \equiv i_{\mathbb{R}u}$, we observe that (16) is in force but (18) fails to be satisfied.

Remark 2.10. Theorem 2.6 and Corollary 2.7 also extend to the infinite dimensional setting Theorem 5 and 8 of [23]. Indeed, taking into account the existence of a basis of neighborhhods of the origin in $\mathbb{R}^d$ constituted by polyhedral sets, one easily checks as in [5], Theorem 1.1' that assumption (16) holds whenever (15) is in force and $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$.

Example 2.11. Let X be a reflexive Banach space and let (f_n) and (g_n) be two sequences in $\Gamma_0(X)$ converging respectively to f and g in the sense of Mosco. Then the sequence $(f_n + g_n)$ converges to $f + g$ for τ_M^+ without any condition. This result fails if we replace the Mosco topology by the slice topology. This is shown by the following slight modification, brought to our attention by G. Beer in a private communication, of an example given in [11]. Let X be a non reflexive space and let $y \in X^*$ which is not norm achieving on the unit ball of X. Pick up $x_0 \in X$ satisfying $\langle x_0, y\rangle = 1$ and $\|x_0\| \geq 1$, and let $H = y^{-1}(1)$. Let $g \in \Gamma_0(X)$ defined by $g(x) = d(x, H) + i_{B(x_0,1)}(x)$ and let $f \in \Gamma_0(X)$ defined by $\operatorname{epi} f = \bigcup_{t \geq 0} t\,(\operatorname{epi} g) \cup (\{0\} \times \mathbb{R})$ (observe that this set is closed from the assuptions made). Setting $\alpha = \inf_{x \in X} f(x) = 0$ and $C = [f \leq \alpha]$ one has $C \neq \emptyset$ and the family of closed convex $C_\epsilon = [f \leq \alpha + \epsilon]$ is such that $C \neq \tau_S^+\text{-}\lim_{\epsilon \downarrow 0} C_\epsilon$. Let us set $A = \operatorname{epi} f$ and $D_\epsilon = X \times \{\alpha + \epsilon\}$. One has $\tau_S\text{-} \lim_{\epsilon \downarrow 0} D_\epsilon = X \times \{\alpha\}$. Moreover $A \cap D_\epsilon = [f \leq \alpha + \epsilon] \times \{\alpha + \epsilon\}$ does not converge for τ_S^+. One observes that in this example the assumption (17) is not satisfied since $D_\epsilon - A \subset X \times [-\epsilon, +\infty]$ thus $sB_{X \times \mathbb{R}} \subset D_\epsilon - A$ imply $s \leq \epsilon$. It follows that $D_\epsilon - A$ cannot contain a fixed ball sB_X for all $\epsilon > 0$.

3. Primal-dual stability

The following result is a quantitative version of [24], Proposition 8.d.

Lemma 3.1. *Let X be a normed vector space and let $f \in \Gamma_0(X)$ be such that there exist $\lambda > \inf_{x \in X} f(x)$, $\rho \geq 0$ satisfying*

$$[f \leq \lambda] \subset \rho B_X. \tag{19}$$

Then for all $\lambda_0 \in]\inf_{x \in X} f(x), \lambda[$ and for all $v \in \frac{\lambda - \lambda_0}{2\rho} B_{X^*}$ one has

$$f^*(v) \leq f^*(0) + \frac{3}{2}(\lambda - \lambda_0).$$

As $f \in \Gamma_0(X)$, it is bounded from below on the bounded subset $[f \leq \lambda]$ and then bounded from below on the whole X. Let us set $\beta = \inf_{x \in X} f = -f^*(0) \in \mathbb{R}$. Let $\lambda_0 \in \mathbb{R}$ be such that $\inf_{x \in X} f(x) < \lambda_0 < \lambda$. Define $h = f^* - \langle x_0, . \rangle + \lambda_0$ where $x_0 \in [f < \lambda_0]$. Observe that $h = g^*$ with $g(x) = f(x + x_0) - \lambda_0$. One has

