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A General Duality Principle for the Sum of Two Operators[1]

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Dedicated to R. T. Rockafellar on his 60th Birthday

A general abstract duality result is proposed for equations which are governed by the sum of two operators (possibly multivalued). It allows to unify a large number of variational duality principles, including the Clarke-Ekeland least dual action principle and the Singer-Toland duality. Moreover, it offers a new duality approach to some central questions in the theory of variational inequalities and maximal monotone operators.

Keywords : Abstract duality principle, Clarke-Ekeland duality, Hamiltonian systems, Fenchel-Moreau-Rockafellar duality, Singer-Toland duality, Robinson normal mappings, maximal monotone operators, convex subdifferential operator, qualification conditions.

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1. Introduction

Most of the equations arising in physics, economics, ... can be written in the following abstract form:

$$Cx := \ Ax + Bx \ni 0 \tag{1.1}$$

where A and $B : X \rightrightarrows Y$ are two (possibly multivalued) operators. The splitting of the operator C, governing equation (1.1), into the sum of two elementary operators A and B

$$C = A + B$$

has usually a deep physical or economical meaning. The reason is that A and B may have very distinct properties. Let us illustrate this by some examples. In some frequent physical

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situations, A and B describe respectively a convection and a diffusion phenomenon. In classical mechanics, A derives from a kinetic energy, while B derives from a potential energy. In decision theory, A derives from a cost or a preference criteria and B from some economical constraints (note that this last situation often leads to multivalued operators).

In this paper, we propose a general abstract dual formulation for Equation (1.1) which unifies most of the duality principles currently known. Indeed, it turns out that (Theorem 3.1) without any restrictive assumption, Equation (1.1) is equivalent to

$$A^{-1}y - B^{-1}(-y) \ni 0. \tag{1.2}$$

As a counterpart to the generality of this duality transformation, let us notice that, even if we start with a classical single valued equation (1.1), Equation (1.2) may be multivalued. The major interest of this transformation is that the operators A^{-1} and/or B^{-1} may enjoy some coerciveness (compactness) properties which are not satisfied by the initial operators A and B. As a result, in contrast with Equation (1.1), the equation (1.2) may be well posed. The key property, which in some sense justifies the duality label for the above transformation, is that when $A = \partial f$ where f is a convex lower semicontinuous proper function, then $A^{-1} = \partial f^*$ where f^* is the classical Legendre-Fenchel conjugate of f. We also observe that the equivalence between Equations (1.1) and (1.2) is nothing else but writing the "symmetrical" decomposition of a vector in the product space $X \times Y$ with respect to the graphs of A and $B(-\cdot)$.

Let us now survey some of the basic situations where this duality principle applies. The Fenchel-Moreau-Rockafellar duality scheme deals with the equation

$$\partial f(x) + \partial g(x) \ni 0$$

where f, g are supposed to be convex lower semicontinuous and proper, while the Singer-Toland duality scheme deals with the equation

$$\partial f(x) - \partial g(x) \ni 0.$$

The Clarke-Ekeland least action dual principle deals with the equation

$$Ax + \partial g(x) \ni 0$$

where A is a linear symmetric operator which is not necessarily positive. Thanks to the abstract duality principle, we shall be able to describe in sections 4.1, 4.2 and 4.3 the corresponding dual variational problems.

In section 4.4, 4.5 and 4.6, we explore new applications of the abstract duality principle to generalized equations arising from the theory of maximal monotone operators and variational inequalities. Indeed, it is in the context of variational inequalities that the duality principle first appeared, see U. Mosco [31].

We consider in section 4.4 the recent theory of normal mappings developed by Robinson ([33] and [34]) and show how these mappings can be naturally introduced via the general duality transformation. We then revisit the Brézis-Crandall & Pazy approach to the maximality of the sum of two maximal operators. We obtain, in the finite dimensional setting, under the following constraint qualification:

$$\bigcup_{\lambda \geq 0} \lambda\Big(D(A) - D(B)\Big) \text{ is a closed linear subspace}$$

that $A+B$ is still a maximal monotone operator (Theorem 4.5). Finally, in the process, we give a new proof of the Brézis-Crandall & Pazy approach. Thanks to the abstract duality principle, we are able to link it with the viscosity approximation method.

Finally, let us say that in our opinion, it is worthwhile to possess such an abstract duality principle, since, the underlying elementary operations may be quite involved when specialized to concrete situations. In such situations, the duality principle may be hidden by technical difficulties. On the other hand, we may expect that this principle does apply to some other situations which have not still been explored.

2. Preliminaries and notations

In this section, we denote by X and Y general linear spaces. For an operator $A : X \rightrightarrows Y$, we note:

$$D(A) := \Big\{ x \in X \,|\, Ax \neq \emptyset \Big\},$$

the *domain* of A,

$$R(A) := \bigcup_{x \in X} Ax,$$

the range of A,

$$\operatorname{graph} A := \Big\{ (x,y) \in X \times Y \,|\, x \in D(A), y \in Ax \Big\}$$

the graph of A, A^{-1} the operator defined by

$$x \in A^{-1}y \Longleftrightarrow y \in Ax,$$

while $\breve{A}$ is defined by $\breve{A}(x) = A(-x)$. Let $\overline{x} \in X$ be fixed. By $A - \overline{x}$ we denote the operator defined by

$$(A - \overline{x})x := Ax - \overline{x} = \Big\{ y - \overline{x} \,|\, y \in Ax \Big\}.$$

Given two operators $A, B : X \rightrightarrows Y$ possibly nonlinear, multivalued, not everywhere defined, the classical notion of sum is the pointwise sum, that is $A+B : X \rightrightarrows Y$ is the operator defined by:

$$\begin{gathered} D(A+B) = D(A) \cap D(B) \\ (A+B)x = Ax + Bx \end{gathered}$$

where the sum $Ax + Bx$ is understood as the Minkowski (also called vectorial) sum:

$$Ax + Bx := \Big\{ y \in Y \,|\, \exists y_1 \in Y, \exists y_2 \in Y \text{ such that } y = y_1 + y_2 \Big\}.$$

The composition of multivalued operators is defined as follows:
Given $A : X \rightrightarrows Y$ and $B : Y \rightrightarrows Z$

$$(BA)(x) := \Big\{ z \in Z \,|\, \exists y \in Ax \text{ with } z \in By \Big\}.$$

