Journal of Convex Analysis
Volume 3 (1996), No.1, 141–151

On the Level Sum of Two Convex Functions on Banach Spaces

S. Traoré, M. Volle
Département de Mathématiques,
Université d'Avignon, 33 rue Louis Pasteur, 84000 Avignon, France.
e-mail: Michel.Volle@univ-avignon.fr

Received May 31, 1994
Revised manuscript received November 29, 1995

Dedicated to R. T. Rockafellar on his 60th Birthday

The general scheme of convex duality theory is used to study the subdifferentiability, the exactness, and the lower semicontinuity of the level sum of two convex functions on a Banach space.

1. Introduction

In this paper we are concerned with the sensitivity properties of the problem of minimizing the upper envelope of two convex functions. Given a real Banach space X with (topological) dual X^*, and two extended-real-valued convex lower semicontinuous (l.s.c.) functions h and k on X, we consider the problem

$$(\mathcal{P}) \quad \text{minimize} \quad h(x) \vee k(x) \quad \text{for} \quad x \in X \,,$$

where $a \vee b$ denotes the maximum of any extended-real numbers a and b, $a \vee b = \max\,(a, b)$. We assume that h and k are proper, which means that they do not take the value $-\infty$ and they are not the constant function $+\infty$.
In order to use the perturbational approach of duality in convex optimization, we associate with $(\mathcal{P})$ the marginal (or value) functional $m\colon\ z \in X \longmapsto m(z) = \inf\limits_{x \in X} \,(h(x-z) \vee k(x))$.
This value function m is related to a kind of infimal convolution in which one has replaced the usual addition by the supremum operator $\vee$. More precisely, by introducing the convex proper l.s.c. function g, $g(x) = h(-x)$ for any $x \in X$, one obtains, for any $z \in X$,

$$m(z) = \inf_{\substack{u \in X \\ x \in X}} \,(g(u) \vee k(x) : u + x = z) \,. \tag{1}$$

Such an operation on functions g and k has been first introduced in Rockafellar's book ([15]) and partially studied in [10], [18], [1], [7], ... More recently, A. Seeger and M. Volle have given new results on this topic ([16]). The mean feature known about this operation is the expression of its strict (lower) level sets (tranches strictes). Denoting by

$$\{f < r\} = \{x \in X : f(x) < r\}$$

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

the strict level set of a function $f \in \overline{\mathbb{R}}^X$ at the level $r \in \mathbb{R}$, one has, for any real number r,

$$\{m < r\} = \{g < r\} + \{k < r\} .$$

To emphasize this important relation we adopt the notation $m = g \underset{t}{+} k$, and call $g \underset{t}{+} k$ the level addition (addition par tranches) of g and k.

This operation appears naturally in various branches of mathematics. For instante, some examples from mathematical economics can be found in ([16]).

At this point we have in mind another operation on functions obtained by adding strict epigraphs. Denoting by,

$$E_s(f) = \{(x, r) \in X \times \mathbb{R} : f(x) < r\}$$

the strict epigraph of a function $f \in \overline{\mathbb{R}}^X$, there exists a function $v \in \overline{\mathbb{R}}^X$ such that,

$$E_s(v) = E_s(g) + E_s(k) .$$

One recognizes here the usual infimal convolution, also called epi-sum, which is explicitly given by $v(z) = \inf\limits_{\substack{u \in X \\ x \in X}} (g(u) + k(x) : u + x = z)$, for any $z \in X$, with the usual convention $+\infty + (-\infty) = -\infty + (+\infty) = +\infty$. Such a function v can be viewed as the value function associated to the problem

$$\text{minimize} \quad h(x) + k(x) \quad \text{for} \quad x \in X .$$

In order to pursue the parallel between v and m we shall adopt in this paper the notation $g \underset{e}{+} k$ for the epi-sum of g and k (cf. [3]): $v = g \underset{e}{+} k$.