$$\inf_{x \in X} g = \beta - \lambda_0 < 0.$$

The set $C = [g \leq \lambda - \lambda_0]$ is a closed convex subset containing 0 and

$$C \subset [f \leq \lambda] - x_0 \subset 2\rho B_X.$$

Let us introduce the gauge function of C

$$j(x) = \inf\{t > 0 : x \in tC\},$$

and let $\Delta = \mathbb{R}_+ z$ be a half-line through the origin. Assuming that $j(z) = +\infty$ we get $g \equiv +\infty$ on $\Delta \setminus \{0\}$. Indeed if $x \in \Delta \setminus \{0\}$ satisfies $g(x) < +\infty$ then $tx \in C$ for all $t > 0$ small enough since $g(tx) \leq tg(x)$ for all $t \in [0,1]$ yielding $z \in \mathbb{R}_+^* C$, a contradiction with the fact that $j(z) = +\infty$. Assuming that j is finite on Δ, we obtain that the function $g - (\lambda - \lambda_0)j$ is convex on Δ thus $g \leq (\lambda - \lambda_0)j$ on $C \cap \Delta$. Observing that $g \geq (\lambda - \lambda_0)j$ on $(X \setminus C) \cap \Delta$, we obtain in both cases that $g \leq (\lambda - \lambda_0)j$ on $C \cap \Delta$ and $g \geq (\lambda - \lambda_0)j$ on $(X \setminus C) \cap \Delta$. It follows that

$$(\lambda - \lambda_0)j + \beta - \lambda \leq g \text{ on } \Delta.$$

Indeed assuming that $x \in (X \setminus C) \cap \Delta$ we get $g(x) \geq (\lambda - \lambda_0)j(x)$ and $\beta - \lambda < 0$, and assuming $x \in C \cap \Delta$ one has

$$(\lambda - \lambda_0)j(x) + \beta - \lambda \leq \lambda - \lambda_0 + \beta - \lambda \leq \beta - \lambda_0 \leq g(x).$$

Setting $W = (\lambda - \lambda_0)C^\circ$, we get $\sigma_W = (\lambda - \lambda_0)\sigma_{C^\circ} = (\lambda - \lambda_0)j$. Taking the conjugate we get $h \leq i_W + \lambda - \beta$ and $\frac{1}{2\rho}B \subset C^\circ$. It follows that

$$f^* \leq \lambda - \lambda_0 + f^*(0) + \langle x_0, . \rangle \text{ on } \frac{\lambda - \lambda_0}{2\rho} B$$

which ends the proof of the lemma taking into account the fact that $x_0 \in \rho B_X$. $\square$

In the sequel, given a function $F \in \operatorname{Conv}(X \times U)$ we shall use the following assumptions

$$sB_{X \times U} \subset X \times \{0\} \cap rB_{X \times U} - [F \leq c] \cap rB_{X \times U} \tag{20}$$

for some $s > 0$, $r \geq 0$, $c \in \mathbb{R}$ and

$$s^* B_{V \times Y} \subset \{0\} \times Y \cap r^* B_{V \times Y} - [F^* \leq c^*] \cap r^* B_{V \times Y} \tag{21}$$

for some $s^* > 0$, $r^* \geq 0$, $c^* \in \mathbb{R}$. It is easy to check that (20) is equivalent to the existence of $s' > 0$, $r' \geq 0$, $c' \in \mathbb{R}$ such that

$$\text{for all } u \in s' B_U, \text{ there exists } x \in r' B_X \text{ such that } F(x,u) \leq c', \tag{22}$$

and (21) is equivalent to the existence of $s^{*\prime} > 0$, $r^{*\prime} \geq 0$, $c^{*\prime} \in \mathbb{R}$ such that

$$\text{for all } v \in s^{*\prime} B_V, \text{ there exists } y \in r^{*\prime} B_Y \text{ such that } F^*(v, y) \leq c^{*\prime}. \tag{23}$$

Here stands our main result.