3. The abstract result

3.1. The Abstract Duality Principle

Throughout this section, X and Y are linear spaces. To begin with we state the main result:

Theorem 3.1. *Let X and Y be two linear spaces and $A, B : X \rightrightarrows Y$ two general operators (possibly multivalued and not everywhere defined).*

(a) *The two generalized equations:*

$$Ax + Bx \ni f \tag{3.3}$$

and

$$A^{-1}y - B^{-1}(f - y) \ni 0 \tag{3.4}$$

are equivalent in the following sense:

If x is a solution of (3.3)*, then there exists $y \in Ax$ that solves* (3.4)*.*

If y is a solution of (3.4)*, then there exists $x \in A^{-1}y$ that solves* (3.3)*.*

(b) *Equation* (3.4) *is equivalent to*

$$y + BA^{-1}y \ni f. \tag{3.5}$$

Proof. (3.3) holds true if and only if there exists $y \in Ax$ such that $Bx \ni f - y$. This amounts to saying that x belongs both to $A^{-1}(y)$ and $B^{-1}(f - y)$, which is (3.4). □

If we take $f = 0$ in Theorem 3.1 we then obtain:

Corollary 3.2. *The two generalized equations:*

$$Ax + Bx \ni 0 \tag{3.6}$$

$$A^{-1}y - B^{-1}(-y) \ni 0. \tag{3.7}$$

are equivalent.

Equation (3.7) *is equivalent to*

$$y + BA^{-1}y \ni 0. \tag{3.8}$$

Remark 3.3. A and B play a symmetric role. By setting $z = -y$ in Equation (3.7) one obtains the following equivalent form:

$$B^{-1}z - A^{-1}(-z) \ni 0. \tag{3.9}$$

Similarly, if we take $z = -y$ in (3.8), we observe that (3.8) is equivalent to

$$z + AB^{-1}z \ni 0. \tag{3.10}$$

Remark 3.4. We also may consider a generalized equation involving the sum of three or more operators. For instance, let us consider $A, B, C : X \rightrightarrows Y$ governing the generalized equation:

$$Ax + Bx + Cx \ni 0. \tag{3.11}$$

Setting $\xi \in Bx, \eta \in Cx$ we obtain

$$Ax + \xi + \eta \ni 0. \tag{3.12}$$

Equation ((3.12) then leads to the system:

$$\begin{cases} A^{-1}(-\xi - \eta) - B^{-1}\xi \ni 0 \\ \\ A^{-1}(-\xi - \eta) - C^{-1}\eta \ni 0. \end{cases}$$

3.2. Geometrical interpretation of the Abstract Duality Principle

In order to give a geometrical interpretation of the Abstract Duality Principle, let us first reformulate the Equation (3.3) as a vectorial decomposition result in the product space $X \times Y$. Indeed we have:

Lemma 3.5. *The two following equations are equivalent:*

$$Ax + Bx \ni f \tag{3.13}$$

$$(0, f) \in \text{graph } A + \text{graph } \check{B}. \tag{3.14}$$

where, in (3.14)*, the sum of graphs is taken in the vectorial sense.*

Proof. $(0, f)$ satisfies (3.14) if and only if

$$(0, f) = (x, y) + (-x, z) \quad \text{with} \quad y \in Ax \quad \text{and} \quad z \in \check{B}(-x),$$

i.e., if and only if $f \in Ax + Bx$. □

The following elementary observation will be useful to perform a geometrical interpretation of the Abstract Duality Principle:

Lemma 3.6. *Suppose given two multivalued operators $S, T : X \rightrightarrows Y$ which are identified with their graphs. Then,*

$$(S + T)^{-1} = S^{-1} + T^{-1}$$

where the sums in both sides of the last equation are taken in the Minkowski sense.

Proof.

$$(x, y) \in S + T \iff (x, y) = (x_S, y_S) + (x_T, y_T) = (x_S + x_T, y_S + y_T).$$

Thus,

$$(y, x) = (y_S + y_T, x_S + x_T) = (y_S, x_S) + (y_T, x_T) \in S^{-1} + T^{-1},$$

establishing the proof. □

At this point, it will be useful to introduce the following notation and terminology. Let us introduce the "symmetry" $\Delta : X \times Y \to Y \times X$ given by $(x, y) \mapsto (y, x)$.

$$(y, x) = (y_S, x_S) + (y_T, x_T) \qquad (y_S, x_S) \in S^{-1},\ (y_T, x_T) \in T^{-1}$$

is called the *symmetrical decomposition* of

$$(x, y) = (x_S, y_S) + (x_T, y_T) \qquad (x_S, y_S) \in S,\ (x_T, y_T) \in T.$$

Geometrical proof of the Abstract Duality Principle:

Using successively Lemma 3.5 and Lemma 3.6 with $S = A$ and $T = \check{B}$, we obtain the following sequence of equivalences:

$$Ax + Bx \ni f \quad \text{i.e.,} \quad \exists y \in Ax, \exists z \in Bx \ \text{ such that } \quad y + z = f$$

$$\Updownarrow \quad \text{(Lemma 3.5)}$$

$$\begin{aligned} &(0, f) \in A + \check{B} \quad \text{with the following decomposition :} \\ &(0, f) = (x, y) + (-x, z) \quad y \in Ax, z \in \check{B}x \end{aligned}$$

$$\Updownarrow \quad \text{(Lemma 3.6)}$$

$$\begin{aligned} &(f, 0) \in A^{-1} + \check{B}^{-1} \quad \text{with the following symmetrical decomposition:} \\ &(f, 0) = (y, x) + (z, -x) \quad x \in A^{-1}y, -x \in -B^{-1}z \end{aligned}$$

$$\Updownarrow$$

$$\begin{cases} A^{-1}y - B^{-1}z \ni 0 \\ y + z = f \end{cases}$$

$$\Updownarrow$$

$$A^{-1}y - B^{-1}(f - y) \ni 0, \quad y \in Ax.$$

Let us summarize the results of this section. The generalized equation governed by $A + B$ can be interpreted as a vector decomposition with respect to the graphs of A and $\check{B}$. This is equivalent by symmetry with respect to Δ to a vector decomposition with respect to the graphs of A^{-1} and $(\check{B})^{-1}$, which again is an equivalent formulation of the dual abstract equation governed by $A^{-1} - B^{-1}(-\cdot)$. This is illustrated by the following picture:

.