It is well known that properties like exactness, subdifferentiability, lower semicontinuity of the epi-sum are strongly related to the duality of the sum of convex functions. Here we address the same basic questions about the level sum: When is $g \underset{t}{+} k$ subdifferentiable? When is $g \underset{t}{+} k$ exact (i.e. the infimum is attained in (1) for any $z \in X$) ? When is $g \underset{t}{+} k$ l.s.c. ? Of course, these questions are linked with the duality for the supremum of g and k and reflect some important feature concerning the minimization problem $(\mathcal{P})$ and its perturbational dual. This is the reason why we have chosen to treat these questions under the light of the powerful theory of convex duality. Let us recall some usual facts about Fenchel conjugacy. To any function $f \in \overline{\mathbb{R}}^X$ (resp. $\varphi \in \overline{\mathbb{R}}^{X^*}$) is associated its Fenchel conjugate $f^* \in \overline{\mathbb{R}}^{X^*}$ (resp. $\varphi^* \in \overline{\mathbb{R}}^X$) which is defined for any $x^* \in X^*$ (resp. $x \in X$) by $f^*(x^*) = \sup\limits_{x \in X} (\langle x^*, x\rangle - f(x))$ (resp $\varphi^*(x) = \sup\limits_{x^* \in X^*} (\langle x^*, x\rangle - \varphi(x^*))$.

A function f (resp. φ), which is proper, coincides with its bi-conjugate f^{**} (resp. φ^{**}) iff it is convex and l.s.c. (resp. convex and weak* l.s.c.). We denote by $\Gamma_0(X)$ the set of convex proper l.s.c. functions on X. In the case when X is reflexive, $\varphi = \varphi^{**}$ amounts to the fact that φ is convex and l.s.c. for the dual norm on X^*, i.e. $\varphi \in \Gamma_0(X^*)$. In the sequel we use the following notation: The domain of f is the set $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$. The subdifferential of f at a point x where f is finite is given by

$\partial f(x) = \{x^* \in X^* : \forall u \in X \ , \ f(u) - f(x) \geq \ \langle x^* \ , \ u - x\rangle \ \}$. One says that f is subdifferentiable at x when $\partial f(x)$ is nonvoid. The cone generated by a nonvoid subset A of a linear space L is denoted by cone (A): cone $(A) = \{\lambda a : \lambda > 0 \ , \ a \in A\} = \underset{\lambda > 0}{\cup} \lambda A$. Note that $0 \in$ cone (A) iff $0 \in A$. If B is another nonvoid subset of L we set $A - B = A + (-B) = \{a - b : a \in A \ , \ b \in B\}$. We denote by I_A the indicator function of a subset $A \subset L : I_A(x) = 0$ if $x \in A, I_A(x) = +\infty$ if $x \in L \setminus A$. Let us recall the following result:

Theorem 1.1. ([2], Theorem I.1). *Let X be a Banach space with dual X^*, and let f_1, f_2 be two functions in $\Gamma_0(X)$ such that*

$$\text{cone}\,(\text{dom}\, f_1 - \ \text{dom}\, f_2) \quad \textit{is a closed linear space in} \quad X \ .$$

We then have, $(f_1 + f_2)^* = f_1^* \underset{e}{+} f_2^*$. In particular, the epi-sum $f_1^* \underset{e}{+} f_2^*$ is convex proper and weak l.s.c.. Moreover, $f_1^* \underset{e}{+} f_2^*$ is exact (at each point of X^*).*

2. Flashback on Convex Perturbational Duality in General Banach Spaces

The theorem below gives a condition ensuring the subdifferentiability of a marginal functional at a fixed point. It can be considered as a straightforward consequence of Theorem 1.1. It can also be deduced from [8] Proposition 3.1, or [20] Proposition 3 (see also [4]).

Theorem 2.1. *Let X and Z be two Banach spaces with respective dual X^* and Z^*, let $F \in \Gamma_0(X \times Z)$, $\overline{z} \in Z$, and let m be the marginal convex function defined on Z by $m(z) = \inf_{x \in X} F(x, z)$. Assume that*

$$\text{cone}\,(\overline{z} - \ \text{dom}\, m) \quad \textit{is a closed linear space} \ . \tag{2}$$

The convex function m is then l.s.c. at $\overline{z}$ and one has:

$$+\infty > m(\overline{z}) = - \min_{z^* \in Z^*} \left(- \langle z^*, \overline{z}\rangle \ + F^*(0, z^*)\right) .$$

Furthermore, if $m(\overline{z}) \in \mathbb{R}$, then m is subdifferentiable at $\overline{z}$.

In the next result we consider the case where the perturbation is defined on the product of two dual Banach spaces. By using Theorem 1.1 again, we obtain a qualification condition ensuring that the corresponding value functional is weak* l.s.c. and exact on the whole dual space.