Theorem 3.2. *Let X, U be normed vector spaces, let $(F_n)_{n\in\mathbb{N}}$, $F \in \Gamma_0(X \times U)$ be such that $F = \tau_S\text{-}\lim_{n\to\infty} F_n$. Assume that there exist $s > 0$, $r \geq 0$, $c \in \mathbb{R}$ with*

$$sB_{X\times U} \subset \bigcap_{n\in\mathbb{N}} \Big(X \times \{0\} \cap rB_{X\times U} - [F_n \leq c] \cap rB_{X\times U}\Big). \tag{24}$$

Assume also that there exist $\lambda > \limsup_{n\to\infty} (\inf_{x\in X} f_n(x))$ and $\rho \geq 0$ such that

$$\bigcup_{n\in\mathbb{N}} [f_n \leq \lambda] \subset \rho B_X \tag{25}$$

where $f_n = F_n(.,0)$. Then

$$F(.,0) = \tau_S\text{-}\lim_{n\to\infty} F_n(.,0), \tag{26}$$

$$F^*(0,.) = \tau_S^*\text{-}\lim_{n\to\infty} F_n^*(0,.), \tag{27}$$

$$\bar{\varphi} = \tau_S\text{-}\lim_{n\to\infty} \bar{\varphi}_n. \tag{28}$$

Proof. It is clear that (26) follows from Lemma 2.5. From Assumption (24), we can apply Theorem 2.4 yielding

$$(F_n + i_{X\times\{0\}})^*(v, 0) = \min_{(v',y')\in V\times Y} F_n^*(v - v', -y') + i_{\{0\}\times Y}(v', y') = \psi_n(v).$$

It follows that $\psi_n \in \Gamma_0(V)$ which implies that $\psi_n = f_n^*$. From assumption (25) we can apply Lemma 3.1. Hence for all $\lambda > \lambda_0 > \limsup_{n\to\infty} (\inf_{x\in X} f_n(x))$ we get

$$\psi_n(v) \leq \psi_n(0) + \frac{3}{2}(\lambda - \lambda_0) \tag{29}$$

for all $v \in s^* B_V$ with $s^* = \frac{\lambda-\lambda_0}{2\rho}$. Let $f = F(.,0)$ and let $v \in \operatorname{dom} f^*$. Using Lemma 2.5 and Proposition 1.2, there exists a sequence (v_n) in V converging to v such that $f^*(v) \geq \limsup_{n\to\infty} f_n^*(v_n)$. For all $x \in X$, one gets $f_n(x) \geq \langle x, v_n\rangle - f_n^*(v_n)$ which shows the existence of $m \in \mathbb{R}$ such that for all n large enough one has $-\inf_{x\in X} f_n(x) < m$. Returning to (29) we have, for all n large enough and for all $v \in s^* B_V$

$$\inf_{y\in Y} F_n^*(v, y) = \psi_n(v) < \lambda^*$$

where $\lambda^* = m + \frac{3}{2}(\lambda - \lambda_0)$. We claim that there exists $r^* \geq 0$ such that for all n large enough, for all $v \in s^* B_V$ there exists $y \in r^* B_Y$ with $F_n^*(v, y) < \lambda^*$. Indeed given $v \in s^* B_V$, let us introduce for all $n \in \mathbb{N}$ the function $\tilde{\phi}_n$ defined for all $u \in U$ by

$$\tilde{\phi}_n(u) = \inf_{x\in X} (F_n(x, u) - \langle x, v\rangle).$$

For all $y \in Y$ we have $\tilde{\phi}_n^*(y) = F_n^*(v, y)$. From (24) there exists for all $u \in sB_U$ an element $x \in rB_U$ such that $F_n(x, u) \leq \lambda$ which entails $\tilde{\phi}_n(u) \leq \lambda + rs^*$ on sB_X. Thus we get $\tilde{\phi}_n \leq \lambda + rs^* + i_{sB_U}$, which by duality implies $F_n^*(v, y) \geq s\|y\| - \lambda - rs^*$. Let $v \in s^*B_V$, there exists $y \in Y$ with $F_n^*(v, y) < \lambda^*$ hence $\|y\| \leq r^*$ with $r^* = \frac{\lambda+\lambda^*+rs^*}{s}$. Thus for all n large enough

$$s^*B_{V\times Y} \subset \{0\} \times Y \cap (r^* + s^*)B_{V\times Y} - [F_n^* \leq \lambda^*] \cap (r^* + s^*)B_{V\times Y}.$$

