Journal of Convex Analysis
Volume 3 (1996), No. 2, 245–254

# On Almost Sure Convergence of Vector Valued Pramarts and Multivalued Pramarts

**Ahmed Choukairi-Dini**
*Université Ibnou-Zohr, Faculté des Sciences,
Departement de Maths, Agadir, Marocco.*

Received January 6, 1994
Revised manuscript received June 20, 1994

Convergence of martingales, amarts, pramarts, mils, have been extentively studied in recent years by a lot of authors: Bellow [3], Blake [4], Chatterji [10], Castaing [5], Castaing and Ezzaki [8], Choukairi ([11], [12], [14]), Daures [15], Bagchi [1], Hiai and Umegaki [20], Hess [19], Egghe ([17], [18]), Millet and Sucheston [26], Lavie [22], Luu ([23], [24], [25]), Slaby [28], Derras [16], Neveu [27], Talagrand [29], and many others. New convergence results for bounded pramarts in $L^1_E$ and in the space $L^1_{wc(E)}$ of integrably bounded multifunction with convex weakly compact values are presented. The main purpose of this paper is to obtain the Linear topology convergence (introduced by Beer [2]) and Mosco-convergence of multivalued pramarts.

*Keywords* : multivalued conditional expectation, pramart, kadec-klee's norm, Mosco topology, linear topology.

*1991 Mathematics Subject Classification:* 60G42, 60G48, 60F15.

## 1. Notations, definitions and preliminaries

Throughout this paper, $(\Omega, F, P)$ denotes a probability space, $E$ a separable Banach space with the dual space $E'$, $\mathrm{cw}(E)$ the family of non-empty weakly compact convex subsets of $E$; $h$ is the Hausdorff distance.

Given $A \in \mathrm{cw}(E)$, the distance function $d(., A)$, and the support function $\delta^\star(., A)$ are defined by:

$$d(x, A) = \inf\{\|x - y\| y \in A\} \quad (x \in E)$$

$$\delta^\star(x', A) = \sup\{\langle x', x\rangle; \quad x \in A\} \quad (x' \in E').$$

Let $(A_n)$ be a sequence in $\mathrm{cw}(E)$. If the two following conditions hold:

(i) $\lim_{n\to\infty} d(x, A_n) = d(x, A) \qquad \forall x \in E$

(ii) $\lim_{n\to\infty} \delta^\star(x', A_n) = \delta^\star(x', A) \qquad \forall x' \in E'$.

We said that $(A_n)$ is convergent in the Linear-topology (see [2]) to $A$. We denote:

$$\tau_L - \lim_{n\to\infty} A_n = A.$$

ISSN 0944-6532 / $ 2.50 

**Lemma 1.1.** (Beer [2], Theorem 5.1) *On* $\mathrm{cw}(E)$, *we have:*

$$\tau_h \supset \tau_L \supset \tau_M,$$

*where $\tau_h$ is the Hausdorff topology, $\tau_L$ is the Linear topology, and $\tau_M$ is the Mosco topology.*

Given a sub-$\sigma$-field $B$ of $F$, and a $F$-measurable multifunction $X$ with values in $\mathrm{cw}(E)$ such that the function $d(o, F(.)) \in L^1(R)$; Hiai and Umegaki [20] showed the existence of $F$-measurable multifunction $E^B(X)$ such that

$$S^1_{E^B(X)}(B) = \mathrm{cl}\{E^B(f); \quad f \in S^1_X(F)\}$$

where $S^1_X(F)$ denotes the set of all $F$-measurable selections of $X$ and $E^B(f)$ is the usual $B$-conditional expectation of $f$ the closure being taken in $L^1_E$. But $S^1_X(F)$ is a convex weakly compact set (see [21]), and $E^B$ is a linear continuous operator, so we have:

$$S^1_{E^B(X)}(B) = \{E^B(f); \quad f \in S^1_X(F)\}.$$

Let $(F_n)$ be an increasing sequence of sub-$\sigma$-fields, of $F$, such that $F = \sigma(\bigcup_n F_n)$. A sequence $(X_n)$ of measurable multifunctions with values in $cW(E)$ is said to be *adapted* to $(F_n)$ if, for any $n$, $X_n$ is $F_n$-measurable. Let $T$ be the set of all bounded stopping times. Let $\tau \in T$, we define the tribe $F_\tau = \{A \in F / A \cap [\tau - n] \in F_n \forall n\}$ where $[\tau = n] = \{\omega \in \Omega, \tau(\omega) = n\}$ and the multifunction $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, then $(X_\tau)_{\tau \in T}$ is adapted to $(F_\tau)$.