Theorem 2.2. *Let X and Z be two Banach spaces with respective duals X^* and Z^*, and let G be a convex proper weak* l.s.c. function on $X^* \times Z^*$. Let m be defined on Z as follows, $z \in Z \longmapsto m(z) := \inf_{x \in X} G^*(x, z)$. Assume that*

$$\text{cone}\,(\text{dom}\, m) \quad \textit{is a closed linear space in} \quad Z \ .$$

Then the convex function, $x^* \in X^* \longmapsto n(x^*) := \inf_{z^* \in Z^*} G(x^*, z^*)$, is proper, weak l.s.c. and exact on X^*, and one has* $n(x^*) = \sup_{x \in X}(\langle x^*, x\rangle - G^*(x, 0))$ for any $x^* \in X^*$.

Proof. By setting $F = G^*$ (hence $F^* = G$) we have, for any $(x^*, z^*) \in X^* \times Z^*$,

$$n(x^*) = (F^* \underset{e}{+} I_{\{0\}\times Z^*})(x^*, z^*) = (F^* \underset{e}{+} (I_{X\times\{0\}})^*)(x^*, z^*). \tag{a}$$

Denoting by P the projection of $X \times Z$ onto Z we have, $P^{-1}(\operatorname{dom} m) = \operatorname{dom} F - X \times \{0\}$. Consequently, the cone generated by $\operatorname{dom} F - \operatorname{dom} I_{X\times\{0\}}$ is a closed linear space in $X \times Z$. In particular,

$$\operatorname{dom} F \cap (X \times \{0\}) \neq \emptyset. \tag{b}$$

From Theorem 1.1 we deduce that,

$$(F + I_{X\times\{0\}})^* = F^* \underset{e}{+} (I_{X\times\{0\}})^*, \tag{c}$$

and that the epi-sum in the right menber above is exact on $X^* \times Z^*$. It then follows from (a), (b), (c) that the convex function n is weak* l.s.c. and exact on X^* and it can't take the value $-\infty$. Moreover, if n were identically $+\infty$ one would have $G \underset{e}{+} I_{\{0\}\times Z^*} \equiv +\infty$, a contradiction with $\operatorname{dom} G \neq \emptyset$. The last formula of the theorem follows from (a) and (c). $\square$

Remark 2.3. For reflexive Banach spaces, the theorem 2.2 admits of course a dual version.

3. Subdifferentiability of the Level Sum of Two Convex Functions

Given two convex proper l.s.c. functions g and k on the Banach space X, let us introduce the function $(x, z) \in X \times X \longmapsto F(x, z) := g(z - x) \vee k(x)$. We then have $F \in \Gamma_0(X \times X)$, and the marginal function, $z \in X \longmapsto m(z) := \inf_{x\in X} F(x, z)$, coincides with the level sum of g and k: $m = g \underset{t}{+} k$. In particular, $\operatorname{dom} m = \operatorname{dom} g + \operatorname{dom} k$. As a consequence of Theorem 2.1 we can state:

Theorem 3.1. *Let X be a Banach space, let g, $k \in \Gamma_0(X)$, and let $\overline{z} \in X$. Assume that*

$$\operatorname{cone}(\operatorname{dom} g + \operatorname{dom} k - \overline{z}) \quad \textit{is a closed linear space}\ . \tag{3}$$

Then, $g \underset{t}{+} k$ is l.s.c. at $\overline{z}$. Furthermore, if $(g \underset{t}{+} k)(\overline{z}) \neq -\infty$, then $g \underset{t}{+} k$ is subdifferentiable at $\overline{z}$.

The next corollary is concerned with the case where the condition (3) is satisfied for all $\overline{z}$ in X:

Corollary 3.2. *Let X be a Banach space and let g and k be two functions in $\Gamma_0(X)$ such that*

$$\operatorname{dom} g + \operatorname{dom} k = X\ . \tag{4}$$

Then $g \underset{t}{+} k$ is l.s.c. on X.

Proof. It is enough to observe that (4) entails (3) for any $\overline{z}$ in X $\square$

Remark 3.3. The conditions (3) and (4) do not entail the exactness of the level sum $g \underset{t}{+} k$: let us take for instance $X = \mathbb{R}$, $g = 0$, $k(x) = e^x$; we then have $g \underset{t}{+} k = 0$ without exactness.

