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Summability of the Solutions of Nonlinear Elliptic Equations With Right Hand Side Measures

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1. Introduction

A classical theorem by Guido Stampacchia states that, if $A(x) = (a_{ij}(x))$ is an elliptic bounded measurable matrix, there exists a solution (obtained by *duality*) $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(A(x)\,\nabla u) = \mu & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is a bounded measure, and Ω is a bounded open subset of $\mathbb{R}^N$.
Of course, the Stampacchia method is restricted to a linear setting. For example, consider the Dirichlet problem for the so-called p-laplacian $(2 - \frac{1}{N} < p < N)$

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u) = \mu & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Then the duality method cannot be used. Moreover the solutions of (1.2) are not, in general, critical points of the convex functional

$$\int_\Omega |\nabla v|^p \;dx - \int_\Omega v\,d\mu.$$

For instance, the minimum can be $-\infty$. We proved by approximation the existence of solutions $u \in W_0^{1,q}(\Omega), q < \frac{N(p-1)}{N-1}$ of nonlinear Dirichlet problems when the right hand side is a bounded measure.

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We begin stating the hypotheses that will hold throughout the paper.
Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$. Let p be a real number such that $1 < p \leq N$.
Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that the following holds:

$$a(x,\xi) \cdot \xi \geq \rho_1 |\xi|^p , \tag{1.3}$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where ρ_1 is a positive constant;

$$|a(x,\xi)| \leq \lambda(x) + \rho_2 |\xi|^{p-1} , \tag{1.4}$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where ρ_2 is a positive constant, and λ belongs to $L^{p'}(\Omega)$;

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0 , \tag{1.5}$$

for almost every $x \in \Omega$, for every ξ and η in $\mathbb{R}^N$, with $\xi \neq \eta$.
Let us define the differential operator

$$A(u) = -\text{div}\,(a(x,\nabla u)) .$$

Thanks to (1.3), (1.4) and (1.5), A is a monotone and coercive differential operator acting between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$; hence, it is surjective.
We look for weak solutions, i.e., for functions u such that

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u) \cdot \nabla v \, dx = \int_\Omega v \, \mu, \ \forall v \in C_0^\infty(\Omega) , \end{cases} \tag{1.6}$$

We recall the following results (see [1] or [2]).

Theorem 1.1.
There exists a solution u of (1.6), and $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N(p-1)}{N-1} = q_0$.

Remark 1.2. If $p > N$, then $\mu \in W^{-1,p'}(\Omega)$, and the existence theorem is a consequence of the classical theorems on the surjectivity of monotone operators.

Remark 1.3. We observe that $q_0 > 1$ if and only if $p > 2 - \frac{1}{N}$.

Remark 1.4. If μ is the Dirac mass, and Ω is a ball, we can check that the result of Theorem 1.1 is optimal.

If μ has a density f with respect to the Lebesgue measure, and $f \in L^1(\Omega)$, we recall the following results.

Theorem 1.5. *If $f \in L^m(\Omega)$, $1 < m < \frac{Np}{Np-p+N} = (p^*)'$, then $u \in W_0^{1,(p-1)m^*}(\Omega)$.*

Remark 1.6. If $m = (p^*)'$, then $L^m(\Omega) \subset W^{-1,p'}(\Omega)$, and $W_0^{1,(p-1)m^*}(\Omega) = W_0^{1,p}(\Omega)$.

Remark 1.7. If m tends to 1, we cannot deduce Theorem 1.1 from Theorem 1.5 (see Remark 1.4, and the counterexample in Section 2 of [3]). In order to obtain a solution in $W_0^{1,q_0}(\Omega)$, a sufficient condition is given by the following theorem.

Theorem 1.8. *If $\int_\Omega |f| \log(1+|f|)\,dx < \infty$, then there exists a solution $u \in W_0^{1,q_0}(\Omega)$. That is there exists a function u such that*

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx = \int_\Omega f\,v\,dx, \ \forall v \in C_0^\infty(\Omega)\,, \end{cases} \tag{1.7}$$

We point out that if $m \in (1,(p^*)')$ then, roughly speaking, the function "$m \to$ summability of $(u,|\nabla u|)$" is increasing. On the other hand, if $m > (p^*)'$, only the function "$m \to$ summability of u" is increasing .
Other results concerning existence, uniqueness, regularity and generalizations can be found in the papers quoted in the references.

2. Regularity results

In this section we present an improvement of the summability of the solution u of the Dirichlet problem (1.6).