**Definition 1.2.** Let $(X_n, F_n)_{n \geq 1}$ be an adapted sequence. We say that $(X_n)n \geq 1$ is an *amart in probability* (shortly pramart) if for every $\epsilon > 0$, there is $\sigma_0 \in T$ such that:

$$\forall \sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_0 \Rightarrow P(\{\|E^{F_\sigma}(X_\tau) - X_\sigma\| > \epsilon) \leq \epsilon.$$

If $E = R$, we say that $(X_n)$ is a *subpramart*, if for every $\epsilon > 0$, there is $\sigma_0 \in T$ such that:

$$\forall \sigma, \tau \in T\colon \quad \tau \geq \sigma \geq \sigma_0 \Rightarrow P(\{X_\sigma - E^{F_\sigma}(X_\tau) > \epsilon\}) \leq \epsilon.$$

Now, let $(X^m_n, F_n)_{n \geq 1} (m \in N^\star)$ be a sequence of real subpramarts. It is called a *uniform sequence of subpramarts* if for every $\epsilon > 0$, there is $\sigma_0 \in T$ such that:

$$\forall \sigma, \tau \in T\colon \quad \tau \geq \sigma \geq \sigma_0 \Rightarrow P(\{\sup_m (X^m_\sigma - E^{F_\sigma}(X^m_t)) > \epsilon\}) \leq \epsilon.$$

1) Every martingale is a pramart, and every uniform amart is a pramart.
2) Every sequence of submartingales is abviously an example of uniform sequence of subpramarts.

**Theorem 1.3.** (Kadec-Klee) *Let $E$ be a separable Banach space. Then on $E$ there exists an equivalent norm $|||.|||$ and a countable norming set $DCE'$ such that if $\langle x', x_n \langle \to \langle x', x \rangle$ for each $x' \in D$ and if $|||x_n||| \to |||x|||$ then $x_n \to x$ in $E$.*

**Proof.** See ([18], Theorem II.2.4.4. p. 45).

**Lemma 1.4.** (Egghe [18]. Lemma VIII.1.15. p. 298) *Let $(X_n^m, F_n)_{n\geq 1}$ be a uniform sequence of positive real subpramarts. Suppose that there is a subsequence $(n_k)k \geq 1$ such that:*

$$\sup_k \int \sup_m X_{n_k}^m < \infty,$$

*then each subpramart $(X_n^m, F_n)_{n\geq 1}$ converges a.e. to an integrable function $X^m$ and we have:*

$$\lim_{n\to+\infty} \sup_m X_n^m = \sup_m X_\infty^m a.e.$$

This lemma is a very important result. It will be used in proof of the main result of this paper (see Theorem 3.3).

## 2. Convergence results

Now, we introduce a class of subsets of $E$, by putting

$$R = \{C \in \mathrm{cf}(E)/C \cap B(0,r) \in \mathrm{cw}(E), \qquad \forall r > 0\}$$

where $B(0,r)$ denotes the closed ball of radius $r$ and $\mathrm{cf}(E)$ the family non-empty closed convex subsets of $E$ (see [19]).
It is clear that $R$ contains the members of $\mathrm{cw}(E)$ which are weakly locally compact. If $E$ is reflexive we have $R = \mathrm{cf}(E)$.