4. Lower Semicontinuity and Exactness of the Level Sum of Two Convex Functions

We first consider the level sum of two convex proper weak* l.s.c. function φ and ψ defined on the dual space X^* of the Banach space X. In order to apply Theorem 2.2 let us introduce the convex proper weak* l.s.c. function G defined on $X^* \times X^*$ by

$$G(x^*, z^*) = \varphi(x^* - z^*) \vee \psi(z^*) \tag{5}$$

for any $(x^*, z^*) \in X^* \times X^*$. Then the marginal function n of Theorem 2.2 is nothing but the level sum $\varphi \underset{t}{+} \psi$. We have to compute the Fenchel conjugate of G. To this end let us introduce some standard notation. For any $x^* \in X^*$ we set $(0\varphi)(x^*) = 0$ if $x^* \in \operatorname{dom}\varphi, +\infty$ if $x^* \in X^* \setminus \operatorname{dom}\varphi$. In this way, $(0\varphi)^*$ coincides with the recession function φ^*0 of φ^* ([11] Theorem 6.8.5). By choosing an arbitrary element a in $\operatorname{dom}\varphi^*$, we then have

$$(\varphi^*0)(x) = \sup_{t>0} \frac{\varphi^*(a+tx) - \varphi^*(a)}{t}, \text{ for any } x \in X.$$

For any positive real number α we set, for all $x \in X$, $(\varphi^*\alpha)(x) = \alpha\varphi^*(\frac{x}{\alpha})$. It follows that $(\alpha\varphi)^* = \varphi^*\alpha$, for any $\alpha \geq 0$.
Denoting by S the line segment, $S = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha \geq 0 \ , \ \beta \geq 0 \ , \ \alpha + \beta = 1\}$, we can now give the expression of the conjugate of the function G defined in (5):

Lemma 4.1. *For any* $(x, z) \in X \times X$,

$$G^*(x, z) = \min_{(\alpha,\beta)\in S} ((\varphi^*\alpha)(x) + (\psi^*\beta)(x+z)) .$$

Proof. A straightforward computation yields

$$G^*(x, z) = \sup_{\substack{x^* \in \operatorname{dom}\varphi \\ z^* \in \operatorname{dom}\psi}} (\langle x^* + z^* \ , \ x\rangle \ + \ \langle z^*, z\rangle \ - (\varphi(x^*) \vee \psi(z^*))) .$$

In the above expression we can replace $-(\varphi(x^*) \vee \psi(z^*))$ by $\min_{(\alpha,\beta)\in S} (-\alpha\varphi(x^*) - \beta\psi(z^*))$, so that:

$$G^*(x, z) = \sup_{\substack{x^* \in \operatorname{dom}\varphi \\ z^* \in \operatorname{dom}\psi}} \ \min_{(\alpha,\beta)\in S} (\langle x^* + z^* \ , \ x\rangle + \ \langle z^*, z\rangle - \alpha\varphi(x^*) - \beta\psi(z^*)) .$$

The classical mini-max Theorem (see e.g. [17] Theorem 4.2') allows us to interchange the "*sup*" and the "*min*" above: $G^*(x, z) = \min_{(\alpha,\beta)\in S} ((\alpha\varphi)^*(x) + (\beta\psi)^*(x+z))$. □

In order to apply Theorem 2.2 we also have to compute the cone generated by the domain of the marginal function,

$$z \in X \longmapsto m(z) = \inf_{x \in X} G^*(x,z) = \inf_{x \in X} \min_{(\alpha,\beta)\in S} ((\varphi^*\alpha)(x) + (\psi^*\beta)(x+z)) . \tag{6}$$

We begin with another lemma in which we denote by $b(A)$ the barrier cone of a nonvoid convex subset A of X^*: $b(A) = \{x \in X : \sup_{x^* \in A} \langle x^*, x\rangle \ < +\infty\}$.

Lemma 4.2. *For any proper function ξ on X^* one has:*

$$\text{cone}\,(\text{dom}\,\xi^*) + b(\text{dom}\,\xi) = \text{cone}\,(\text{dom}\,\xi^*) .$$

Proof. As $b(\text{dom}\,\xi)$ contains the origin, it suffices to prove the inclusion $\subset$. Let $x_1 \in \text{dom}\,\xi^*$, $\lambda > 0$, $x_2 \in b(\text{dom}\,\xi)$. We have to prove that $x := \lambda x_1 + x_2$ belongs to the cone generated by dom ξ^*. More precisely, we are going to prove that $\lambda^{-1}x$ belongs to dom ξ^*: $\xi^*(\lambda^{-1}x) = \xi^*(x_1 + \lambda^{-1}x_2) = \sup_{x^* \in \text{dom}\,\xi} (\langle x^*, x_1\rangle - \xi(x^*) + \lambda^{-1}\langle x^*, x_2\rangle)$. Hence