Theorem 2.1. *The solution u of* (1.6) *given by* Theorem 1.1 *is such that, for every $\beta > \frac{1}{p-1}$,*

$$\frac{u}{[\log(2+|u|)]^\beta} \in L^{q_0^*}(\Omega)\,, \tag{2.1}$$

$$\frac{|\nabla u|}{[\log(2+|u|)]^\beta} \in L^{q_0}(\Omega)\,. \tag{2.2}$$

Proof. Let $\{f_k\}$ be a sequence of smooth functions converging to μ in the sense of the measures. Consider the approximate problems

$$\begin{cases} -\mathrm{div}\,(a(x,\nabla u_k)) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

that have a unique solution $u_k \in W_0^{1,p}(\Omega)$. Define, for $\beta > \frac{1}{p-1}$ and for $s \in \mathbb{R}$,

$$\psi(s) = \frac{s}{\log(2+|s|)^\beta}\,,$$

and, for $n \in \mathbb{N}$,

$$\varphi_n(s) = \begin{cases} 0 & \text{if } 0 \le s \le n, \\ s-n & \text{if } n < s < n+1, \\ 1 & \text{if } s > n+1, \\ -\varphi_n(-s) & \text{if } s < 0. \end{cases}$$

The use of $\varphi_n(u_k)$ as test function in (2.3), and the boundedness of $\{f_k\}$ in $L^1(\Omega)$, imply that

$$\int_{B_n} |\nabla u_k|^p \, dx \le c_1 , \tag{2.4}$$

where

$$B_n = \{x \in \Omega : n \le |u_k| < n+1\}$$

(observe that B_n is the set in Ω where the derivative of $\varphi_n(u_k)$ is different from zero). Then, if $\alpha > 1$,

$$\begin{aligned} \int_\Omega \frac{|\nabla u_k|^p}{(2+|u_k|)\,[\log(2+|u_k|)]^\alpha}\,dx &= \sum_{n=0}^{\infty} \int_{B_n} \frac{|\nabla u_k|^p}{(2+|u_k|)\,[\log(2+|u_k|)]^\alpha}\,dx \\ &\le c_1 \sum_{n=0}^{\infty} \frac{1}{(2+n)\,[\log(2+n)]^\alpha} \le c_2 . \end{aligned}$$

Sobolev and Hölder inequalities yield, setting $g(u_k) = (2+|u_k|)\,[\log(2+|u_k|)]^\alpha$ for the sake of simplicity,

$$\begin{aligned} c_3 \left(\int_\Omega |\psi(u_k)|^{q_0^*}\,dx \right)^{\frac{q_0}{q_0^*}} &\le \int_\Omega |\nabla \psi(u_k)|^{q_0}\,dx = \int_\Omega |\nabla u_k|^{q_0}\,|\psi'(u_k)|^{q_0}\,dx \\ &= \int_\Omega \frac{|\nabla u_k|^{q_0}}{g(u_k)^{\frac{q_0}{p}}}\,|\psi'(u_k)|^{q_0}\,g(u_k)^{\frac{q_0}{p}}\,dx \\ &\le \left(\int_\Omega \frac{|\nabla u_k|^p}{g(u_k)}\,dx \right)^{\frac{q_0}{p}} \left(\int_\Omega |\psi'(u_k)|^{\frac{pq_0}{p-q_0}}\,g(u_k)^{\frac{q_0}{p-q_0}}\,dx \right)^{1-\frac{q_0}{p}} \\ &\le c_2^{\frac{q_0}{p}} \left(\int_\Omega |\psi'(u_k)|^{\frac{pq_0}{p-q_0}}\,(2+|u_k|)^{q_0^*}\,[\log(2+|u_k|)]^{\frac{\alpha q_0}{p-q_0}}\,dx \right)^{1-\frac{q_0}{p}} . \end{aligned}$$

Now we point out that

$$\begin{aligned} |\psi'(s)|^{\frac{pq_0}{p-q_0}}\,(2+|s|)^{q_0^*}\,[\log(2+|s|)]^{\frac{\alpha q_0}{p-q_0}} &\le [\log(2+|s|)]^{\alpha q_0^* - \beta p q_0^*}\,(2+|s|)^{q_0^*} \\ &\le c_4\,(1+|\psi(s)|^{q_0^*}) , \end{aligned}$$

if $\alpha q_0^* - \beta p q_0^* \le -\beta q_0^*$; the previous inequality is verified if $\alpha \le \beta(p-1)$, and this explains the bound on β given in the statement. Thus

$$\int_\Omega |\psi(u_k)|^{q_0^*}\,dx \le c_5 , \qquad \int_\Omega |\nabla \psi(u_k)|^{q_0}\,dx \le c_6 .$$

Then the methods of [1], [2] allow us to pass to the limit in the approximate equations, and we can prove that there exists a solution u of (1.6) such that both (2.1) and (2.2) hold true. $\square$

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Leere Seite
366