**Theorem 2.1.** *Let $E$ be a separable Banach space. Let $(X_n)$ be a pramart with values in $E$ which satisfies the following two conditions:*

(i) $\sup_{\tau\in T} \int |X_\tau| < \infty$ *(i.e. $(X_n)$ of class $(B)$).*
(ii) *There exists a $F$-measurable multifunction $L$ with values in $R$ such that $X_n(\omega) \in L(\omega)$ for every $n \geq 1$, and every $\omega \in \Omega$.*

*Then $(X_n)$ converges strongly a.e.*

**Proof.** Using a result of Millet and Sucheston (see [18], Theorem VII.2.8) and reduction Lemma (see Castaing [5], Lemma 2.2), we can suppose that:

$$g = \sup_n |X_n| \in L^1_R(F),$$

so, we have:

$$X_n(\omega) \in L(\omega) \cap g(\omega)B \qquad \forall\omega \in \Omega, \forall n \geq 1.$$

$B$ denotes the closed unit ball.
Further let us define the multifunction:

$$\Gamma(\omega) = L(\omega) \cap g(\omega)B.$$

Then $\Gamma$ is an $F$-measurable integrably bounded multifunction with values in $cW(E)$, and the set $S^1_\Gamma(F)$ is convex and $\sigma(L^1_E, L^\infty_{E'_s})$-compact (see Klei and Assani [21]). $E'_s$ is the weak dual.
Since $X_n \in S^1_\Gamma(F)$, there is a subsequence $(X_{n_k})_{k\geq 1}$ which converges to $X_\infty \in L^1_E(F)$ with $\sigma(L^1_E, L^\infty_{E'_s})$-topology.
Now, let $D$ be a countable norming set in $E'$ such in Theorem 1.3 and $x' \in D$. Since $(\langle x', X_n\rangle, F_n)_{n\geq 1}$ is an $L^1$-bounded real pramart, it converges to $\langle x', X_\infty\rangle$ almost surely: this can be viewed as a consequence of Hess's Lemma ([19], Lemma 5.2). This Lemma simply has to be particularized to the single-valued case.
Applying Lemma 1.4 to the uniform sequence of positive real subpramarts $(|\langle x', X_n\rangle|, F_n)_{n\geq 1}$ $(x' \in D)$ we obtain:

$$\lim_{n\to\infty} \sup_{x'\in D} |\langle x', X_n(\omega)\rangle| = \sup_{x'\in D} |\langle x', X_\infty(\omega)\rangle|.$$

Therefore:

$$\lim_{n\to\infty} |||X_n(\omega)||| = |||X_\infty(\omega)||| \quad \text{a.e.}$$

And Theorem 1.3 gives the desired conclusion.

**Remark 2.2.** In [26], assuming that $E$ has Radon-Nikodym property, Millet and sucheston proved variants of Theorem 2.1.

**Remark 2.3.** Using a vector version of Brooks and Chacon's result (see Castaing and Clauzure [7]), we can extend Theorem 2.1 to $L^1$-bounded pramarts without class $(B)$. The next result concerns this.

**Theorem 2.4.** *Let $E$ be a separable Banach space. Let $(X_n)$ be a pramart with values in $E$ which satisfies the following conditions:*

(i) $\sup_n \int |X_n| < \infty$,

(ii) *there exists an $F$-measurable multifunction $L$ with values in $R$ such that $X_n(\omega) \in L(\omega)$ for every $n \geq 1$, and every $\omega \in \Omega$.*

*Then $(X_n)$ converges strongly a.e.*

**Proof.** First observe that for every $x' \in E'$, the real pramart $(\langle x', X_n\rangle)$ converges a.e. to an integrable function. Now applying Theorem 3.3 in [7], we can find a decreasing sequence $(B_p)p \geq 1$ in $F$ such that $P(\bigcap_p B_p) = 0$, a subsequence $(X_{n_k})k \geq 1$ of $(X_n)$ and $X_\infty \in L^1_E(F)$ such that $(X_{n_k} \mid B^c_p)_k$ is uniformly integrable, and

$$\lim_{k\to\infty} \int_A \langle u(\omega), X_{n_k}(\omega)\rangle dP(\omega) = \int_A \langle u(\omega), X_\infty(\omega)\rangle dP(\omega)$$

for any $u \in L^\infty_{E'_s}(F)$ and $A \in B^c_p \cap F$. Where $B^c_p$ denotes the complementary of $B_p$.
For each $p$, the sequence $(X_{n_k} \mid B^c_p)_k$ is uniformly integrable; hence for every $x' \in E'$, for every $A \in B_p \cap F$ we have:

$$\lim_k \int_A \langle x', X_{n_k}\rangle = \int_A \lim_k \langle x', X_{n_k}\rangle = \int_A \langle x', X_\infty\rangle.$$

