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The Graves Theorem Revisited[1]

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Dedicated to R. T. Rockafellar on his 60th Birthday

In 1950 Graves proved the following theorem: If the function f from a Banach space X into a Banach space Y is strictly differentiable at x_0 and the strict derivative $\nabla f(x_0)$ is onto, then f is open with linear rate around x_0. Under fairly general assumptions, the latter property is equivalent to either the metric regularity or to the Aubin property of the inverse. In this paper, we prove that the Graves theorem is a consequence of the following general result: the openness with linear rate of a locally closed set-valued map F around a point (x_0, y_0) of its graph is invariant with respect to a perturbation of the form $f + F$ provided that the strict derivative of f at x_0 is zero.

In 1934 L. A. Lyusternik [19] published the following fundamental geometric result: if a function f from Banach space X into a Banach space Y is Fréchet differentiable near x_0, its derivative ∇f is continuous at x_0, and $\nabla f(x_0)$ is onto, then the tangent manifold to $f^{-1}(0)$ at x_0 is exactly $x_0+\mathrm{Ker}\nabla f(x_0)$. That is, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{dist}(x, f^{-1}(0)) \le \epsilon\|x - x_0\|$ whenever $x \in (x_0+\mathrm{Ker}\nabla f(x_0))$ and $\|x - x_0\| \le \delta$. Based on this result Lyusternik obtained an abstract version of the Lagrange multiplier rule. In the 60s and 70s the Lyusternik theorem was extended to a very general framework, for a survey see Dmitruk et al. [6].

In 1950 L. M. Graves [12] obtained an open mapping theorem for nonlinear mappings. As we will see below, this result is a generalization of the original version of the Lyusternik theorem.

Let us recall that a map f from a topological space X to a topological space Y is *open at* x if for every open $V \subset X$ with $x \in V$ there exists an open $W \subset Y$ with $f(x) \in W$ such that $W \subset f(V)$. The following version of the Banach open mapping principle can be extracted from the literature, see e.g. [2] or [27].

Theorem 1.1. (Banach open mapping theorem). *Let X and Y be Banach spaces and $A : X \to Y$ be a linear and continuous map. Then the following are equivalent:*

(i) $A(X) = Y$;

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(ii) *A is open (at every point);*
(iii) *There exists a constant M such that for every $y \in Y$ there exists $x \in X$ with $y = A(x)$ and*

$$\| x \| \leq M \| y \| .$$

Let us denote by $B_a(x)$ the closed ball centered at x with radius a. Up to some minor adjustments in notation, the original formulation and proof of the Graves theorem are as follows.

Theorem 1.2. (Graves [12]). *Let X, Y be Banach spaces and let f be a continuous function from X to Y defined in $B_\epsilon(0)$ for some $\epsilon > 0$ with $f(0) = 0$. Let A be a continuous and linear operator from X onto Y and let M be the corresponding constant from* Theorem 1.1 (iii). *Suppose that there exists a constant $\delta < M^{-1}$ such that*

$$\| f(x_1) - f(x_2) - A(x_1 - x_2) \| \leq \delta \| x_1 - x_2 \| \tag{1}$$

whenever $x_1, x_2 \in B_\epsilon(0)$. Then the equation $y = f(x)$ has a solution $x \in B_\epsilon(0)$ whenever $\| y \| \leq c\epsilon$, where $c = M^{-1} - \delta$.

Proof. A sequence x_n is constructed convergent to x in the following way: take $y \in Y$, $\| y \| \leq c\epsilon$, and let $x_0 = 0$. Since A is surjective, by condition (iii) in Theorem 1.1 there exists $x_1 \in X$ such that

$$A(x_1) = y \text{ and } \| x_1 \| \leq M \| y \| \leq \epsilon.$$

Suppose that for $n > 1$ we are given $x_i, i = 1, \cdots, n-1$, satisfying

$$A(x_i) = y - f(x_{i-1}) + A(x_{i-1}) \text{ and } \| x_i - x_{i-1} \| \leq M(M\delta)^{i-1} \| y \| .$$

Then

$$\begin{aligned} \| x_i \| &\leq \sum_{j=1}^{i} \| x_j - x_{j-1} \| \\ &\leq M \| y \| \sum_{j=1}^{i} (M\delta)^{j-1} \leq M \| y \| /(1 - M\delta) = \|y\|/c \leq \epsilon. \end{aligned}$$

