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On a Gap Phenomenon for Isoperimetrically Constrained Variational Problems

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Dedicated to R. T. Rockafellar on his 60th Birthday

We consider functionals of the calculus of variations subjected to constraints of the form

$$\int_\Omega g(x, u)\, dx = 1.$$

We identify the relaxed problem and we show that, when a lack of compactness occurs, the constraint may relax to a gap term.

1. Introduction

The term *Lavrentiev phenomenon* refers to a surprising result first demonstrated by Lavrentiev in 1926 [29]. There he presented an example showing that it is possible for the variational integral of a two-point Lagrange problem which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of C^1 admissible functions that *strictly exceeds* its minimum value on the full admissible class. The global C^1 regularity constraint on the admissible functions was thereby shown to incur an *infimum gap* in comparison with the relaxed problem in which this constraint has been removed. Since that time there have been additional works devoted to analyzing this gap phenomenon (see References), of which the paper [12] is most closely related to the work we present here.
The present article was stimulated by a result of Lezenina and Sobolevskii [30] in which a gap phenomenon having some analogy to the Lavrentiev phenomenon for free problems was encountered in an isoperimetric variational problem associated with a singular elliptic equation. We here demonstrate that for a large class of isoperimetrically constrained

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variational problems there is such a gap phenomenon, which has a natural interpretation as a relaxation effect, as was shown in Buttazzo and Mizel [12] to be also true of free problems for which the Lavrentiev phenomenon occurs. We prefer here to consider only the gap effect deriving from the *lack of compactness* and so, to avoid possible interactions with the gap due to regularity of admissible functions, we consider our problems as defined on the whole class of absolutely continuous functions. The result is that, after relaxation, the initial constraint becomes a kind of penalization term which, in several cases, can be explicitly computed.

2. The Relaxation Result

Let Ω be the interval $]0,1[$; we consider

$W^{1,1}(\Omega)$ the space of all absolutely continuous functions on Ω;

$\mathcal{A}$ the class of all functions $u \in W^{1,1}(\Omega)$ with $u(0)=0$;

$f(x,s,\zeta)$ a nonnegative Borel function from $\Omega \times \mathbb{R} \times \mathbb{R}$ into $\overline{\mathbb{R}}$ which is lower semicontinuous in (s,ζ) and convex in ζ;

$g(x,s)$ a nonnegative Borel function from $\Omega \times \mathbb{R}$ into $\overline{\mathbb{R}}$ which is lower semicontinuous in s.

For every $u \in \mathcal{A}$ we define

$$F(u) = \int_\Omega f(x,u,u')\,dx$$

$$G(u) = \int_\Omega g(x,u)\,dx$$

$$H(u) = \begin{cases} F(u) & \text{if } G(u)=1 \\ +\infty & \text{otherwise} \end{cases}$$

and we denote by $\overline{H}$ the greatest functional on $\mathcal{A}$ which is sequentially lower semicontinuous with respect to weak $W^{1,1}_{loc}(\Omega)$ convergence and less than or equal to H. By the assumptions made on the integrand f, the functional F turns out to be sequentially lower semicontinuous, so that

$$\overline{H}(u) \geq F(u) \qquad \forall u \in \mathcal{A}.$$

On the other hand, the equality $\overline{H} = H$ is possible only if the constraint $G(u)=1$ is preserved in the relaxation, which does not occur in general when the integrand $g(x,s)$ has singularities which prevent the compactness of embeddings. We want to write $\overline{H}$ in the form

$$\overline{H}(u) = F(u) + L(u)$$

and to characterize the gap L explicitly. It is clear that, when the compactness condition

$$u_h \to u \text{ in } w-W^{1,1}_{loc}(\Omega),\ F(u_h) \leq c \quad \Rightarrow \quad G(u_h) \to G(u)$$

is fulfilled, then $\overline{H} = H$, that is

$$L(u) = \begin{cases} 0 & \text{if } G(u)=1 \\ +\infty & \text{otherwise.} \end{cases}$$

In general, however, only the inequality $G(u) \leq 1$ holds in the relaxed functional, by Fatou's lemma. Thus $L(u) = +\infty$ whenever $G(u) > 1$, $L(u) = 0$ whenever $G(u) = 1$, but $L(u)$ may take finite nonzero values on functions $u \in \mathcal{A}$ such that $G(u) < 1$.
In order to characterize explicitly the gap functional L we introduce the following notations:

$$\begin{aligned} &F_x(u) = \int_0^x f(t, u, u')\, dt \\ &G_x(u) = z_u(x) = \int_0^x g(t, u)\, dt \\ &V(x, s, z) = \inf\{F_x(u) \;:\; u \in \mathcal{A},\; u(x) = s,\; G_x(u) = z\} \\ \\ &W(x, s, z) = \liminf_{(\xi,\eta)\to(s,z)} V(x, \xi, \eta). \end{aligned}$$

The representation result for L is then the following.