Then for every $x' \in D$, every $p$, there is a negligible set $N_{p,x'}$ in $B_p^c$ such that for $\omega \in B_p^c \mid N_{p,x'}$

$$\lim_k \langle x', X_{n_k} \rangle = \langle x', X_\infty \rangle.$$

Since $\lim_p P(B_p^c) = 1$, for fixed $x' \in D$, we have

$$\lim_n \langle x', X_n \rangle = \langle x', X_\infty \rangle \quad \text{a.s.}$$

thus, we can finish as in the proof of Theorem 2.1. □

**Remark 2.5.** In ([5], Theorem 3.5) Castaing proved results similar to ours, when $(X_n)$ is a martingale.

## 3. The main result

In this section we shall present a convergence result for multivalued pramarts with values in $\mathrm{cw}(E)$.

**Definition 3.1.** Let $(X_n)$ be an adapted sequence of multifunctions. We say that $(X_n)$ is *a pramart* (resp. *w-pramart*) if for every $\epsilon > 0$, there is $\sigma_0 \in T$ such that:

$$\forall \sigma, \tau \in T \qquad \tau \geq \sigma \geq \sigma_0 \ \Rightarrow\ P(\{h(E^{F_\sigma}(X_\tau), X_\sigma) > \epsilon\}) \leq \epsilon$$

(resp. $(\delta^\star(x', X_n))_{n \geq 1}$ is a real pramart $\forall x' \in E'$).

We begin by simple Lemma

**Lemma 3.2.** *Let $F \neq \emptyset$, $H \subset F$, and $f\colon F \to R$ such that there exists $x_0 \in H$ with $f(x_0) = 0$. Then $\sup_{x \in H} f(x) = \sup_{x \in H} f^+(x)$ where $f^+ = \sup(f, 0)$.*

The next Theorem is the main result of this paper. It concerns the $\tau_L$-convergence and Mosco-convergence of multivalued pramarts.

**Theorem 3.3.** *Let $E$ be a separable Banach space. Let $(X_n, F_n)_{n \geq 1}$ an adapted sequence with values in $\mathrm{cw}(E)$. Suppose that:*

$$\overline{\mathrm{co}}\Big(\bigcup_n X_n(\omega)\Big) \in \mathrm{cw}(E). \quad a.e.$$

1) *If $(X_n)$ is a w-pramart such that $\sup_n \int |\delta^\star(x', X_n)| < \infty$ for every $x' \in E'$. Then there exist a measurable multifunction $X_\infty\colon \Omega \to \mathrm{cw}(E)$ and a negligible subset $N$ of $\Omega$ such that:*

$$\lim_{n \to \infty} \delta^\star(x', X_n(\omega)) = \delta^\star(x', X_\infty(\omega)) \qquad \forall \omega \in \Omega \mid N; \quad \forall x' \in E'.$$

2) *If $(X_n)$ is a pramart such that $\sup_n \int d(0, X_n) < \infty$, then $S^1_{X_\infty}(F) = \emptyset$, and*

$$\tau_L - \lim_{n \to \infty} X_n(\omega) = X_\infty(\omega) \quad a.e.$$

*In particular, this yields:*

$$M - \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega) \quad a.e.$$

3) *If the hypothesis in* 2) *is replaced by the stronger one:*

$$\sup_n \int |X_n| < \infty$$

*then* $X_\infty$ *is integrably bounded.*

**Proof.** 1) We denote by $D_1^\star$ a countable subset which is dense for the Mackey topology in the closed unit ball $B^\star$ of $E'$.
$D^\star$ will denote the set of all rational linear combinations of members of $D_1^\star$. It is clear that $D^\star$ is a countable dense subset of $E'$ for the Mackey topology.