By Theorem 1.1 one can choose x_n such that

$$A(x_n) = y - f(x_{n-1}) + A(x_{n-1}) \tag{2}$$

and

$$\| x_n - x_{n-1} \| \leq M \| y - f(x_{n-1}) \| .$$

Since $y = A(x_{n-1}) - A(x_{n-2}) + f(x_{n-2})$, from (1) we have

$$\| x_n - x_{n-1} \| \leq M\delta \| x_{n-1} - x_{n-2} \| .$$

Hence,

$$\| x_n - x_{n-1} \| \leq M(M\delta)^{n-1} \| y \| .$$

The induction step is complete. Thus x_n is a Cauchy sequence, hence convergent to some x with $\| x \| \leq \epsilon$. Passing to the limit in (2) with $n \to \infty$ we obtain $y = f(x)$. □

Observe that Graves does not assume differentiability of the map f. If we suppose that *for every* $\delta > 0$ there exists $\epsilon > 0$ such that (1) is satisfied for every $x_1, x_2 \in B_\epsilon(0)$, then A is the *strict derivative* of f at 0, $A = \nabla f(0)$, formally introduced in 1961 by Leach [17] (for a discussion see Nijenhuis [21]). Note that the strict differentiability of f at x_0 implies that f is continuous (even Lipschitz) in a neighborhood of x_0. Then the Graves theorem can be stated as follows: if f is strictly differentiable at 0 and its derivative is onto, then there exists $\epsilon > 0$ and $c > 0$ such that for every $y \in Y$ with $\| y \| \leq c\epsilon$ there exists a $x \in X$ such that $\| x \| \leq \epsilon$ and $y = f(x)$. In other words, $B_{c\epsilon}(0) \subset f(B_\epsilon(0))$.

It is clear that the assumption $f(0) = 0$ is superfluous. Let $x_0 \in X$, let f be strictly differentiable at x_0 and let the strict derivative $\nabla f(x_0)$ be onto. Let $\delta > 0$ be so small that $\delta M < 1$ where M is the constant in condition (iii) of Theorem 1.1 for the strict derivative $\nabla f(x_0)$. Then there exists an open neighborhood U of x_0 such that (1) holds whenever $x_1, x_2 \in U$. Let $\bar{x} \in U$ and let $\epsilon > 0$ be such that $B_\epsilon(\bar{x}) \subset U$. If $\tilde{f}(x) = f(x + \bar{x}) - f(\bar{x})$, then $\tilde{f}(0) = 0$ and $\tilde{f}$ satisfies (1) in $B_\epsilon(0)$. Applying Theorem 1.2 we obtain that $B_{c\epsilon}(0) \subset \tilde{f}(B_\epsilon(0))$, where $c = M^{-1} - \delta$. Thus $B_{c\epsilon}(f(\bar{x})) \subset f(B_\epsilon(\bar{x}))$ and we obtain the following result which we call the *Graves theorem*:

Theorem 1.3. *Let X and Y be Banach spaces, let $x_0 \in X$ and let $f : X \to Y$ be a function which is strictly differentiable at x_0. Suppose that $\nabla f(x_0)$ is onto. Then there exist a neighborhood U of x_0 and a constant $c > 0$ such that for every $x \in U$ and $\tau > 0$ with $B_\tau(x) \subset U$,*

$$B_{c\tau}(f(x)) \subset f(B_\tau(x)). \tag{3}$$

The property (3) is stronger than the openness at a point; it is *uniform* with respect to x in a neighborhood of x_0 with the same constant c determining a proportion between the sizes of the neighborhoods of x and $f(x)$. This property is called *openness with linear rate around a point* or *covering in a neighborhood.* Note that for linear maps the openness at a point and the openness with linear rate around any point coincide. Thus the Graves theorem generalizes the implication (i) $\Rightarrow$ (ii) in the Banach open mapping theorem (Theorem 1.1) in two ways: first, it gives a sufficient condition for openness of a nonlinear function and second, it shows that the surjectivity of the strict derivative actually implies openness with linear rate *around* the reference point[2]. In further lines we prove a symmetric open mapping theorem for set-valued maps from which follows a complete analogue of the Banach open mapping theorem for nonlinear maps: a function f, strictly differentiable at x_0, is open with linear rate around x_0 if and only if $\nabla f(x_0)$ is onto.