$\xi^*(\lambda^{-1}x) \le \xi^*(x_1) + \lambda^{-1} \sup_{x^* \in \text{dom}\,\xi} \langle x^*, x_2\rangle$. Now, $x_1 \in \text{dom}\,\xi^*$ and $x_2 \in b(\text{dom}\,\xi)$, so that $\xi^*(\lambda^{-1}x) \ < +\infty$. □

On the other hand, as a result of (6), it is not difficult to see that cone (dom m) can be written as the union of three convex cones (with vertex at the origin of X), namely,

$$\text{cone}(\text{dom}\,m) = C_1 \cup C_2 \cup C_3$$

where $C_1 = b(\text{dom}\,\psi) - \text{cone}(\text{dom}\,\varphi^*)$, $C_2 = \text{cone}(\text{dom}\,\psi^*) - \text{cone}(\text{dom}\,\varphi^*)$, $C_3 = \text{cone}(\text{dom}\,\psi^*) - b(\text{dom}\,\varphi)$. Since φ and ψ are assumed to be convex proper and weak* l.s.c. , these cones are nonvoid, but they do not necessarily contain the origin. Anyway, $C_1 \cup C_2 \cup C_3$ is included in the closure of the sum $C_1 + C_2 + C_3$. Now, by Lemma 4.2, one has

$$C_1 + C_2 + C_3 = \text{cone}(\text{dom}\,\psi^*) - \text{cone}(\text{dom}\,\varphi^*) = C_2 .$$

Therefore,

$$C_2 \subset \text{cone}(\text{dom}\,m) \subset \overline{C_1 + C_2 + C_3} = \overline{C_2} .$$

Forcing C_2 to be closed we then obtain:

$$\text{cone}(\text{dom}\,m) = \text{cone}(\text{dom}\,\psi^*) - \text{cone}(\text{dom}\,\varphi^*) .$$

Now we are in position to apply Theorem 2.2 in the case where the function G is given by (5):

Theorem 4.3. *Let φ and ψ be two convex proper weak* l.s.c. functions on the dual space X^* of a Banach space X. Assume that*

$$\text{cone}\,(\text{dom}\,\varphi^*) - \text{cone}\,(\text{dom}\,\psi^*) \quad \textit{is a closed linear space in} \quad X . \tag{7}$$

Then the level sum $\varphi \underset{t}{+} \psi$ is convex proper weak l.s.c. and exact on X^*.*

Remark 4.4. As in Theorem 2.2, there exists an obvious dual version of Theorem 4.3 for reflexive Banach spaces (that we shall use later on in the paper).

5. On the Condition "cone (dom φ^*) − cone (dom ψ^*) Closed Linear Space"

Let φ, ψ be two convex proper weak* l.s.c. functions on the dual space X^* of the Banach space X. Denoting by $(\mathrm{dom}\,\varphi^*)^-$ the negative polar cone of dom φ^* in X, we know that,

$$\{\varphi 0 \le 0\} = (\mathrm{dom}\,\varphi^*)^- = (\mathrm{cone}\,(\mathrm{dom}\,\varphi^*))^-$$

(remember that $\varphi 0$ is the recession function of φ). It follows that the condition (7) entails,

$$\{\varphi 0 \le 0\} \cap (-\{\psi 0 \le 0\}) \quad \text{is a linear space in} \quad X^* . \tag{8}$$

Notice that the above set is automatically weak* closed. In fact, one easily sees that (8) is equivalent to,

$$c\ell(\mathrm{cone}\,(\mathrm{dom}\,\varphi^*) - \mathrm{cone}\,(\mathrm{dom}\,\psi^*)) \quad \text{is a linear space} . \tag{9}$$

When X is a finite dimensional space, the conditions (7), (8), (9) are equivalent. This is due to the fact that in such spaces a convex subset which is dense is forced to be the whole set. We then have from Theorem 4.3 the following corollary:

Corollary 5.1. *Let φ and ψ be two functions in $\Gamma_0(\mathbb{R}^n)$ such that*

$$\{\varphi 0 \le 0\} \cap (-\{\psi 0 \le 0\}) \quad \textit{is a linear space.}$$

Then the level sum $\varphi \underset{t}{+} \psi$ belongs to $\Gamma_0(\mathbb{R}^n)$ and is exact on $\mathbb{R}^n$.