By using the symmetry $\Delta : (x, y) \to (y, x)$ we obtain:

4. Applications

We first review some classical situations where the abstract duality principle applies (Fenchel-Moreau-Rockafellar, Clarke-Ekeland, Singer-Toland), then explore some new applications.

4.1. Fenchel-Moreau-Rockafellar convex duality

Let X be a normed space with continuous dual X^*. Let us first review some classical basic facts from convex analysis.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be given. We recall that

$$\mathrm{Dom} f := \Big\{ x \in X \mid f(x) < +\infty \Big\}$$

denotes the *domain* of f and that f is said to be *proper* if $\mathrm{Dom} f$ is nonempty. We denote by $\Gamma_0(X)$ the set of those functionals $f : X \to \mathbb{R} \cup \{+\infty\}$ which are lower semicontinuous convex and proper. Let $f \in \Gamma_0(X)$. The *subdifferential* of f at $x \in \mathrm{Dom} f$ is the set

$$\partial f(x) := \Big\{ x^* \in X^* \mid \langle x^*, x\rangle = f(x) + f^*(x^*) \Big\},$$

where $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is the Legendre-Fenchel conjugate of f and is given by

$$f^*(x^*) := \sup \Big\{ \langle x^*, x\rangle - f(x) \mid x \in X \Big\}.$$

Given $f \in \Gamma_0(X)$, the inverse of the subdifferential of f is related to the subdifferential of its conjugate f^*, namely,

$$(\partial f)^{-1} = \partial f^*.$$

Let $f, g \in \Gamma_0(X)$ be given. When applying Theorem 3.1 with $A := \partial f$ and $B := \partial g$, we obtain:

Theorem 4.1. *The two problems $(\mathcal{P})$ and $(\mathcal{D})$*

$$(\mathcal{P}) \qquad \partial f(u) + \partial g(u) \ni 0$$

and

$$(\mathcal{D}) \qquad (\partial g)^{-1}(u^*) - (\partial f)^{-1}(-u^*) \ni 0$$

are equivalent in the following sense:

If u solves $(\mathcal{P})$, then there exists $u^* \in \partial g(u)$ solving $(\mathcal{D})$ and conversely if u^* solves $(\mathcal{D})$, then there exists some $u \in (\partial g)^{-1}(u^*)$ which solves $(\mathcal{P})$.

If we set $\check{f}(u) = f(-u)$, then $-(\partial f)^{-1}(-\cdot) = \partial \check{f}^*$, and $(\mathcal{D})$ can be rewritten as

$$(\mathcal{D}) \qquad \partial g^*(u^*) + \partial \check{f}^*(u^*) \ni 0.$$

We do not need any qualification assumption for this result.

Let us recall (cf. [12]) that for a convex subset $C \subset X$, the *strong quasi interior* of C denoted by sqi C is the set of those $x \in C$ for which $\bigcup_{\lambda \geq 0} \lambda(C - x)$ is a closed subspace. When X is finite dimensional, the strong quasi interior of C coincides with the *relative interior* of C denoted by ri C, and is equal to the interior of C in the affine hull of C. Under a qualification assumption (we use here the recent version given in [6]) we have the following duality theorem:

Theorem 4.2. *Let X be a reflexive Banach space and let $f, g \in \Gamma_0(X)$ be given. Provided the constraint qualification*

$$(\mathcal{CQ}) \qquad 0 \in \operatorname{sqi}\Big(D(\partial f) - D(\partial g)\Big)$$

is satisfied, the following statements hold:

(i) *For all* $x^* \in X^*$, $(f+g)^*(x^*) = \min\limits_{x_1^*+x_2^*=x^*} f^*(x_1^*) + g^*(x_2^*)$;

(ii) *For all* $x \in \operatorname{Dom}(f+g)$ $\quad \partial(f+g)(x) = \partial f(x) + \partial g(x)$;

(iii) $\inf\limits_{x \in X} \Big(f(x) + g(x)\Big) = \max\limits_{x^* \in X^*} \Big(f^*(x^*) + g^*(-x^*)\Big)$.

Corollary 4.3. *Let X be a Banach space and let $f, g \in \Gamma_0(X)$ be given. Suppose that the constraint qualification $(\mathcal{CQ})$ is verified and that u minimizes the function $f + g$ on X, then*

$$\text{there exists } u^* \in \partial f(u) \quad \text{such that} \quad \partial f^*(u^*) - \partial g^*(-u^*) \ni 0$$

and

$$\inf_{x \in X} \Big(f(x) + g(x)\Big) = \max_{x^* \in X^*} \Big(f^*(x^*) + g^*(-x^*)\Big).$$

4.2. The dual least action principle of Clarke-Ekeland

Let specialize the abstract duality scheme in order to cover various situations including both Hamiltonian systems (with convex Hamiltonian H)) and some P.D.E arising in the study of nonlinear wave equations. Our presentation is close to Brézis ([14]) and is valid in a slightly more general framework. Let H be an Hilbert space, and $A \subseteq H \to H$ be an unbounded linear operator which is self-adjoint, $A = A^*$. Note that A is not supposed to be positive and the quadratic form

$$Q(x) = \frac{1}{2}\langle Ax, x\rangle$$

is not in general a convex function. Let $F : H \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous proper function and ∂F its associated subdifferential operator.

Let us consider the equation

$$Ax + \partial F(x) \ni 0. \tag{4.1}$$

This equation has a variational structure. The solutions of (4.1) correspond (at least formally) to the critical points of the functional

$$\Phi(x) := \frac{1}{2}\langle Ax, x\rangle + F(x). \tag{4.2}$$

In general, the functional Φ is indefinite, that is, it is unbounded both from above and below. This is related to the fact that the linear operator A has an infinite sequence of eigenvalues going from $-\infty$ to $+\infty$. Critical points of Φ cannot be obtained by a direct global minimization (or a maximization) and the direct Tonelli method of calculus of variations does not apply.

Let us show how the abstract duality principle is interesting in this situation. Without any assumption, by setting

$$y \in \partial F(x)$$

(4.1) turns out to be equivalent to

$$A^{-1}y + (\partial F)^{-1}(y) \ni 0,$$

that is

$$A^{-1}y + \partial F^*(y) \ni 0. \tag{4.3}$$

Indeed, A^{-1} is a multivalued operator since Ker A does not in general reduce to $\{0\}$.