From Lemma 2.2 we derive

$$\operatorname{epi}(F^* + i_{\{0\}\times Y}) \subset \liminf_{n\to\infty} \Big(\operatorname{epi}(F_n^* + i_{\{0\}\times Y})\Big),$$

which easily leads to

$$\operatorname{epi} F^*(0, .) \subset \liminf_{n\to\infty} (\operatorname{epi} F_n^*(0, .)).$$

On the other hand Lemma 1.1 ensures that $\operatorname{epi} \bar{\varphi} \subset \liminf_{n\to\infty} \operatorname{epi} \bar{\varphi}_n$. Hence we get

$$F^*(0, .) = \tau_S\text{-}\lim_{n\to\infty} F_n^*(0, .) \text{ and } \bar{\varphi} = \tau_S\text{-}\lim_{n\to\infty} \bar{\varphi}_n$$

and the proof of the theorem is complete. □

In the particular case of the Fenchel duality, we get the following corollary.

Corollary 3.3. *Let X, U be normed vector spaces, let $A \in L(X, U)$ and let $(f_n)_{n\in\mathbb{N}}$, $f \in \Gamma_0(X)$, $(g_n)_{n\in\mathbb{N}}$, $g \in \Gamma_0(U)$ be such that*

$$\tau_S\text{-}\lim_{n\to\infty} f_n = f \text{ and } \tau_S\text{-}\lim_{n\to\infty} g_n = g.$$

Assume that there exist $s > 0$, $r \geq 0$, $c \in \mathbb{R}$ such that

$$sB_U \subset \bigcap_{n\in\mathbb{N}} \Big([g_n \leq c] \cap rB_U - A([f_n \leq c] \cap rB_X)\Big). \tag{30}$$

Assume also that there exist $\lambda > \limsup_{n\to\infty} \Big(\inf_{x\in X}(f_n(x) + g_n(A(x)))\Big)$ and $\rho \geq 0$ such that

$$\bigcup_{n\in\mathbb{N}} \Big([f_n + (g_n \circ A) \leq \lambda]\Big) \subset \rho B_X.$$

Then

$$\tau_S\text{-}\lim_{n\to\infty} (f_n + (g_n \circ A)) = f + (g \circ A)$$

and

$$\tau_S^*\text{-}\lim_{n\to\infty} (f_n^* \circ (-A^*) + g_n^*) = f^* \circ (-A^*) + g^*.$$

Proof. Setting $F(x, u) = f(x) + g(A(x) + u)$ and $F_n(x, u) = f_n(x) + g_n(A(x) + u)$, we have (see the proof of Theorem 2.6)

$$\tau_S\text{-}\lim_{n\to\infty} F_n = F.$$

Moreover one has

$$F^*(v,y) = f^*(-A^*(y)+v) + g^*(y) \text{ and } F_n^*(v,y) = f_n^*(-A^*(y)+v) + g_n^*(y),$$

thus the corollary follows from Theorem 3.2. □

4. Example of application

In this section, we apply the results of the preceding one to the convergence of the adjoint space in some singularly perturbed optimal control problem described in [1], p. 336. Let $\Omega \subset \mathbb{R}^d$ be a bounded open subset with a smooth boundary Γ such that Ω is locally on the same side of Γ. Let us set $Z = L^2(\Gamma)$, $V = H^1(\Omega)$, $H = L^2(\Omega)$, let $\epsilon > 0$ and let $a_\epsilon(.,.)$ be the continuous and elliptic bilinear form on V defined by

$$a_\epsilon(y_1,y_2) = \epsilon \int_\Omega Dy_1.Dy_2\,dx + \int_\Omega y_1 y_2\,dx.$$