Theorem 2.1. *Assume the following conditions are fulfilled:*
(2.1) *for every $\delta > 0$ there exists a function $a_\delta \in L^1(\Omega)$ and a function $\theta_\delta : \mathbb{R} \to \mathbb{R}$ with $\theta_\delta(r)/r \to +\infty$ as $r \to +\infty$ such that*

$$f(x, s, \zeta) \geq \theta_\delta(|\zeta|) - a_\delta(x) \qquad \forall x \in]\delta, 1[,\ \forall s \in \mathbb{R},\ \forall \zeta \in \mathbb{R};$$

(2.2) *$g(x, s)$ is continuous in s, and for every $\delta > 0$ there exists a function $\gamma_\delta(x, t)$ increasing in t and integrable in x such that*

$$g(x, s) \leq \gamma_\delta(x, |s|) \qquad \forall x \in]\delta, 1[,\ \forall s \in \mathbb{R}.$$

Then for every $u \in \mathcal{A}$ with $G(u) \leq 1$ we have

$$L(u) \geq \limsup_{x\to 0} W\left(x, u(x), 1 - \int_x^1 g(t, u)\, dt\right) \tag{2.3}$$

$$L(u) \leq \liminf_{x\to 0} V\left(x, u(x), 1 - \int_x^1 g(t, u)\, dt\right). \tag{2.4}$$

Hence

$$L(u) = \lim_{x\to 0} V\left(x, u(x), 1 - \int_x^1 g(t, u)\, dt\right)$$

if $V(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R} \times \mathbb{R}$.

Proof. Consider any sequence (u_h) in $\mathcal{A}$ with $G(u_h) = 1$ and $u_h \to u$ weakly in $W^{1,1}_{loc}(\Omega)$; then for every $\delta > 0$ we have

$$\begin{aligned} F(u_h) &= \int_0^\delta f(x, u_h, u_h')\, dx + \int_\delta^1 f(x, u_h, u_h')\, dx \geq \\ &\geq V\left(\delta, u_h(\delta), 1 - \int_\delta^1 g(x, u_h)\, dx\right) + \int_\delta^1 f(x, u_h, u_h')\, dx \geq \\ &\geq W\left(\delta, u_h(\delta), 1 - \int_\delta^1 g(x, u_h)\, dx\right) + \int_\delta^1 f(x, u_h, u_h')\, dx. \end{aligned}$$

Passing to the limit as $h \to +\infty$ gives, by using (2.1) and (2.2)

$$\overline{H}(u) \geq W\left(\delta, u(\delta), 1 - \int_\delta^1 g(x,u)\,dx\right) + \int_\delta^1 f(x,u,u')\,dx$$

and, as $\delta \to 0$,

$$\overline{H}(u) \geq \limsup_{\delta \to 0} W\left(\delta, u(\delta), 1 - \int_\delta^1 g(x,u)\,dx\right) + F(u)$$

so that inequality (2.3) is proved.

In order to prove inequality (2.4) let $x_h \to 0$ be such that

$$\liminf_{x \to 0} V\left(x, u(x), 1 - \int_x^1 g(t,u)\,dt\right) = \lim_{h \to +\infty} V\left(x_h, u(x_h), 1 - \int_{x_h}^1 g(t,u)\,dt\right) < +\infty \tag{2.5}$$

and let $v_h \in \mathcal{A}$ be such that

$$v_h(x_h) = u(x_h), \qquad \int_0^{x_h} g(t, v_h)\,dt = 1 - \int_{x_h}^1 g(t,u)\,dt \tag{2.6}$$

$$\int_0^{x_h} f(t, v_h, v_h')\,dt \leq \frac{1}{h} + V\left(x_h, u(x_h), 1 - \int_{x_h}^1 g(t,u)\,dt\right). \tag{2.7}$$