The original version of the Lyusternik theorem can be derived from Theorem 1.3. Let $f(0) = 0$ and $\nabla f(0)X = Y$, and let the neighborhood U of $x_0 = 0$ and the constant c be as in Theorem 1.3. Let $\epsilon > 0$. There exists $\delta > 0$ such that $B_{\epsilon\|x\|}(x) \subset U$ for every $x \in B_\delta(0)$ and, moreover, $\|f(x)\| \leq c\epsilon\|x\|$ for every $x \in B_\delta(0) \cap \mathrm{Ker}\nabla f(0)$. Taking any $x \in B_\delta(0) \cap \mathrm{Ker}\nabla f(0)$ and applying Theorem 1.3 with $\tau = \epsilon\|x\|$ we obtain that there exists $\tilde{x}$ with $f(\tilde{x}) = 0$ and such that $\|\tilde{x} - x\| \leq \epsilon\|x\|$. Hence $\mathrm{dist}(x, f^{-1}(0)) \leq \epsilon\|x\|$ for any $x \in B_\delta(0) \cap \mathrm{Ker}\nabla f(0)$; that is, $\mathrm{Ker}\nabla f(0)$ is tangent to $f^{-1}(0)$ at 0. We should

[2] Our discussion here clearly contradicts the opinion of the authors of [6] who claim that Graves proved openness at a point.

note, however, that the iteration (2) in the proof of Graves is similar to the one used by Lyusternik.

Generally, a set-valued map F from a metric space X to subsets of a metric space Y is open with linear rate around a point $(x_0, y_0) \in \operatorname{graph} F$ with a constant c if there exist positive numbers a, b, ρ and c such that for any $x \in B_a(x_0), y \in B_b(y_0)$, with $y \in F(x)$, and $\tau \in [0, \rho]$,

$$B_{c\tau}(y) \subset F(B_\tau(x)).$$

Here we denote by graph F the set $\{(x, y) : x \in X, y \in F(x)\}$. Dmitruk et al. [6] noted that the openness with linear rate for functions is equivalent to another basic property in nonlinear analysis, known as *distance estimate* or *metric regularity.* A map $F : X \to Y$ is metrically regular around (x_0, y_0) with $(x_0, y_0) \in \operatorname{graph} F$, if there exist positive numbers α, β, ϵ and c such that

$$\operatorname{dist}(x, F^{-1}(y)) \leq c \operatorname{dist}(y, F(x))$$

whenever $x \in B_\alpha(x_0), y \in B_\beta(y_0), \operatorname{dist}(y, F(x)) \leq \epsilon$. Let us recall that the inverse of a map F is defined as $F^{-1}(y) = \{x \in X : y \in F(x)\}$.

The following definition was introduced by Aubin [1]: A set-valued map F from Y to the subsets of X is *pseudo - Lipschitz* around $(y_0, x_0) \in \operatorname{graph} F$ with modulus M if there exist constants a and b such that for every $y', y'' \in B_b(y_0)$

$$e(F(y') \cap B_a(x_0), F(y'')) \leq M d(y', y'').$$

Here $e(A, B) = \sup\{\operatorname{dist}(x, B) : x \in A\}$ is the excess from the set B with respect to the set A. In a recent paper [10] we propose to call this property the *Aubin property.* If $a = \infty$, then the map F is Lipschitz continuous in $B_b(y_0)$ with respect to the Hausdorff metric $h(A, B) = \max\{e(A, B), e(B, A)\}$. Characterizations of the Aubin property are obtained in Rockafellar [26] and Mordukhovich [20]. After the work of Borwein-Zhuang [4] and Penot [22] it became clear that, in a very general setting, the openness with linear rate of a set-valued map F around some point $(x_0, y_0) \in \operatorname{graph} F$, the metric regularity of F around (x_0, y_0), and the Aubin property of F^{-1} around (y_0, x_0) are equivalent properties. There are a number of open mapping, Lyusternik-type and metric regularity results in the literature which we do not discuss here, for references see [3], [4], [5], [11], [13], [14], [15], [16], [18], [23], [24], [28], [29].

Consider the iteration (2), where the linear and bounded map A appears on the left and on the right side of the equality. For the left A we use the openness with linear rate while for the right A we employ the assumption that A is the strict derivative of f at 0. Suppose that the strict derivative of f at 0 is zero. Then the right A disappears and the only property of A needed is the openness with linear rate. We can then replace A by a general set-valued map, say F, and the iteration (2) becomes

$$y \in f(x_{n-1}) + F(x_n).$$

Note that the condition $\nabla f(0) = 0$ is not essential; it can be eliminated by "hiding" the derivative of f in the map F. Furthermore, X may be any complete metric space and Y a linear normed space. Using this observation, we obtain below a general open mapping theorem for set-valued maps.

Let X be a complete metric space with a metric ρ and let Y be a linear metric space with an invariant metric d. A function $f : X \to Y$ is called *strictly stationary* at the point x_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x', x'' \in B_\delta(x_0)$

$$d(f(x'), f(x'')) \le \epsilon\rho(x', x'').$$

If d is associated with a norm, then a function f is strictly stationary at x_0 if and only if it is strictly differentiable at x_0 and $\nabla f(x_0) = 0$. For a map $F : X \to Y$ we say that graph F is *locally closed* around a point $(x_0, y_0) \in \operatorname{graph} F$ if there exists $\mu > 0$ such that the set graph $F \cap B_\mu((x_0, y_0))$ is closed.