Let us denote by $A_\epsilon \in L(V,V^*)$ the linear isomorphism defined for all $y \in V$ by $a_\epsilon(y,.) = A_\epsilon(y)$. Given $g \in H$ and $y \in V$, one has $A_\epsilon(y) = g$ if and only if

$$\begin{cases} -\epsilon\Delta y + y = g & \text{on } \Omega \\ \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma \end{cases} \tag{31}$$

where ν is the outward normal vector to Γ at x. The optimal control problem we are dealing with is

$$\min \frac{1}{2}\|\gamma_0 y - z_d\|_Z^2 + \frac{N}{2}\|v\|_H^2 \tag{32}$$

over $v \in H$, $y \in V$ related by (31) where $\gamma_0 y \in H^{\frac{1}{2}}(\Gamma)$ denotes the trace of y on Γ and $z_d \in Z$ is given. In order to dualize our problem, we use the general method for duality in optimal control introduced in [8]. Let us introduce the function $F_\epsilon \in \Gamma_0(Z \times H \times H)$ defined by

$$F_\epsilon(z,v,f) = \begin{cases} \frac{1}{2}\|z - z_d\|_Z^2 + \frac{N}{2}\|v\|_H^2 + i_{C_\epsilon}(y,v,f) & \text{if } z = \gamma_0 y \\ \\ +\infty & \text{otherwise} \end{cases}$$

where

$$C_\epsilon = \{(y,v,f) \in V \times H \times H : A_\epsilon(y) = v + f\}.$$

Problem (32) is equivalent to

$$\min_{(z,v)\in Z\times H} F_\epsilon((z,v),0).$$

Let us recall that given $p \in H$ such that $\Delta p \in H$ the trace $\gamma_0 p = p \mid \Gamma$ and $\gamma_1 p = \frac{\partial p}{\partial \nu} \mid \Gamma$ are defined respectively in $H^{-\frac{3}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ (see [22], Vol. 1, Théorème 6.5, p. 187-189) and that, for all $y \in H^2(\Omega)$ one has

$$-\int_\Omega p\Delta y\,dx = -\int_\Omega y\Delta p\,dx + \langle \gamma_1 p, \gamma_0 y\rangle_{H^{-\frac{3}{2}}(\Gamma),H^{\frac{3}{2}}(\Gamma)} - \langle \gamma_0 p, \gamma_1 y\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)}.$$

From the preceding Green's formula the Lagrangian associated with F_ϵ is easily shown to be defined on $Z \times H \times H$ by

$$L_\epsilon(z, v, p) =$$

$$\begin{cases} \frac{1}{2}\|z - z_d\|_Z^2 + \langle p, v\rangle + \frac{N}{2}\|v\|_H^2 - \langle \epsilon\gamma_1 p, z\rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)} - i_{D_\epsilon}(p) & \text{if } z \in \gamma_0(A^{-1}(H)) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$D_\epsilon = \{p \in H : -\epsilon\Delta p + p = 0\}.$$

It follows that a pair $((\bar{z}_\epsilon, \bar{v}_\epsilon), \bar{p}_\epsilon) \in Z \times H \times H$ is a saddle-point for L_ϵ if and only if there exists $\bar{y}_\epsilon \in V$ such that

$$\begin{gathered} \begin{cases} -\epsilon\Delta\bar{y}_\epsilon + \bar{y}_\epsilon = \bar{v}_\epsilon & \text{on } \Omega \\ \dfrac{\partial \bar{y}_\epsilon}{\partial \nu} = 0 & \text{on } \Gamma, \end{cases} \\ \bar{z}_\epsilon = \gamma_0\bar{y}_\epsilon \\ \bar{p}_\epsilon + N\bar{v}_\epsilon = 0, \\ \begin{cases} -\epsilon\Delta\bar{p}_\epsilon + \bar{p}_\epsilon = 0 & \text{on } \Omega \\ \epsilon\gamma_1\bar{p} = \bar{z}_\epsilon - z_d & \text{on } \Gamma. \end{cases} \end{gathered} \tag{33}$$