In order to go further and obtain the Clarke-Ekeland least dual action principle, we will make some additional assumptions:

1) the range $R(A)$ of A is a closed subspace of H.
2) F^* is a convex continuous function with Dom $F^* = H$.

As a consequence of Assumption 1), H admits an orthogonal decomposition $H = R(A) \oplus \operatorname{Ker} A$. Then, it should be observed that for each $y \in R(A)$, there is a unique $x \in R(A)$ such that $y = Ax$. If we note $x = \mathcal{A}^{-1}y$, then $\mathcal{A}^{-1}$ is a well defined bounded operator from $R(A)$ into itself and

$$A^{-1}y = \mathcal{A}^{-1}y + \operatorname{Ker} A. \tag{4.4}$$

Using the observation (4.4), we note that Equation (4.3) is equivalent to finding $y \in R(A)$ such that

$$\mathcal{A}^{-1}(y) + \partial F^*(y) \in \operatorname{Ker} A. \tag{4.5}$$

Equation (4.5) has a variational structure. Its solutions correspond to the critical points of the functional Ψ given by:

$$\Psi(y) := \frac{1}{2}\langle \mathcal{A}^{-1}y, y\rangle + F^*(y) \tag{4.6}$$

subject to the constraint:

$$y \in R(A).$$

In order to obtain the above result we have used the equality

$$\partial(F^* + I_{R(A)}) = \partial F^* + \partial I_{R(A)} = \partial F^* + \operatorname{Ker} A.$$

Note that the first above equality is a consequence of the Moreau-Rockafellar Theorem on the subdifferential of a sum of convex functions, which applies here because of the continuity assumption on F^*. Note also that in (4.5) the second member of the equation is a Lagrange multiplier of the constraint $y \in R(A)$.

Let us summarize the above results in the following statement:

Theorem 4.4. [Abstract Clarke-Ekeland duality principle] *Under the hypothesis* 1), 2) *given in this section, if* $\overline{y}$ *is a critical point of* Ψ*, then there exists some* $z \in \operatorname{Ker} A$ *such that* $\overline{x} = \mathcal{A}^{-1}(-\overline{y}) + z$ *is a critical point of* Φ.

Let us briefly describe in two classical situations why the dual variational problem (4.6) may be easier to study than the primal variational problem (4.1).

a) Finite dimensional Hamiltonian systems:

A basic problem in mechanics (classical and celestial) is the study of periodic solutions of the Hamiltonian system:

$$J\dot{u}(t) + \nabla H(t, u(t)) = 0$$

where J is the symplectic matrix, $u = (p, q) \in \mathbb{R}^n \times \mathbb{R}^n$ and H is the classical Hamiltonian. Equivalently,

$$\begin{cases} \dot{p} - H_q = 0 \\ \dot{q} + H_p = 0. \end{cases}$$

We set

$$\mathcal{H} = L^2(0, T) \times L^2(0, T),$$

$$Au = J\dot{u}$$

with

$$D(A) = \Big\{ u \in \mathcal{H} \mid \dot{u} \in \mathcal{H}, u(0) = u(T) \Big\}$$

$$R(A) = \Big\{ u \in \mathcal{H} \mid \int_0^T u(t)dt = 0 \Big\},$$

and

$$F(u) = \int_0^T H(t, u(t))dt, \text{ i.e.,} (\nabla F)(u)(t) = \nabla H(t, u(t)).$$

Therefore,

$$Au + \nabla F(u) \ni 0.$$

The Hamiltonian action Φ can be written

$$\Phi(u) = \int_0^T \frac{1}{2}\langle J\dot{u}, u\rangle + H(t, u(t))dt.$$

Let us explain briefly how the abstract duality principle leads naturally to the dual least action principle. First note that J and the mapping $u \to \dot{u}$ are antisymmetric and so A is symmetric. On the other hand, the range of A is the subspace of $\mathcal{H}$ of those functions u with zero mean value, which is clearly closed.

Let us describe the functional Ψ given by Theorem 4.4 and formula (4.6). An elementary computation yields, $\mathcal{A}^{-1}v = -\int_0^t Jv$. Hence,

$$\Psi(v) = \int_0^T [\frac{1}{2}\langle -\int_0^t Jv, v(t)\rangle + H^*(t, v(t))]dt. \tag{4.7}$$

So we are looking for the critical points of Ψ on the subset of zero mean value functions. Noticing that such functions can be equivalently represented as derivatives of periodic functions, we are lead to consider the critical points of the functional

$$\Psi(\dot{\xi}) = \int_0^T [\frac{1}{2}\langle - \int_0^t J\dot{\xi}, \dot{\xi}(t)\rangle + H^*(t, \dot{\xi}(t))]dt \tag{4.8}$$

on the subspace of functions ξ which are T-periodic. An elementary transformation yields the Clarke-Ekeland dual least action principle associated to the functional that we still denote by Ψ :

$$\Psi(v) = \int_0^T [\frac{1}{2}\langle J\dot{v}, v(t)\rangle + H^*(t, \dot{v}(t))]dt. \tag{4.9}$$

b) Nonlinear wave equations (see [14]).

Let us consider the nonlinear vibrating string equation:

$$\text{(NVSE)} \qquad \begin{cases} u_{tt} - u_{xx} + \beta(u) = 0 & 0 < x < \pi, \;\; t \in \mathbb{R} \\ u(x,t) = 0 & x = 0, \;\; x = \pi, \;\; t \in \mathbb{R} \\ u(x, t+\pi) = u(x,t) & 0 < x < \pi, \;\; t \in \mathbb{R} \end{cases}$$

where β is a continuous nondecreasing function on $\mathbb{R}$ such that $\beta(0) = 0$. We set $j(x) := \int_0^x \beta(s)ds$, that is $\nabla j = \beta$.

(NVSE) can be viewed as an infinite dimensional Hamiltonian system. By setting $p = u$ and $q = u_t$, it becomes

$$\begin{cases} p_t - H_q = 0 \\ q_t + H_p = 0 \end{cases}$$

where the Hamiltonian H is given by:

$$H(p,q) = \frac{1}{2}\int_0^\pi (p_x)^2 + \int_0^\pi j(p)dx + \frac{1}{2}\int_0^\pi q^2 dx$$

on the space $H_0^1(0,\pi) \times L^2(0,\pi)$.