Suppose that $(X_n)$ is a $w$-pramart such that $\sup_n \int |\delta^\star(x', X_n)| < \infty$ for every $x' \in D^\star$. Since $(\delta^\star(x', X_n))_{n\geq 1}$ is a bounded real pramart, we deduce that it admits a limit. Therefore, the Hess's Lemma ([19], Lemma 5.2) shows that there is a measurable multifunction $X_\infty\colon \Omega \to \mathrm{cw}(E)$ and a negligible subset $N_1$ such that:

$$\lim_{n\to\infty} \delta^\star(x', X_n(\omega)) = \delta^\star(x', X_\infty(\omega)) \qquad \forall x' \in E', \quad \forall \omega \in \Omega \mid N_1.$$

2) Now suppose that $(X_n)$ is a pramart such that $\sup_n \int d(0, X_n) < \infty$. Let $D$ be a countable subset which is dense for the norm topology in $E$, and fix $x \in D$.
For every $x' \in E'$, $n \geq 1$, $\omega \in \Omega$, we put:

$$\varphi_n(\omega, x') = \langle x', x\rangle - \delta^\star(x', X_n(\omega)).$$

Let us prove that $((\varphi_n^+(., x'), F_n)_{n\geq 1, x'\in D^\star}$ is a uniform sequence of positive real subpramarts (see Definition 1.2), where $\varphi_n^+ = \sup(\varphi_n, 0)$.
Given $\tau, \sigma \in T$ with $\tau \geq \sigma$. By Jensen's inequality there exists a negligible subset $N_{x',x}$ such that:

$$|E^{F_\sigma}(\varphi_\tau)(\omega, x')| \leq E^{F_\sigma}(|\varphi_\tau|(\omega, x')) \qquad \forall \omega \in \Omega | N_{x',x}.$$

Note that $N_x = \bigcup_{x'\in D_1^\star} N_{x',x}$, then for every $\omega \in \Omega \mid N_x$:

$$\begin{aligned}
\varphi_\sigma^+ - E^{F_\sigma}(\varphi_\tau^+) &= \frac{1}{2}[\varphi_\sigma + |\varphi_\sigma| - E^{F_\sigma}(\varphi_\tau) - E^{F_\sigma}(|\varphi_\tau|)] \\
&= \frac{1}{2}[\varphi_\sigma - E^{F_\sigma}(\varphi_\tau) + |\varphi_\sigma| - E^{F_\sigma}(|\varphi_\tau|)] \\
&\leq \frac{1}{2}[\varphi_\sigma - E^{F_\sigma}(\varphi_\tau) + |\varphi_\sigma - E^{F_\sigma}(\varphi_\tau)] \\
&= [\varphi_\sigma - E^{F_\sigma}(\varphi_\tau)]^+.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{x'\in D_1^\star} &[\varphi_\sigma^+(\omega,x') - E^{F_\sigma}(\varphi_\tau^+)(\omega,x')] \\
&\le \sup_{x'\in D_1^\star} [\delta^\star(x',E^{F_\sigma}(X_\tau)(\omega)) - \delta^\star(x',X_\sigma(\omega))]^+ \\
&\le h(X_\sigma(\omega),E^{F_\sigma}(X_\tau)(\omega)).
\end{aligned}$$

Since $(X_n)$ is a pramart, there exists $\sigma_0 \in T$ such that:

$$\forall \tau,\sigma \in T\colon \tau \ge \sigma \ge \sigma_0$$

we have:

$$P(\sup_{x'\in D_1^\star} [(\varphi_\sigma^+(.,x') - E^{F_\sigma}(\varphi_\tau^+)(.,x')] \ge \epsilon)) \ \le \ \epsilon$$

for every $\epsilon > 0$.

On the other hand, by 1) we have:

$$\lim_{n\to\infty} \varphi_n^+(\omega,x') = \varphi^+(\omega,x')$$

where $\varphi(\omega,x') = \langle x,x'\rangle - \delta^\star(x',X_\infty(\omega))$.