We now compare the condition (7) with the more familiar condition,

$$\mathrm{cone}\,(\mathrm{dom}\,\varphi^* - \mathrm{dom}\,\psi^*) \quad \text{is a closed linear space in} \quad X . \tag{10}$$

It is known that (10) is of interest since it implies the weak* lower semicontinuity and the exactness of the epi sum $\varphi \underset{e}{+} \psi$.

Here is an example where (7) is satisfied but not (10): let φ and ψ be defined on $\mathbb{R}$ by $\varphi(r) = r$, $\psi(r) = \frac{1}{2} r \vee 0$. On one hand we have, cone(dom φ^*) − cone(dom ψ^*) = $\mathbb{R}$ and, on the other hand, cone (dom φ^* − dom ψ^*) $=]0, +\infty[$. Here the level sum of φ and ψ is exact and we have, $(\varphi \underset{t}{+} \psi)(r) = r/3$ if $r \ge 0$, $(\varphi \underset{t}{+} \psi)(r) = 0$ if $r \le 0$. However, the epi sum $\varphi \underset{e}{+} \psi$ is identically $-\infty$. Next result furnishes an important class of functions for which the conditions (7) and (10) are equivalent:

Proposition 5.2. *Assume that the functions φ and ψ are bounded from below on X^*. The conditions (7) and (10) are then equivalent.*

Proof. Let us show that cone (dom φ^*) − cone (dom ψ^*) = cone (dom φ^* − dom ψ^*) .

The inclusion $\supset$ is clear. On the other hand, as φ and ψ are bounded from below we have $0 \in \operatorname{dom}\varphi^* \cap \operatorname{dom}\psi^*$. For any $\lambda x_1 - \mu x_2$ in cone $(\operatorname{dom}\varphi^*) -$ cone $(\operatorname{dom}\psi^*)$ we then have $u_1 := \frac{\lambda}{\lambda+\mu} x_1 \in \operatorname{dom}\varphi^*$ and $u_2 := \frac{\mu}{\lambda+\mu} x_2 \in \operatorname{dom}\psi^*$, so that $\lambda x_1 - \mu x_2 = (\lambda+\mu)(u_1 - u_2)$ belongs to cone $(\operatorname{dom}\varphi^* - \operatorname{dom}\psi^*)$. □

In the finite dimensional case, (10) is stronger than (7):

Proposition 5.3. *Let φ and ψ be defined on $\mathbb{R}^n$. Then condition* (10) *entails condition* (7).

Proof. Assume that (10) is satisfied. We have just to prove that cone $(\operatorname{dom}\varphi^*) -$ cone $(\operatorname{dom}\psi^*)$ is a linear space. To this end, let $\lambda x_1 - \mu x_2$ in the above set: it suffices to show that $\mu x_2 - \lambda x_1$ belongs to this set. Now there exist $\nu > 0$, $u_1 \in \operatorname{dom}\varphi^*$, $u_2 \in \operatorname{dom}\psi^*$ such that $(\lambda+\mu)(x_2 - x_1) = \nu(u_1 - u_2)$. We then have,

$$\begin{aligned}\mu x_2 - \lambda x_1 &= \mu x_1 + \nu u_1 - \lambda x_2 - \nu u_2 \\ &= (\mu+\nu)[\frac{\mu}{\mu+\nu}x_1 + \frac{\nu}{\mu+\nu}u_1] - (\lambda+\nu)[\frac{\lambda}{\lambda+\nu}x_2 + \frac{\nu}{\lambda+\nu}u_2].\end{aligned}$$

As a result of the convexity of the sets dom φ^* and dom ψ^*, the elements inside the brackets belong respectively to dom φ^* and dom ψ^*, and the proof is complete. □

6. Some Examples

Let us apply Theorem 4.3 in the case where the functions φ and ψ are the indicator functions of two nonvoid weak*-closed convex subsets C and D of $X^* : \varphi = I_C$, $\psi = I_D$. Here the level sum of φ and ψ coincides with the indicator of the vectorial sum $C + D$ of C and D: $I_C \underset{t}{+} I_D = I_{C+D}$.