Let us now explain how this problem enters in the setting of the abstract duality theorem 4.4.

We set $\mathcal{H} = L^2(0,\pi) \times L^2(0,T), \Omega = (0,\pi) \times (0,T)$. Let us define

- $Au = u_{tt} - u_{xx}$ acting on the functions which are T-periodic in t and satisfying the Dirichlet boundary condition in x.

and

- $Bu = \beta(u)$.

Clearly, $B = \nabla F$ where $F(u) = \int_\Omega j(u)$ is a convex lower semicontinuous function. On the other hand, A is a self-adjoint operator such that (take $T = 2\pi$)

$$\text{Ker } A = \Big\{ \phi(x,t) \,|\, \phi(x,t) = p(x+t) - p(t-x) \;\; \text{with } p \;\; 2\pi \text{ periodic} \Big\}$$

and its range is the closed orthogonal subspace.

The (NVSE) equation can be written equivalently as

$$Au + Bu \ni 0.$$

When B is onto (which implies that F^* is everywhere defined) Theorem 4.4 applies and (NVSE) is equivalent to finding critical points of

$$\Psi(v) = \frac{1}{2}\int_\Omega A^{-1}v \cdot v + \int_\Omega j^*(v)$$

subject to the constraint $v \in R(A)$. Note that A^{-1} is a compact operator. When B fails to be onto, one can overcome this difficulty by considering the approximate problem

$$Au_\epsilon + Bu_\epsilon + \epsilon u_\epsilon = 0,$$

then get estimates on u_ϵ and pass to the limit as $\epsilon \to 0$ (see [14] and the bibliography therein for further details, where existence results are obtained for (NVSE) by this way).

4.3. Singer-Toland duality

The class of so-called $DC-$functions which consists of functions which are difference of convex functions has received a great attention in the last decade. According to Toland [42], x is a *critical point* of $f - g$, if

$$\partial f(x) - \partial g(x) \ni 0.$$

By virtue of Theorem 3.1, this is equivalent to say that there exists $x^* \in X^*$ such that

$$\partial g^*(x^*) - \partial f^*(x^*) \ni 0,$$

i.e., x^* is a critical point of $g^* - f^*$.

Let us give a nice application in the theory of plasma equilibrium which shows the power of this duality scheme. Let us consider the free boundary problem $(\mathcal{P})$ introduced by Temam ([41]):

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u = \lambda \cdot u^+ \text{ in } \Omega \\ u = \text{unknown constant on the boundary } \Gamma \text{of } \Omega \\ \displaystyle\int_\Gamma \frac{\partial u}{\partial n} = I \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ and $\lambda > 0$, I are given parameters.

It has been established in [41] that u solves $(\mathcal{P})$ if and only if u is a critical point of the functional E defined on the direct sum $H = H_0^1(\Omega) \otimes \mathbb{R}$ of $H_0^1(\Omega)$ and $\mathbb{R}$ by

$$E(v) = \frac{1}{2}|\nabla v|^2_{L^2(\Omega)^N} - \frac{\lambda}{2}|v^+|^2_{L^2(\Omega)^N} + Iv(\Gamma).$$

Write E as a DC-functional, i.e., as the difference $E = G - F$ of the two convex functionals F and G defined by

$$F(u) := \frac{\lambda}{2}|u^+|^2_{L^2(\Omega)} + \frac{1}{2}|u(\Gamma)|^2 - Iu(\Gamma)$$

and

$$G(u) := \frac{1}{2}|\nabla u|^2_{L^2(\Omega)} + \frac{1}{2}|u(\Gamma)|^2.$$

Note that $G = G^*$, F and G are $\mathcal{C}^1$ on H which makes the computation of the dual problem quite easy to handle.

By using the Singer-Toland duality, a simple calculation of the Fenchel-Legendre conjugates of F and G leads to the Beresticki-Brézis variational formulation of problem $(\mathcal{P})$ ([11]), which consists of finding the critical points of a DC-functional Ψ. By applying again the Singer-Toland duality to Ψ, A. Damlamian ([23]) proved that in fact the two variational formulations of problem $(\mathcal{P})$ are equivalent. The reader interested by this application should consult [23], [17] and the references therein.

4.4. Normal mappings in the sense of Robinson

In this section, we deal with a special class of mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ called by S. Robinson *normal maps*. Let C be a nonempty closed convex set in $\mathbb{R}^n$, and let f be a continuous function from $\mathbb{R}^n$ to itself. A problem which arises frequently in optimization and equilibrium analysis is that of finding a point x such that

$$f(\Pi_C(x)) + (x - \Pi_C(x)) = 0, \tag{4.10}$$

where Π_C is the Euclidean projector on C.

Let us recall that the *normal cone* to a convex set C at x is given by

$$N_C(x) := \begin{cases} \Big\{ z \in X \mid \langle z, c - x\rangle \leq 0, \forall c \in C \Big\} & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

It is well known that $N_C(v) = \partial I_C(v)$, where $\partial I_C(v)$ is the subdifferential of the indicator function I_C of C defined by

$$I_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

Furthermore, $(I + N_C)^{-1}(v)$ is nothing but the projection of v over C, i.e.,

$$(I + N_C)^{-1}(v) = \Pi_C(v). \tag{4.11}$$

As quoted by Robinson [33] or [34], Equation (4.10) can be derived from a generalized equation

$$0 \in f(u) + N_C(u) \tag{4.12}$$

where f and C are as above. Indeed the two equations (4.10) and (4.12) can be easily derived one from the other by using the abstract duality principle (we use here the version (3.10)).