Define

$$u_h(x) = \begin{cases} v_h(x) & \text{if } x \in]0, x_h[\\ u(x) & \text{if } x \in [x_h, 1[; \end{cases}$$

we have $u_h \in \mathcal{A}$, $G(u_h) = 1$ by (2.6), and $u_h \to u$ weakly in $W^{1,1}_{loc}(\Omega)$ by (2.1). Furthermore, by (2.5) and (2.7)

$$\begin{aligned} \overline{H}(u) &\leq \liminf_{h \to +\infty} F(u_h) = \\ &= \liminf_{h \to +\infty} \left[\int_0^{x_h} f(t, v_h, v_h')\,dt + \int_{x_h}^1 f(t,u,u')\,dt\right] \leq \\ &\leq \liminf_{h \to +\infty} \left[\frac{1}{h} + V\left(x_h, u(x_h), 1 - \int_{x_h}^1 g(t,u)\,dt\right) + F(u)\right] = \\ &= \liminf_{x \to 0} V\left(x, u(x), 1 - \int_x^1 g(t,u)\,dt\right) + F(u). \end{aligned}$$

Hence inequality (2.4) is also proved. □

Remark 2.2. Similar results with similar proofs hold if we replace the class $\mathcal{A}$ with $W^{1,1}(\Omega)$ or $W^{1,1}_0(\Omega)$.

Remark 2.3. The assumptions of Theorem 2.1 ensure that the lack of compactness may occur only at the origin. In a similar way we can treat problems in which the lack of compactness occurs at a finite number of points, but we do not know the form of the relaxed functional $\overline{H}$ when the function $g(x,s)$ may allow more general "singular sets".

3. The Hamilton-Jacobi Approach

In order to provide an explicit computation, in some cases, of the gap $L(u)$, it is useful to present a dynamic programming type of result by studying the link between the value function $V(x,s,z)$ and the solutions of the associated Hamilton-Jacobi equation. In the following we denote by $f^*(x,s,\cdot)$ the Fenchel conjugate of $f(x,s,\cdot)$. We will also suppose that

$$g(x,s) > 0 \qquad \text{for a. e. } x \in \Omega \text{ and for all } s \neq 0; \tag{3.1}$$

this implies that for every $x \in \Omega$ and $s \neq 0$

$$V(x,s,0^+) = +\infty.$$

Theorem 3.1. *Suppose that $U(t,s,z)$ is a positive C^1 solution on $D = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ of the Hamilton-Jacobi equation*

$$\begin{cases} U_t + g(t,s)U_z + f^*(t,s,U_s) = 0 \\ U(t,s,0^+) = +\infty \end{cases} \tag{3.2}$$

such that

$$\lim_{\substack{(t,s,z)\to(0,0,0)\\(t,s,z)\in D_j}} U(t,s,z) = 0 \qquad \forall j \in \mathbb{N} \tag{3.3}$$

where (D_j) is an expanding sequence of bounded subdomains of D such that $(0,0,0) \in \overline{D_j}$, U is C^1-bounded on D_j, and $\cup_j D_j = D$.
Putting

$$\mathcal{B} = \left\{ u \in \mathcal{A} \ : \ \liminf_{x\to 0} U(x,u(x),z_u(x)) = 0 \right\},$$

one has the following inequality for every $u \in \mathcal{B}$

$$F_x(u) \geq U(x,u(x),z_u(x)) \qquad \forall x \in [0,1]. \tag{3.4}$$

Furthermore, if for some $(x,s,z) \in D$ there is a $u \in \mathcal{B}$ such that $u(x) = s$, $z_u(x) = z$, and (3.4) holds with equality, then u is a minimizer for the problem

$$\min\{F_x(u) \ : \ u \in \mathcal{B},\ u(x) = s,\ z_u(x) = z\}. \tag{$\mathcal{P}_{x,s,z}$}$$

Proof. Fix $u \in \mathcal{B}$; then for every x, y with $0 < y < x \leq 1$ the graph $(t,u(t),z_u(t))$, $t \in [y,x]$ lies in D_j for some $j \in \mathbb{N}$. Then by the chain rule for the function U on $D_j \cap ([y,x] \times \mathbb{R}^+ \times \mathbb{R}^+)$ we get

$$U(x,u(x),z_u(x)) - U(y,u(y),z_u(y)) = \int_y^x [U_t + g(t,u(t))U_z + u'(t)U_s]\,dt.$$