The condition (7) involves the barrier cones of C and D and can be written

$$b(C) - b(D) \quad \text{is a closed linear space in} \quad X \,. \tag{11}$$

What Theorem 4.3 says in this case is that, under the condition (11), the set $C + D$ is weak*-closed. In this way we recapture the closedness criterion given in [19], Theorem 1. Let us now consider the symmetric case in which the functions φ and ψ in Theorem 4.3 are the support functions $\sigma_C = (I_C)^*$ and $\sigma_D = (I_D)^*$ of two nonvoid closed convex subsets C and D of X: for any $x^* \in X^*$, $\sigma_C(x^*) = \sup_{x\in C} \langle x^*, x\rangle$, $\sigma_D(x^*) = \sup_{x\in D} \langle x^*, x\rangle$.

The level sum $\varphi \underset{t}{+} \psi$ is then sublinear. Moreover, as a consequence of Theorem 4.3, we get:

Corollary 6.1. *Let C and D be two nonvoid closed convex subsets of the Banach space X with dual X^*. Assume that*

$$\text{cone}\,(C) - \text{cone}\,(D) \qquad \textit{is a closed linear space in} \quad X \,.$$

Then the level sum of the support functions of C and D is sublinear proper weak l.s.c. and exact on W.*

We end this section by considering the level sum of convex quadratic functionals. Let A and B be two bounded linear symmetric positive semi definite operators on an Hilbert space H, and let q_A and q_B be the corresponding convex quadratic forms: for any $x \in H$, $q_A(x) = \frac{1}{2}\langle Ax, x\rangle$, $q_B(x) = \frac{1}{2}\langle Bx, x\rangle$. It is known (see e.g. [5] Thm I.34,[12] Prop. II.1) that the domain of the conjugate of q_A coincides with the range of the square root $A^{\frac{1}{2}}$ of A. Now, A and $A^{\frac{1}{2}}$ are symmetric operators which admit the same kernel. Consequently, $R(A)$ and $R(A^{\frac{1}{2}})$ have the same closure; more precisely, $R(A) \subset R(A^{\frac{1}{2}}) \subset \overline{R(A)}$. We then have,

$$R(A) + R(B) \subset R(A^{\frac{1}{2}}) + R(B^{\frac{1}{2}}) \subset \overline{R(A) + R(B)}\,. \tag{12}$$

Now we can state:

Proposition 6.2. *Let q_A and q_B be two convex quadratic forms on an Hilbert space H. Assume that $R(A) + R(B)$ is closed. Then the level sum $q_A \underset{t}{+} q_B$ is convex, finite valued, continuous, and exact on H. In particular, the level sum of two convex quadratic forms on an euclidean space is convex, finite valued, and exact.*

Proof. We apply Theorem 4.3 with $\varphi = q_A$, $\psi = q_B$: this is possible for cone $(\operatorname{dom}\varphi^*)-$ cone $(\operatorname{dom}\psi^*) = R(A^{\frac{1}{2}}) - R(B^{\frac{1}{2}}) = R(A) + R(B)$ is closed (by virtue of (12) and as $R(A) + R(B)$ is closed by hypothesis). Therefore, $q_A \underset{t}{+} q_B$ belongs to $\Gamma_0(H)$ and is exact on H. As $\operatorname{dom}(q_A \underset{t}{+} q_B) = \operatorname{dom} q_A + \operatorname{dom} q_B = H$, $q_A \underset{t}{+} q_B$ is finite-valued. The continuity of $q_A \underset{t}{+} q_B$ follows from [6] Cor.2.5. □

7. Duality for the Problem of Minimizing the Max of Two Convex Functions

We return to the minimization problem $(\mathcal{P})$ mentioned in the introduction:

$$(\mathcal{P}): \ \text{minimize}\, h(x) \vee k(x) \qquad \text{for} \quad x \in X$$

where X is a Banach space (with dual X^*) and h, k belong to $\Gamma_0(X)$. By introducing the perturbation F $(x, z) \in X \times X \longmapsto F(x, z) = h(x - z) \vee k(x) = g(z - x) \vee k(x)$ the general scheme of convex duality ([14], [9], [6], [11], ...) leads to the dual problem:

$$(Q) \quad \text{maximize} \ - F^*(0, z^*) \quad \text{for} \quad z^* \in X^*\,.$$

By using Lemma 4.1 (or its dual version) we can write this dual problem as follows (recall that $g(x) = h(-x)$)

$$(Q) \quad \text{maximize} \quad - \min_{(\alpha,\beta)\in S} ((h^*\alpha)(-z^*) + (k^*\beta)(z^*)) \quad \text{for} \quad z^* \in X^*\,.$$