More generally, let $A : H \rightrightarrows H$ be a general abstract operator from an Hilbert space H into itself, and $C \subset H$ be a closed convex nonempty subset of H. We consider the variational inequality

$$Au + N_C(u) \ni 0 \tag{4.13}$$

that is, we look for some $u \in C$ such that there exists $u^* \in Au$ such that

$$\langle u^*, v-u\rangle \geq 0 \quad \text{for every } v \in C.$$

Let us notice that, (it is the trick), (4.13) is equivalent to (add and subtract u to the equation)

$$-u + Au + u + N_C(u) \ni 0. \tag{4.14}$$

By setting

$$\mathcal{A}u = (-I + A)(u),$$

$$\mathcal{B}u = (I + N_C)(u)$$

we are faced with an equation of the type

$$\mathcal{A}u + \mathcal{B}u \ni 0. \tag{4.15}$$

Then we apply Theorem 3.1 (abstract duality principle), to obtain the equivalent equation:

$$x + \mathcal{A}\mathcal{B}^{-1}x \ni 0. \tag{4.16}$$

(One passes from (4.15) to (4.16) by taking $x \in \mathcal{B}u$). Noticing that $\mathcal{B}^{-1} = \Pi_C$ we finally obtain

$$(x - \Pi_C(x)) + A(\Pi_C(x)) \ni 0. \tag{4.17}$$

Note that the variable transformation can be written as

$$x \in \mathcal{B}u \iff u = \Pi_C x$$

and

$$x \in -\mathcal{A}u \iff x \in u - Au.$$

Equation (4.17) then appears as a more general case of Equation (4.10).

In order to conclude this subsection, it should be noticed that a large variety of problems of equilibrium analysis other than optimization problems are formulated as the generalized equation (4.12).

4.5. The pointwise sum of maximal monotone operators

Let us first recall some basic definitions and properties concerning maximal monotone operators. Let X be a linear normed space with continuous dual X^*.

$A : X \rightrightarrows X^*$ is said to be *monotone,* if
for all $(x_1, y_1) \in \operatorname{graph} A, (x_2, y_2) \in \operatorname{graph} A$ we have $\langle y_2 - y_1, x_2 - x_1\rangle \geq 0$.

A is declared *maximal monotone,* if the graph of A is maximal for the inclusion in the set of graphs of monotone operators. It is well known that if A is maximal monotone, then A^{-1} and $A - \overline{x}$ are also maximal monotone.

4.5.1 The Attouch-Riahi-Théra Theorem in finite dimensions revisited

It is a well known fact that, without any qualification condition, the pointwise sum of maximal monotone operators can fail to be a maximal monotone operator. Several examples of this fact are given in [2]. When X is reflexive, one qualification condition which is widely used is the Rockafellar constraint qualification (we refer to [39] for a proof of this result):

$$\operatorname{Int}(D(A)) \cap D(B) \neq \emptyset. \tag{4.18}$$

Recently, in the setting of reflexive Banach spaces, using the Brézis-Crandall-Pazy condition and the Banach-Steinhaus Theorem, Attouch, Riahi & Théra [6] gave a weaker constraint qualification:

$$\bigcup_{\lambda \geq 0} \lambda\Big(D(A) - D(B)\Big) \text{ is a closed linear subspace.} \tag{4.19}$$

The main concern of this section is to recapture this result in the finite dimensional setting by using the duality principle (Theorem 3.1). The result reads as follows:

Theorem 4.5. *Let $A, B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be maximal monotone operators such that*

$$\overline{D(A)} \cap \overline{D(B)} \neq \emptyset, \tag{4.20}$$

and

$$\bigcup_{\lambda \geq 0} \lambda\Big(D(A) - D(B)\Big) \text{ is a closed linear subspace.}$$

Then, the pointwise sum $A + B$ is a maximal monotone operator.

Before to prove this result we need to introduce some additional notations. If T is a maximal monotone operator, we denote by f^T_∞ the *recession function* of T which is defined by:

$$f^T_\infty(y) := \sup \Big\{ \langle y, x\rangle \mid x \in \overline{R(T)} \Big\},$$

i.e., f^T_∞ is the support function of the closed convex set $\overline{R(T)}$.

Let A be a maximal monotone operator. We say that A verifies the Brézis-Haraux condition if:

$$(\mathcal{B} - \mathcal{H}) \quad \forall f \in R(A),\ \forall y \in D(A), \qquad \sup_{(z,h) \in \operatorname{graph} A} \langle h - f, y - z\rangle < +\infty.$$

The following result will be useful:

Theorem 4.6. [Brézis-Haraux] [16, Theorem 4] *Let H be a Hilbert space and let $A, B : H \rightrightarrows H$ be two maximal monotone operators such that*

(i) *B satisfies the Brézis-Haraux condition;*
(ii) *$D(A) \subset D(B)$;*
(iii) *$A + B$ is maximal monotone.*

Then, $f^{A+B}_\infty = f^A_\infty + f^B_\infty$.

It should be remarked that a resolvent $J^A_\lambda := (I + \lambda A)^{-1}$ always satisfies the Brézis-Haraux condition.

The following result, in conjunction with the duality principle, plays a key role in the proof of Theorem 4.5.

Theorem 4.7. [Attouch-Chbani & Moudafi] *Let H be a Hilbert space, $\mathcal{C} : H \rightrightarrows H$ be a maximal monotone operator and $f^{\mathcal{C}}_{\infty}$ its recession function. Let us assume* (i), (ii) *:*

(i) **Compactness assumption:** $\forall t_n \to +\infty, \forall v_n \overset{w}{\to} v$ *with* $\mathcal{C}(t_n v_n)$ *bounded, we have* $v_n \to v$ *strongly.*

(ii) **Compatibility condition :**

(iia) $f^{\mathcal{C}}_{\infty} \geq 0$

(iib) *$Ker\ f^{\mathcal{C}}_{\infty}$ is a subspace.*

Then the equation

$$\mathcal{C}u \ni 0$$

has at least one solution.