Hence, by approaching the liminf as $y \to 0^+$ and using the definition of $\mathcal{B}$ and property (3.3) we can write

$$U(x, u(x), z_u(x)) - F_x(u) = \int_0^x [U_t + g(t, u)U_z + u'U_s - f(t, u, u')]\, dt.$$

Now, by the Hamilton-Jacobi equation (3.2), the integrand is nonpositive everywhere on $]0, x]$, whence (3.4) follows. The conclusion that u is a minimizer of problem $(\mathcal{P}_{x,s,z})$ when equality holds in (3.4) is now immediate. □

Remark 3.2. When problem (3.2) has multiple solutions the above result applies to the maximal solution, in particular, as the only candidate for the value function of the problem.

4. Some Examples

In this section we present some examples of isoperimetrically constrained one-dimensional variational problems possessing a gap. We start by proving a homogeneity property of the value functions Y of $(\mathcal{P}_{x,s,z})$ and V of $(\hat{\mathcal{P}}_{x,s,z})$, the analogue of $(\mathcal{P}_{x,s,z})$ on the full admissible class $\mathcal{A}$.

Theorem 4.1. *Suppose that the integrands f and g satisfying* (2.1) *and* (2.2) *have the form*

$$f(t, s, \zeta) = t^\alpha |s|^\gamma h\left(\frac{t\zeta}{s}\right) \qquad \forall s \neq 0$$

$$g(t, s) = t^\beta |s|^\delta$$

with $\alpha, \beta \in \mathbb{R}$ and $\gamma, \delta > 0$. Then the value function V associated with the problems $(\hat{\mathcal{P}}_{x,s,z})$ satisfies

$$V(x, s, z) = x^{\alpha+1}|s|^\gamma V(1, 1, zx^{-\beta-1}|s|^{-\delta}) \qquad \forall s \neq 0. \tag{4.1}$$

Likewise, the value function Y associated with $(\mathcal{P}_{x,s,z})$ satisfies

$$Y(x, s, z) = x^{\alpha+1}|s|^\gamma Y(1, 1, zx^{-\beta-1}|s|^{-\delta}) \qquad \forall s \neq 0.$$

Proof. Given $u \in \mathcal{A}$ with $u(x) = s$ and $z_u(x) = z$ define $v(t) = u(tx)$ so that

$$v'(t) = xu'(tx), \qquad z_v(1) = x^{-\beta-1} z_u(x), \qquad F_1(v) = x^{-\alpha-1} F_x(u).$$

It follows from these relations that

$$V(x, s, z) = x^{\alpha+1} V(1, s, zx^{-\beta-1}). \tag{4.2}$$

On the other hand, given $\lambda \in \mathbb{R}$ define $w(t) = \lambda u(t)$ so that

$$w'(t) = \lambda u'(t), \qquad z_w(x) = |\lambda|^\delta z_u(x), \qquad F_x(w) = |\lambda|^\gamma F_x(u).$$

These relations yield

$$V(x,s,z) = |\lambda|^{-\gamma} V(x, \lambda s, z|\lambda|^{\delta}). \tag{4.3}$$

The validity of (4.1) now follows from (4.2) and (4.3). The proof for Y is similar. □

As a consequence of Theorems 3.1 and 4.1 we obtain the following result.

Theorem 4.2. *Let f and g be as in* Theorem 4.1. *Then every solution U of* (3.2) *having the form*

$$U(t,s,z) = t^{\alpha+1} s^{\gamma} R(zt^{-\beta-1} s^{-\delta})$$

for some positive function $R \in C^1(\mathbb{R}^+)$, is such that R is a positive solution of the ordinary differential equation

$$\begin{cases} (\alpha+1)R(w) - (\beta+1)wR'(w) + R'(w) + h^*(\gamma R(w) - \delta w R'(w)) = 0 \\ R(0^+) = +\infty. \end{cases} \tag{4.4}$$

In particular, for every $u \in \mathcal{B}$ satisfying $u(x) = s$ and $z_u(x) = z$ one has

$$F_x(u) \geq x^{\alpha+1} |s|^{\gamma} R(zx^{-\beta-1}|s|^{-\delta}),$$

and equality ensures that u is a minimizer for $(\mathcal{P}_{x,s,z})$.