Theorem 1.4. *Let F be a map from X to subsets of Y, let $y_0 \in F(x_0)$, and let* graph F *be locally closed around (x_0, y_0). Let $f : X \to Y$ be a function which is strictly stationary at x_0 and let L be a positive number. Then the following are equivalent:*

(i) *The map F is open with linear rate around (x_0, y_0) with a constant greater than L;*
(ii) *The map $f + F$ is open with linear rate around $(x_0, y_0 + f(x_0))$ with a constant greater than L.*

Proof. Let F be open with linear rate around (y_0, x_0) with a constant $c > L$; that is, for some $a > 0, b > 0$ and $\gamma > 0$ and for every $x \in B_a(x_0)$, every $y \in B_b(y_0)$ with $y \in F(x)$ and every $\tau \in [0, \gamma]$,

$$B_\tau(y) \subset F(B_{M\tau}(x)),$$

where $M = 1/c$. That is, for every $\tilde{y} \in B_\tau(y)$ there exists $\tilde{x} \in F^{-1}(\tilde{y})$ such that $\rho(\tilde{x}, x) \le M\tau$. Take a and b smaller if necessary such that the set graph $F \cap \{B_a(x_0) \times B_b(y_0)\}$ is closed. Let $M < M^+ < 1/L$ and let $\epsilon > 0$ be such that

$$M\epsilon < 1 \quad \text{and} \quad \frac{M}{1 - \epsilon M} \le M^+.$$

Choose $\alpha > 0$ such that

$$\alpha \le a, \quad \alpha < b/\epsilon,$$

and

$$d(f(x'), f(x'')) \le \epsilon\rho(x', x'')$$

whenever $x', x'' \in B_\alpha(x_0)$. Let $\beta > 0$ be such that

$$M^+\beta \le \alpha/4 \quad \text{and} \quad 2\beta \le b - \epsilon\alpha,$$

and let κ satisfy

$$0 < \kappa < \min\{\beta, \gamma\}.$$

Let $(x, y) \in \operatorname{graph}(f + F), x \in B_{\alpha/2}(x_0), y \in B_\beta(y_0 + f(x_0))$, let $\tau \in [0, \kappa]$, and let $y' \in B_\tau(y)$. We show that there exists $x' \in (f + F)^{-1}(y')$ such that $\rho(x', x) \le M^+\tau$; that is, (ii) holds with the constant $c^+ = 1/M^+, c > c^+ > L$. Denote $x_1 = x$. We have

$$\begin{aligned} d(y - f(x_1), y_0) &= d(y - f(x_1) + f(x_0), y_0 + f(x_0)) \\ &\le d(y, y_0 + f(x_0)) + d(f(x), f(x_0)) \\ &\le \beta + \epsilon\alpha \le b. \end{aligned}$$

Clearly, $y' - f(x_1) \in B_\tau(y - f(x_1))$. From the openness with linear rate of F there exists $x_2 \in X$ such that

$$y' - f(x_1) \in F(x_2) \quad \text{and} \quad \rho(x_2, x_1) \leq M\tau.$$

Suppose that there exist a natural number $n \geq 3$ and points $x_2, x_3, \cdots, x_{n-1}$ in X with the following properties:

$$y' - f(x_{i-1}) \in F(x_i)$$

and

$$\rho(x_i, x_{i-1}) \leq M\tau(M\epsilon)^{i-2}$$

for $i = 2, \cdots, n-1$. Then

$$\begin{aligned} \rho(x_i, x_0) &\leq \rho(x_1, x_0) + \sum_{j=2}^{i} \rho(x_j, x_{j-1}) \leq \alpha/2 + M\tau \sum_{j=2}^{i} (M\epsilon)^{j-2} \\ &\leq \alpha/2 + \frac{M\tau}{1 - \epsilon M} \leq \alpha/2 + M^+\tau \leq \alpha/2 + M^+\beta \leq \alpha, \end{aligned}$$

and hence

$$\begin{aligned} d(y' - f(x_i), y_0) &= d(y' - f(x_i) + f(x_0), y_0 + f(x_0)) \\ &\leq d(y', y) + d(y, y_0 + f(x_0)) + d(f(x_i), f(x_0)) \\ &\leq \tau + \beta + \epsilon\alpha \leq \kappa + \beta + \epsilon\alpha \leq 2\beta + \epsilon\alpha \leq b. \end{aligned}$$