Proof of Theorem 4.5. We want to prove that $A+B$ is a maximal monotone operator. Equivalently, according to the Minty Theorem, we must show that $I+A+B$ is surjective. This amounts to saying that

$$\forall h \in H, \ \exists u \in H \ \text{ such that } u + Au + Bu \ni h. \tag{4.21}$$

Equation (4.21) being equivalent to

$$(I+A)u + (B-h)u \ni 0, \tag{4.22}$$

thanks to Theorem 3.1, we derive that Equation (4.21) is equivalent to

$$\forall h \in H, \exists u \in H \ \text{ such that } \ (I+A)^{-1}y - (B-h)^{-1}(-y) \ni 0, \tag{4.23}$$

with the change of variable $y \in (I+A)u$. Let us notice that, in Equation (4.23), the operator $\mathcal{A} := (I+A)^{-1}$ is a maximal monotone operator which is a contraction everywhere defined, while $\mathcal{B} := -(B-h)^{-1}$ is a maximal monotone operator. As a result, $\mathcal{C} := \mathcal{A}+\mathcal{B}$ is necessarily a maximal monotone operator. Therefore, the problem of the maximal monotonicity of $A+B$ can be equivalently viewed as a generalized equation governed by the maximal monotone operator $\mathcal{C}$:

$$(\mathcal{E}) \qquad \text{Find } y \in H \ \text{ such that } \ \ 0 \in \mathcal{C}y.$$

Let us notice that $R(\mathcal{A}) = D(A)$ while $R(\mathcal{B}) = -D(B)$. Therefore,

$$f^{\mathcal{A}}_{\infty}(y) = \sup \left\{ \langle y, v \rangle \mid v \in \overline{D(A)} \right\} \tag{4.24}$$

while

$$f^{\mathcal{B}}_{\infty}(y) = \sup \left\{ \langle y, v \rangle | \mid v \in -\overline{D(B)} \right\}. \tag{4.25}$$

(4.24) and (4.25) combined with Theorem 4.6 yield:

$$f^{\mathcal{C}}_{\infty}(y) = f^{\mathcal{A}+\mathcal{B}}_{\infty}(y) = \sup \left\{ \langle y, v-w \rangle \mid v \in \overline{D(A)}, w \in \overline{D(B)} \right\} \tag{4.26}$$

and therefore $f^{\mathcal{C}}_{\infty}$ is the support function of $\overline{D(A)} - \overline{D(B)}$, i.e.,

$$f^{\mathcal{C}}_{\infty} = s(\overline{D(A)} - \overline{D(B)}; \cdot). \tag{4.27}$$

We have to verify that the two conditions in the Attouch-Chbani-Moudafi Theorem are satisfied since the compacity condition automatically holds in finite dimension. By virtue of (4.27), the condition $f^{\mathcal{C}}_{\infty} \geq 0$ equivalently reads as

$$s(\overline{D(A)} - \overline{D(B)}; \cdot) \geq 0. \tag{4.28}$$

One may eventually observe that (4.28) holds true if in particular the condition

$$\overline{D(A)} \cap \overline{D(B)} \neq \emptyset \tag{4.29}$$

is satisfied.

Since

$$f^{\mathcal{C}}_{\infty}(y) = 0 \iff \langle y, w \rangle \leq 0, \quad \forall w \in \overline{D(A) - D(B)},$$

$\mathrm{Ker} f^{\mathcal{C}}_{\infty}$ is a linear subspace if $\bigcup_{\lambda \geq 0} \lambda \Big(D(A) - D(B) \Big)$ is a closed subspace, and the proof is complete. □

Remark 4.8. $D(A) - D(B)$ is not convex in general.

Condition (4.19) reads equivalently as

$$0 \in \mathrm{sqi}\, \Big(D(A) - D(B) \Big). \tag{4.30}$$

4.5.2 The case of subdifferential operators

Let us now suppose that $A = \partial f$, and $B = \partial g$, with $f, g \in \Gamma_0(H)$. We know from the Rockafellar Theorem [38] that A and B are maximal monotone operators on H and from the Brøndsted-Rockafellar Theorem, that for each $f \in \Gamma_0(H), D(\partial f)$ is dense in Dom f. Also Theorem 4.5 implies the following one:

Theorem 4.9. *Let $f, g \in \Gamma_0(H)$ such that*

$$\mathrm{Dom} f \cap \mathrm{Dom} g \neq \emptyset, \tag{4.31}$$

and

$$0 \in \mathrm{sqi}\, \Big(D(\partial f) - D(\partial g) \Big). \tag{4.32}$$

Then,

$$\partial f + \partial g = \partial (f + g).$$

Proof. It follows from Theorem 4.5 that $\partial f + \partial g$ is a maximal monotone operator. Since $\partial f + \partial g \subset \partial (f + g)$ and since $\partial (f + g)$ is a maximal monotone operator, equality follows. □

Let us remark that, in the above statement, the qualification assumption insuring that $\partial(f+g) = \partial f + \partial g$ is expressed with the help of the difference $D(\partial f) - D(\partial g)$ of the domains of the subdifferentials ∂f and ∂g. This makes a sharp contrast with the usual qualification assumptions which involve the difference of the domains $\mathrm{Dom} f - \mathrm{Dom} g$ of f and g.

This suggests that the sets $\mathrm{Dom} f - \mathrm{Dom} g$ and $D(\partial f) - D(\partial g)$ have very similar properties. A complete discussion is beyond the scope of the present work and will be presented elsewhere. We shall just mention the following result:

Theorem 4.10. [2] *Let X be a reflexive Banach space and let $f, g : X \to]-\infty, \infty]$ be proper lower semi-continuous convex functions on X. Then the following holds:*

$$\mathrm{Int}\Big(\mathrm{Dom} f - \mathrm{Dom} g\Big) = \mathrm{Int}\Big(D(\partial f) - D(\partial g)\Big)$$

Proof. We just need to show that

$$\mathrm{Int}\Big(\mathrm{Dom}\, f - \mathrm{Dom}\, g\Big) \subset \mathrm{Int}\Big(D(\partial f) - D(\partial g)\Big), \tag{4.33}$$

since the reverse inclusion is clearly true. First, observe that for a given $h \in \Gamma_0(X)$, the domains $\mathrm{Dom}\, h$ and $D(\partial h)$ remain unchanged when adding to h a convex continuous function. So, by subtracting from f and g some continuous affine minorant we can assume that f and g are nonnegative. Similarly, by adding $\|\cdot\|^2$ to f and/or g we may assume that f and/or g is coercive.