A slight adaptation of the proof given in [1], p. 338-340, shows that the sequence (F_ϵ) congerges in the Mosco sense as ϵ goes to 0 on $Z \times H$ to the function $F \in \Gamma_0(Z \times H \times H)$ defined by

$$F((z, v), f) = \frac{1}{2}\|z - z_d\|_Z^2 + \frac{N}{2}\|v + f\|_H^2.$$

The Lagrangian associated to F is defined for all $((z, v), p) \in Z \times H \times H$ by

$$L((z, v), p) = \langle p, v\rangle + \frac{1}{2}\|z - z_d\|_Z^2 - \frac{1}{2N}\|p\|_H^2.$$

The unique saddle-point of L is clearly $((z_d, 0), 0)$. The sequence of functions $(F_\epsilon(., 0))$ is clearly equicoercive thus assumption (25) of Theorem 3.2 is in force. Moreover assumption (24) is also satisfied since $F_\epsilon(0, -f, f) \leq \frac{1}{2}\|z_d\|_Z^2 + \frac{N}{2}\|f\|_H^2$ yielding

$$\sup_{\epsilon > 0} \varphi_\epsilon(f) \leq \frac{1}{2}\|z_d\|_Z^2 + \frac{N}{2}\|f\|_H^2$$

where $\varphi_\epsilon(f) = \inf_{(z,v) \in Z \times H} F_\epsilon(z, v, f)$ is the value function associated to F_ϵ; which implies (25), (see (22)). From Theorem 3.2 and from the variational properties of Mosco convergence (see [1]) we obtain that the sequence of adjoint states $(\bar{p}_\epsilon)$ converges to 0 in w- $L^2(\Omega)$. In fact one can prove, using our duality method, that the sequence $(\bar{p}_\epsilon)$

converges to 0 in s-$L^2(\Omega)$. Indeed some straightforward computations based on the generalized Green's formula quoted above show that the dual functional is defined on H by

$$F_\epsilon^*((0,0),p) = \frac{1}{2}\|\epsilon\gamma_1 p\|_Z^2 + \langle \epsilon\gamma_1 p, z_d\rangle_Z + \frac{1}{2N}\|p\|_H^2 + i_{\tilde{D}_\epsilon}(p)$$

where

$$\tilde{D}_\epsilon = \{p \in H : -\epsilon\Delta p + p = 0,\ \gamma_1 p \in Z\}.$$

As the duality gap is equal to 0, we get taking into account (33)

$$\begin{aligned}\tfrac{1}{2}\|\bar{z}_\epsilon - z_d\|_Z^2 + \tfrac{N}{2}\|\bar{v}_\epsilon\|_H^2 &= -\tfrac{1}{2}\|\epsilon\gamma_1\bar{p}_\epsilon\|_Z^2 - \langle\epsilon\gamma_1\bar{p}_\epsilon, z_d\rangle_Z - \tfrac{1}{2N}\|\bar{p}_\epsilon\|_H^2 \\ &= -\tfrac{1}{2}\|\bar{z}_\epsilon\|_Z^2 + \tfrac{1}{2}\|z_d\|_Z^2 - \tfrac{1}{2N}\|\bar{p}_\epsilon\|_H^2\end{aligned}$$

thus $(\bar{p}_\epsilon)$ converges to 0 in s-$L^2(\Omega)$ relying on the fact (see [1], p. 340) that

$$\lim_{\epsilon\to 0}\frac{1}{2}\|\bar{z}_\epsilon - z_d\|_Z^2 + \frac{N}{2}\|\bar{v}_\epsilon\|_H^2 = 0.$$

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