We denote by r (resp. s) the value of $(\mathcal{P})$ (resp. (Q)). By construction, the marginal function m associated with the perturbation F is nothing but $g \underset{t}{+} k$. We then have classically, $s = (g \underset{t}{+} k)^{**}(0) \leq (g \underset{t}{+} k)(0) = r$. Moreover, the set of dual optimal solution is known to coincide with $\partial(g \underset{t}{+} k)^{**}(0)$. With this in mind, it directely follows from Theorem 3.1 (with $\overline{z} = 0$) the existence of dual optimal solutions. More precisely:

Theorem 7.1. *Let X be a Banach space with dual X^*, let h and k be two functions in $\Gamma_0(X)$, and let r, s be as above. Assume that $r \in \mathbb{R}$ and that*

$$\text{cone}\,(\text{dom}\, h - \text{dom}\, k) \quad \textit{is a closed linear space} .$$

Then, $r = s$ and (Q) has optimal solutions: there exist $z^* \in X^*$, and $(\alpha, \beta) \in S$ such that

$$(h^*\alpha)(-z^*) + (k^*\beta)(z^*) = -s .$$

In the reflexive case, the dual version of Theorem 4.3 provides a condition ensuring the existence of optimal primal solutions together with a zero duality gap:

Theorem 7.2. *Let X be a reflexive Banach space and let h, k, r, s, be as above. Assume that,*

$$\text{cone}\,(\text{dom}\, h^*) + \text{cone}\,(\text{dom}\, k^*) \quad \textit{is a closed linear space.}$$

Then, $r = s \in \mathbb{R} \cup \{+\infty\}$ and $(\mathcal{P})$ has an optimal solution: there exists $x \in X$ such that

$$r = h(x) \vee k(x) .$$

References

[1] S. Abdulfattah, M. Soueycatt: Analyse épi/hypographique, Séminaire d'Analyse Convexe, Montpellier, Exposé no. 13, 1991.

[2] H. Attouch, H. Brézis: Duality for the sum of convex functions in general Banach spaces, Aspects of Math. and its Appl., J. A. Barroso ed., 125–133, North Holland, Amsterdam, 1986.

[3] H. Attouch, R. Wets: Epigraphical Analysis, Ann. Inst. Poincaré Analyse Non Linéaire 6 (1989) 73–100.

[4] D. Azé, Duality for the sum of convex functions in general normed spaces, Arkiv Math. 62 (1994) 554–561.

[5] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics no. 580, Springer-Verlag, 1977.

[6] I. Ekeland, R. Teman: Analyse Convexe et Problèmes variationnels, Dunod, 1974.

[7] A. Elqortobi: Inf-convolution quasi-convexe des fonctionnelles positives, Rech. Oper. 26 (1992) 301–311.

[8] V. Jeyakumar: Duality and infinite dimensional optimization, Nonlinear Anal. 15, 12 (1990) 1111–1122.

[9] J.-L. Joly, P.-J. Laurent: Stability and duality in convex minimization problems, R.I.R.O. 5ème année, R-2 (1971) 3–42.

[10] A.G. Kusraev, S.S. Kutateladze: Subdifferential Calculus, Nauka Publishing House, Novosibirsk, 1987 (In Russian).

[11] P.-J. Laurent: Approximation et optimisation, Herman, 1972.

[12] M.-L. Mazure: Equations de convolution et formes quadratiques, Ann. Mat. Pura ed Appl. (4) 158 (1991) 75–97.

[13] J.-J. Moreau: Fonctionnelles convexes,Collège de France, 1966.

[14] R.T. Rockafellar: Duality and stability in extremum problems involving convex functions, Pac. J. of Math. 21 (1967) 167–187.

[15] R.T. Rockafellar: Convex analysis, Princeton Univ. Press, Princeton, N.J., 1970.

[16] A. Seeger, M. Volle: On a convolution operation obtained by adding level sets : classical and new results, Rech. Oper. 29, 2 (1995) 131–154.

[17] M. Sion: On general minimax theorems, Pacific J. Math. 8 (1959) 171–176.

[18] M. Volle: Contributions à la dualité et à l'épi-convergence, Thèse d'Etat, Université de Pau, 1986.

[19] M. Volle: Some applications of the Attouch-Brezis condition to closedness criterions, optimization, and duality, Séminaire d'Analyse Convexe de Montpellier, 1992, Exposé no. 16.

[20] C. Zalinescu: On some open problems in convex analysis, Archiv der Math. 53 (1992) 566–571.

HIER :

Leere Seite
152