So let us start with some

$$x_0 \in \mathrm{Int}\Big(\mathrm{Dom} f - \mathrm{Dom} g\Big)$$

and let us notice that by setting $f_{x_0}(v) := f(x_0 + v)$, then

$$0 \in \mathrm{Int}\Big(\mathrm{Dom} f_{x_0} - \mathrm{Dom} g\Big).$$

Since $D(\partial f_{x_0}) = D(\partial f) - \{x_0\}$, we should observe that if we are able to prove that

$$0 \in \mathrm{Int}\Big(D(\partial f_{x_0}) - D(\partial g)\Big)$$

then necessarily,

$$x_0 \in \mathrm{Int}\Big(D(\partial f) - D(\partial g)\Big).$$

Therefore, without any restriction, we shall assume that $x_0 = 0$. We want to prove that if

$$0 \in \mathrm{Int}\Big(\mathrm{Dom} f - \mathrm{Dom} g\Big) \tag{4.34}$$

[2] Thanks are due to Jonathan Borwein for having pointed out to our attention the idea of a simple proof of this result

then

$$0 \in \text{Int}\left(D(\partial f) - D(\partial g)\right). \tag{4.35}$$

To that end, let us introduce the functional p given by

$$p(x) := \inf\left(f(x+v) + g(x) |\, v \in X\right)$$

and notice, (that is the key), that

$$\text{Dom}p = \text{Dom}f - \text{Dom}g. \tag{4.36}$$

Moreover, it easily follows from the (reduced) assumptions on f and/or g that p is a convex lower semicontinuous proper function, which is nonnegative and $p(0) < \infty$. Note that the lower semicontinuity of p follows from the coerciveness assumption on f and/or g. So, assumption (4.34) is equivalent to

$$0 \in \text{Int Dom}p. \tag{4.37}$$

Since, for every convex lower semicontinuous proper function p on X, Int Dom $p =$ Int $D(\partial p)$ (see R. T. Rockafellar [37]), we infer

$$0 \in \text{Int } D(\partial p). \tag{4.38}$$

We complete the proof by computing ∂p, which is a standard argument. First, since X is reflexive and f and/or g is coercive, for any $x \in X$ there exists an element $v(x) \in X$ such that

$$p(x) = f(x + v(x)) + g(v(x)). \tag{4.39}$$

Let us suppose that $x \in D(\partial p)$, that is to say that there exists some $x^* \in X^*$ such that for all $v \in X$

$$p(v) \geq p(x) + \langle x^*, v - x\rangle,$$

i.e., for all $u, v \in X$

$$f(v+u) + g(u) \geq f(x + v(x)) + g(v(x)) + \langle x^*, v - x\rangle. \tag{4.40}$$

Then take successively, $u = v(x)$ and $v + u = x + v(x)$ in (4.40) to infer

$$x^* \in \partial f(x + v(x))$$

$$-x^* \in \partial g(v(x)).$$

Hence, there exists some $v(x) \in X$ such that

$$x = (x + v(x)) - v(x) \in D(\partial f) - D(\partial g),$$

that is

$$D(\partial p) \subset D(\partial f) - D(\partial g). \tag{4.41}$$

Combining (4.37) and (4.41), we finally obtain $0 \in \text{Int}\left(D(\partial f) - D(\partial g)\right)$, that is (4.35) and establishes the proof. □

4.6. Brézis-Crandall-Pazy viewed as a dual viscosity method

In this subsection, we consider two maximal monotone operators $A, B : H \rightrightarrows H$ and we denote by B_λ the Yosida approximate of index λ of B:

$$B_\lambda := \frac{1}{\lambda}(I - J_\lambda^B)$$

where

$$J_\lambda^B := (I + \lambda B)^{-1}$$

is the *resolvent* of index λ of B. For each $\lambda > 0$ we denote by u_λ the solution of the equation

$$u_\lambda + Au_\lambda + B_\lambda u_\lambda \ni f \tag{4.42}$$

which exists thanks to the maximal monotonicity of the operator $A + B_\lambda$, which follows from the fact that B_λ is everywhere defined and continuous (see for instance [13]).

As a final application of the abstract duality principle, let us give a simple proof of the Brézis-Crandall-Pazy Theorem:

Theorem 4.11. [Brézis-Crandall & Pazy] *Let H be a Hilbert space and $A, B : H \rightrightarrows H$ two given maximal monotone operators. The equation*

$$u + Au + Bu \ni f$$

has a solution if and only if the family $\{ B_\lambda u_\lambda | \, \lambda \to 0 \}$ remains bounded. When this condition is satisfied, the family $\{ u_\lambda | \, \lambda > 0 \}$ norm-converges to u as λ goes to 0 and the family $\{ B_\lambda u_\lambda | \, \lambda > 0 \}$ norm-converges as λ goes to 0 to $(f - Au - Bu)^0$ which is the element of minimal norm of the convex set $f - Au - Bu$.

Proof. We know from the Minty Theorem that $A + B$ is maximal monotone if and only if

$$\forall f \in H \;\; \exists u \in H \;\text{ such that }\;\; u + Au + Bu \ni f. \tag{4.43}$$

Hence, by applying the abstract duality Theorem (Theorem 3.1), Equation (4.43) is equivalent to

$$\exists \xi \in H \;\text{ such that }\;\; B^{-1}\xi - (I + A)^{-1}(f - \xi) \ni 0. \tag{4.44}$$

Set $C := B^{-1} - (I + A)^{-1}(f - \cdot)$ which is a maximal monotone operator since $(I + A)^{-1}$ is a maximal monotone operator everywhere defined and continuous. The solvability of (4.44) is therefore equivalent to the solvability of the generalized equation governed by the maximal monotone operator C:

$$C\xi \ni 0. \tag{4.45}$$

According to [1], the solvability of (4.45) is equivalent to the fact that the viscosity approximate solution ξ_λ of

$$\lambda\xi_\lambda + C\xi_\lambda \ni 0 \tag{4.46}$$

or equivalently of,

$$\lambda\xi_\lambda + B^{-1}\xi_\lambda - (I + A)^{-1}(f - \xi_\lambda) \ni 0 \tag{4.47}$$

remains bounded as λ goes to zero.

Applying again the abstract duality principle, we observe that $\xi_\lambda = B_\lambda u_\lambda$, where u_λ satisfies Equation (4.42), i.e.,

$$u_\lambda + Au_\lambda + B_\lambda u_\lambda \ni f.$$

The condition:

$$\text{the family } \{\xi_\lambda = B_\lambda u_\lambda \mid \lambda \to 0\} \quad \text{remains bounded}$$

is precisely the Brézis-Crandall-Pazy condition for the maximality of $A + B$. Furthermore, it is a well known result, consult for instance [1] for a survey of viscosity methods, that $\xi_\lambda = B_\lambda u_\lambda$ norm converges to $\Pi_{C^{-1}(0)}$, the element of minimal norm of the set $C^{-1}(0)$, as λ goes to zero. □

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