Applying Egghe's Lemma (see Lemma 1.4) to the uniform sequence of positive real sub-pramarts $((\varphi_n^+(.,x'),F_n)n \ge 1, x' \in D_1^\star$ we obtain for every $\omega \in \Omega \mid N_1 \cup N_x$

$$\lim_{n\to\infty} \sup_{x'\in D_1^\star} \varphi_n^+(\omega,x') = \sup_{x'\in D_1^\star} \varphi^+(\omega,x').$$

If we put $N = N_1 \cup \left[\bigcup_{x\in D} N_x\right]$, Lemma 3.2 show that:

$$\begin{aligned}
\lim_{n\to\infty} d(x,X_n(\omega) &= \lim_{n\to\infty} \sup_{x'\in D_1^\star} \varphi_n(\omega,x') \\
&= \lim_{n\to\infty} \sup_{x'\in D_1^\star} \varphi_n^+(\omega,x') \\
&= \sup_{x'\in D_1^\star} \varphi^+(\omega,x') \\
&= \sup_{x'\in D_1^\star} \varphi(\omega,x') = d(x,X_\infty(\omega))
\end{aligned}$$

for every $x \in D$, and $\omega \in \Omega \mid N$.

But the sequence $(d(.,X_n(\omega)))_{n\ge 1}$ being equicontinuous, for each $\omega \in \Omega \mid N$, we deduce that:

$$\lim_{n\to\infty} d(x,X_n(\omega)) = d(x,X_\infty(\omega))$$

remains valid for any $x \in E$.

So

$$\tau_L - \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega) \quad \text{a.e.}$$

In particular, this yields:

$$M - \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega) \quad \text{a.e.}$$

On the other hand, Fatou's Lemma shows that:

$$\int d(0, X_\infty) \le \lim_{n\to\infty} \int d(0, X_n) \le \sup \int d(0, X_n) < \infty.$$

So $S^1_{X_\infty}(F) \neq \emptyset$.

3) is an obvious consequence of Hess's Lemma ([19], Lemma 5.2). □

**Remark 3.4.** .Consider a subset $K$ of $\mathrm{cw}(E)$ which is separable for the topology generated by the Hausdorff distance. If the values of pramart $(X_n)$ lie in $K$, then Bagchi [1] proved that:

$$\lim_{n\to\infty} h(X_n(\omega), X_\infty(\omega)) = 0 \quad \text{a.e.}$$

**Remark 3.5.** Since every martingale (and uniform amart) is a pramart, Theorem 3.3 generalises some results which were obtained before by Choukairi ([11], Theorem 2.11 and [12] Theorem 4.3) Lavie ([22], Theorem 4.1) and Hess ([19], Prop. 5.7).

**Remark 3.6.** Now, it is natural to ask the following questions:

1) Is it possible to prove a version of Theorem 3.3 for unbounded pramart, or multivalued mil (martingale in limit)?
2) We know that any multivalued martingale $(F_n)$ has a martingale selection $(f_n)$ (i.e. $f_n(\omega) \in F_n(\omega)$ a.e.).
   Now, it is natural to ask the same question with multivalued pramart and mil.

Many of this problems were formulated by Professor C. Castaing during his visit to the University of Sciences at Marrakech.

**Acknowledgment.** The author would like to express many thanks to Prof. C. Castaing for suggesting the problem in Section 3. He is also grateful to the referee.

## References

[1] S. Bagchi: On a.s. convergence of class of multivalued asymptotic martingales. Ann. Inst. H. Poincaré, t. 21. No. 4 (1985).

[2] G. Beer: Support and distance functionals for convex sets. Numer. Funct. Analy. and Optimis. 10 (162) 15–30 (1989).

[3] A. Bellow: Uniform amarts: a class of asymptotic martingales for which strong almost sure convergence obtains. Z. Wahrsch. verw. Geb. 41, 177–191 (1978).

[4] L.H. Blake: A generalisation of martingales and consequent convergence theorems. Pacifis J. Math. 35 (1970) 279–283.

[5] C. Castaing: Quelques résultats de convergence des suites adaptées. Séminaire d'Analyse Convexe. Montpellier, Expose no. 2 (1987).