Corollary 4.3. *Let f and g be as in* Theorem 4.1 *with $\alpha = \beta$, $\gamma = \delta$ and assume that $h \in C^1(\mathbb{R})$, is strictly convex, and satisfies $h(\zeta)/\zeta \to +\infty$ as $|\zeta| \to +\infty$. Then equation* (4.4) *becomes*

$$\begin{cases} h^*\Big(\gamma(R(w) - wR'(w))\Big) + (\alpha+1)(R(w) - wR'(w)) + R'(w) = 0 \\ R(0^+) = +\infty. \end{cases} \tag{4.5}$$

The maximal solution of (4.5) *on $]0, +\infty[$ is given by the convex function*

$$R(w) = wh\left(\frac{1}{\gamma w} - \frac{\alpha+1}{\gamma}\right).$$

corresponding to the C^1 solution of (3.2)

$$U(x,s,z) = zh\left(\frac{s^{\gamma} x^{\alpha+1}}{\gamma z} - \frac{\alpha+1}{\gamma}\right). \tag{4.6}$$

Moreover, the unique $\mathcal{B}$-optimal trajectory (u, z_u) for $(\mathcal{P}_{1,s,z})$ corresponding to the above solution is given by

$$u(t) = st^{(|s|^{\gamma}/z - \alpha - 1)/\gamma}, \qquad z_u(t) = zt^{|s|^{\gamma}/z}. \tag{4.7}$$

Finally, if for some $a > 0$

$$\frac{1}{th^{-1}(t)} \in L^1(a, +\infty) \tag{4.8}$$

where h^{-1} denotes the inverse of the restriction of h to some interval $[A,+\infty[$ on which it is strictly monotone, then

$$V(x,s,z) = Y(x,s,z) = U(x,s,z)$$

and the trajectory in (4.7) *is* $(\hat{\mathcal{P}}_{1,s,z})$ *optimal.*

Proof. By virtue of superlinearity and strict convexity of h, we have for every ζ

$$h^*(h'(\zeta)) = \zeta h'(\zeta) - h(\zeta) \qquad \text{and} \qquad (h^*)'(\zeta) = (h')^{-1}(\zeta) \tag{4.9}$$

so that the supremum yielding $h^*\Big(\gamma(R(w) - wR'(w))\Big)$ in (4.5) is attained at (assuming existence of a minimizer u for problem $(\mathcal{P}_{1,s,z})$)

$$\frac{tu'(t)}{u(t)} = (h')^{-1}\Big(\gamma(R(w) - wR'(w))\Big) = (h^*)'\Big(\gamma(R(w) - wR'(w))\Big). \tag{4.10}$$

On the other hand, by formally differentiating the implicit Clairaut type equation (4.5) one obtains (we recall that, by Rockafellar [36], Theorem 26.5, h^* is C^1)

$$R''(w)\Big[1 - (\alpha+1)w - \gamma w (h^*)'\Big(\gamma(R(w) - wR'(w))\Big)\Big] = 0.$$

Since $R'' = 0$ is inconsistent with $R(0^+) = +\infty$, we have

$$(h^*)'\Big(\gamma(R(w) - wR'(w))\Big) = \frac{1}{\gamma w} - \frac{\alpha+1}{\gamma}. \tag{4.11}$$

Then, from (4.10) and (4.11) we obtain

$$u'(t) = \frac{u(t)}{\gamma t}\left(\frac{1}{w} - \alpha - 1\right). \tag{4.12}$$

Moreover, inserting (4.11) into (4.5), and taking into account (4.9) gives as singular solution the convex function

$$R(w) = wh\left(\frac{1}{\gamma w} - \frac{\alpha+1}{\gamma}\right)$$

which proves (4.6) by Theorem 4.2. Now, recalling that $w = zt^{-1-\alpha}|s|^{-\gamma}$, while $z_u'(t) = g(t,u(t)) = t^\alpha |u(t)|^\gamma$, we obtain by (4.12)

$$\begin{aligned} w'(t) &= z_u' t^{-\alpha-1}|u|^{-\gamma} - (\alpha+1)z_u t^{-\alpha-2}|u|^{-\gamma} - \gamma z_u t^{-\alpha-1}|u|^{-\gamma-1}\operatorname{sgn}(u)u' = \\ &= \frac{1}{t} - \frac{\alpha+1}{t}w - \frac{\gamma w}{t}\left(\frac{1}{\gamma w} - \frac{\alpha+1}{\gamma}\right) = 0. \end{aligned}$$

That is, w is constant along an optimal trajectory. In particular, $w(t) = w(1) = z|s|^{-\gamma}$, so that (4.7) follows from (4.12).