Note that

$$d(y' - f(x_{n-1}), y' - f(x_{n-2})) \leq \epsilon\rho(x_{n-1}, x_{n-2}) \leq \epsilon M\tau(M\epsilon)^{n-3}$$

and

$$\epsilon M\tau(M\epsilon)^{n-3} \leq \gamma.$$

Hence, from the openness with linear rate of F there exists x_n such that

$$y' - f(x_{n-1}) \in F(x_n) \tag{4}$$

and

$$\rho(x_n, x_{n-1}) \leq M[\epsilon M\tau(M\epsilon)^{n-3}] = M\tau(M\epsilon)^{n-2}. \tag{5}$$

By induction, we obtain an infinite sequence $\{x_n\}$ which satisfies (4) and (5) for all natural n. The sequence x_n is a Cauchy sequence, hence convergent to some $x' \in B_\alpha(x_0)$. Since graph F is locally closed around (x_0, y_0) and f is continuous in $B_\alpha(x_0)$ we conclude that $x' \in (F + f)^{-1}(y')$. Moreover,

$$\begin{aligned} \rho(x_n, x) &\leq \sum_{i=2}^{n} \rho(x_i, x_{i-1}) \\ &\leq M\tau \sum_{i=2}^{n} (M\epsilon)^{i-2} \leq M^+\tau. \end{aligned}$$

Passing to the limit we obtain

$$\rho(x', x) \leq M^+\tau.$$

The implication (i) $\Rightarrow$ (ii) is proved.
Let (ii) hold; that is, the map $G = f + F$ is open with linear rate around $(y_0 + f(x_0), x_0)$ with a constant $c > L$. The function $-f$ is strictly stationary at x_0 and graph G is locally closed around $(y_0 + f(x_0), x_0)$. From the first part of the proof, $-f + G = F$ is open with linear rate around (y_0, x_0) with a constant $c^+ > L$. This completes the proof. $\square$

Theorem 1.4 shows that a perturbation of order higher than one does not contribute to the openness with linear rate around a point; that is, *the openness with linear rate around a point is invariant under higher-order perturbations.* This is not an unexpected result. Cominetti [5] showed that, given two continuous functions f and g from a Banach space X to a Banach space Y and a closed convex set C in X, the metric regularity of $f + C$ is equivalent to the metric regularity of $g + C$, provided that the strict derivative of the difference $f - g$ at the reference point is zero. In a previous paper [7], using a generalization of a fixed point theorem from [15], we proved that the Aubin (pseudo-Lipschitz) property of the inverse of a set-valued map is invariant under higher-order perturbations. A different proof of this result is given in [10]. In finite-dimensional spaces this result can be also obtained by the Mordukhovich characterization of the Aubin property [20].

Let Y be a linear normed space and let $f : X \to Y$ be strictly differentiable at x_0. Denote $\phi(x) = f(x) - \nabla f(x_0)(x - x_0)$ and $\mathcal{F}(x) = \nabla f(x_0)(x - x_0) + F(x)$. The function ϕ is strictly stationary at x_0. Applying Theorem 1.4 to the maps ϕ and $\mathcal{F}$ we obtain that the map $f + F$ is open with linear rate around (x_0, y_0) if and only if the (partial) linearization $f(x_0) + \nabla f(x_0)(\cdot - x_0) + F(\cdot)$ is open with linear rate around (x_0, y_0). In other words, the openness with linear rate around a point of a linearization of a map is inherited by the the map and vice versa. From this version of Theorem 1.4 we obtain a symmetric form of the Graves theorem. Let X be a Banach space and Y be a linear normed space and let $f : X \to Y$ be strictly differentiable at x_0. From the Banach open mapping theorem the map $f(x_0) + \nabla f(x_0)(\cdot - x_0)$ is open with linear rate around x_0 if and only if $\nabla f(x_0)X = Y$. Hence the surjectivity of $\nabla f(x_0)$ is equivalent to the openness with linear rate of f around x_0. This result can be also obtained from [5] taking into account that the openness with linear rate is equivalent to the metric regularity. We note that the usual openness, as defined before Theorem 1.1, is not invariant under higher-order perturbations, in the sense of the present paper.

The key hypothesis in Theorem 1.4, already present in the original formulation of the Graves theorem, is that the perturbing function f has strict derivative equal to zero. Recently, Robinson [25] extended this idea by introducing the so-called "strong approximation" to nonsmooth functions. By using this concept and suitably modifying the above analysis one can obtain implicit-function-type theorems for generalized equations. Results of this kind and applications to stability analysis in optimization are presented in the recent papers [8], [9] and [10].

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