[6] C. Castaing: Sur la décomposition de Slaby, applications aux problèmes de convergence en probabilites, èconmie mathématiques. Théorie du contrôle Minimisation. S.A.C. Montpellier. Expose no. 3 (1989).

[7] C. Castaing, P. Clauzure: Version vectorielle et multivoguè d'un résultat de Brooks-Chacon. S.A.C. Montpellier. Exposé no. 4 (1988).

[8] C. Castaing, F. Ezzaki: Some convergence results for multivalued martingales in the limit. S.A.C. Montpellier. Exposé no. 1 (1990).

[9] C. Castaing, M. Valadier: Convex Analysis and measurable multifunctions. Lecture notes in Maths 580. Springer, 1977.

[10] S.D. Chatterji: Martingale convergence and the Radon-Nikodym theorem in Banach space. Math. SCand. 22, 21–41.

[11] A. Choukairi-Dini: $M$-Convergence et régularité des martingales multivoques. Epi-martingale. S.A.C. Montpellier (1987) et Journal of Multivariate Analysis. Vol. 33. No. 1 April (1990).

[12] A. Choukairi-Dini: $M$-Convergence des martingales. Application aux amarts uniformes et aux intégrandes (II). S.A.C. Montpellier (1989). Exposé no. 7. Et, C.R.A.S. Paris. t. 309, Série I (1989) 889–892.

[13] A. Choukairi-Dini: Sur la convergence d'esperance conditionnelle de multi applications. T.S.A.C. Montpellier. Exposé no. 10 (1987).

[14] A. Choukairi-Dini: $B$-Convergence des suites adaptées et ensembles de Radon-Nikodym (à paraître) C.R.A.S. Paris, 1994.

[15] J.P. Daures: Version multivoque du théorème de Doob. Ann. Inst. H. Poincaré 9, no. 2, 167–176.

[16] A. Derras: Thèse. Montpellier, 1990.

[17] L. Egghe: Strong convergence of pramarts in Banach spaces. Can. J. Math. 33, no. 2, 357–361.

[18] L. Egghe: Stopping times techniques in Analysis. Cambridge University Press. Cambridge (1984). London Math. Soc. Lecture Notes ser. Vol. 100.

[19] C. Hess: On multivalued martingales whose values may be unbounded: martingale selectors and Mosco-convergence. S.A.C. Montpellier. Expose no. 8 (1989) et Journal of Multivariate. Analysis vol. 39, no. 1, Octobre (1991).

[20] F. Hiai, H. Umegaki: Integrales, conditional expectations and martingales of multivalued functions. J. of Multivariate. Analysis 7 (1977) 149–182.

[21] H.A. Klei, I. Assani: Parties décomposables de $L^1_E$. C.R.A.S. Paris. t. 294, Série I.P. (1982) 533–536.

[22] M. Lavie: Fonction caractéristique d'un ensemble alèatoire. Applications a la convergence de martingales et de sommes d'ensembles aléatoires. C.R.A.S. Paris. t. 315. Série I. (1992) 399–402.

[23] D.Q. Luu: On convergence of multivalued asymptotic martingales. S.A.C. Montpellier.

Expose no. 5 (1984).

[24] D.Q. Luu: Représentation of multivalued (regular) uniform amarts. S.A.C. Montpellier. Expose no. 9 (1984).

[25] D.Q. Luu: Applications of set-valued Radon-Nikodym theorems to convergence of multivalued $L^1$-amarts. Math. Scand. 5.4. (1984) 101–113.

[26] A. Millet, L. Sucheston: Convergence of class of amarts indexed by directed sets. Can. J. Math. 31 (1979) 1033–1046.

[27] J. Neveu: Convergence presque sûre des martingales multivoques. Ann. Inst. H-Poincaré. Sect. B 8.1–7.

[28] M. Slaby: Strong convergence of vector valued pramarts and subpramarts. Probability and Math. Statistics. vol. 5, Fas. 2 (1985).

[29] M. Talagrand: Some structure results for martingales in the limit and pramarts. The annals of probability. vol. 13, no. 4 (1985).