Finally, we show that when (4.8) is satisfied, then $Y = V$ by proving that for each $u \in \mathcal{A} \setminus \mathcal{B}$ we have $z_u(x) = +\infty$ for every $x \in \Omega$. By definition of $\mathcal{B}$ it follows that for some positive b, σ one has

$$U(x, u(x), z_u(x)) = z_u(x) h \left(\frac{x^{\alpha+1}|u(x)|^\gamma}{\gamma z_u(x)} - \frac{\alpha+1}{\gamma} \right) \geq b \qquad \forall x \in]0, \sigma[.$$

If we suppose $z_u(1) < +\infty$, then by taking σ sufficiently small, we can ensure that the argument of h lies in the semi-axis $[A, +\infty[$ where h is invertible. Thus, since $z_u'(x) = x^\alpha |u(x)|^\gamma$, we have the relation

$$\frac{x z_u'(x)}{z_u(x)} \geq \gamma h^{-1} \left(\frac{b}{z_u(x)} \right) + \alpha + 1 \qquad \forall x \in]0, \sigma[.$$

Since $z_u(0^+) = 0$ and h is superlinear, we may deduce from this (by decreasing σ) that

$$\frac{z_u'(x)}{z_u(x) h^{-1}(b/z_u(x))} \geq \frac{\gamma}{2x} \qquad \forall x \in [0, \sigma[. \tag{4.13}$$

Denoting by $k(z_u)$ the coefficient of z_u' in (4.13) we have by (4.8) that $k \in L^1(0, \varepsilon)$ for ε sufficiently small. Letting $K(t) = \int_0^t k(s)\, ds$, it follows from the local absolute continuity of z_u by the chain rule (cf. e.g. [34]) and from (4.13) that

$$K(z_u(x)) - K(z_u(y)) \geq \frac{\gamma}{2} \log(x/y) \qquad 0 < y < x < \sigma$$

which gives, as $y \to 0^+$, $z_u(x) = +\infty$ for every x small enough. □

We now examine a special subclass of the integrands discussed in Corollary 4.3 for which an explicit estimate is available.

Corollary 4.4. *Let $p > n$ and let, with the notation of* Corollary 4.3, *$h(\zeta) = |\zeta|^p$, $\alpha = n - 1 - p$, $\gamma = p$, so that*

$$f(t, \zeta) = t^{n-1} |\zeta|^p, \qquad g(t, s) = t^{n-p-1} |s|^p.$$

Consider the variational problem

$$\min \{ F_1(u) \ : \ u \in \mathcal{B},\ u(1) = s,\ z_u(1) = z \}. \tag{$\mathcal{P}_{1,s,z}$}$$

Then an optimal trajectory is given by

$$u(t) = s t^{(|s|^p/z - n + p)/p}, \qquad z_u(t) = z t^{|s|^p/z}$$

corresponding to a minimal cost given by

$$V(x, s, z) = Y(x, s, z) = \frac{z}{p^p} \left| \frac{|s|^p}{z x^{p-n}} - n + p \right|^p, \qquad (x, s, z) \in D. \tag{4.14}$$

In particular,

$$L(u) = \left(1 - \frac{n}{p}\right)^p (1 - G(u)).$$

Proof. The expression of $L(u)$ follows from (4.14) by noticing that, thanks to the implication

$$w/x \in L^1(0,1), \qquad w' \in L^1(0,1) \quad \Rightarrow \quad \lim_{x\to 0} w(x) = 0,$$

we have, with $w(x) = x^{n-p}|u(x)|^p$,

$$\lim_{x\to 0} \frac{|u(x)|^p}{x^{p-n}} = 0.$$

□

Corollary 4.5. *Let $d(t) = \operatorname{dist}(t, \partial\Omega)$ and consider the problem*

$$\min\left\{ \int_0^1 |u'|^p\, dt \;:\; u \in W_0^{1,1}(0,1),\; \int_0^1 \frac{|u(t)|^p}{|d(t)|^p}\, dt = z \right\}. \qquad (\mathcal{P}_{1,z})$$

Then, if V is the value function of Corollary 4.4 *with $n = 1$, the infimum m is given by $m = 2V(1/2, 0^+, z/2)$ and there is a minimizing sequence which converges to $u_0 \equiv 0$. Therefore*

$$L(u) = \left(1 - \frac{1}{p}\right)^p (1 - G(u)).$$

Proof. By the symmetry in t the infimum m can be obtained as

$$\begin{aligned} m &= \inf_{s,u,v} \left\{ \int_0^{1/2} |u'|^p\, dt + \int_0^{1/2} |v'|^p\, dt \;:\; u, v \in \Gamma_s \right\} = \\ &= \inf_{s,z_1,z_2} \{ V(1/2, s, z_1) + V(1/2, s, z_2) \;:\; z_1 + z_2 = z \} \end{aligned}$$

where Γ_s is the class of functions such that $u(0) = v(0) = 0$, $u(1/2) = v(1/2) = s$, $z_u(1/2) + z_v(1/2) = z$. By the convexity of $V(x, s, \cdot)$ we get $z_1 = z_2 = z/2$ so that

$$m = \inf_s 2V(1/2, s, z/2) = 2V(1/2, 0^+, z/2) = (1 - 1/p)^p z$$

and a minimizing sequence (as $s \to 0^+$) is

$$u_s(x) = \begin{cases} s(2x)^{(2^{-p}s^p/z + p - 1)/p} & \text{if } 0 \le x \le 1/2 \\ s(2-2x)^{(2^{-p}s^p/z + p - 1)/p} & \text{if } 1/2 \le x \le 1. \end{cases}$$

□

Theorem 4.6. *Let f and $\tilde{f}$ be integrands satisfying* (2.1)*, and let g and $\tilde{g}$ be integrands satisfying* (2.2)*. Suppose furthermore that as $t \to 0$ we have $f \approx \tilde{f}$ and $g \approx \tilde{g}$ in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that (putting $a_\varepsilon = 1 + \varepsilon$)*

$$\frac{\tilde{f}(t,s,\zeta)}{f(t,s,\zeta)}, \ \frac{\tilde{g}(t,s)}{g(t,s)} \in \,]a_\varepsilon^{-1}, a_\varepsilon[\tag{4.14}$$

for every $x \in]0,\delta[$ and every $s, \zeta \in \mathbb{R} \setminus \{0\}$. Suppose in addition that the value function V corresponding to f and g is lower semicontinuous in (s,ζ) and satisfies

$$|V(x,s,z_1) - V(x,s,z_2)| \leq \omega(|z_1 - z_2|)k(x,s) \tag{4.15}$$

where ω is a continuity modulus and $k(x,u(x))$ is bounded for every u such that $F(u) < +\infty$. Then for every $u \in \mathcal{A}$ satisfying

$$\tilde{G}(u) = G(u) < 1$$

we have that the gap terms relative to f, g and to $\tilde{f}, \tilde{g}$ verify

$$\tilde{L}(u) \geq L(u) \qquad \forall u \in \mathcal{A}.$$

Proof. Given $\varepsilon > 0$ it follows from (4.14) that for any sequence (u_h) in $\mathcal{A}$ one has

$$\begin{cases} a_\varepsilon^{-1} F_x(u_h) \leq \tilde{F}_x(u_h) \leq a_\varepsilon F_x(u_h) \\ a_\varepsilon^{-1} G_x(u_h) \leq \tilde{G}_x(u_h) \leq a_\varepsilon G_x(u_h) \end{cases} \tag{4.16}$$

for every $x < \delta$. Now, given (x,s,z) select $u_h \in \mathcal{A}$ with $u_h(x) = s$, $\tilde{G}_x(u_h) = z$, and $\tilde{F}_x(u_h) \to \tilde{V}(x,s,z)$. Possibly passing to subsequences we may assume that $G_x(u_h) = z_h \to z'$ and that $F_x(u_h)$ converges as well. By use of (4.16) we obtain easily $a_\varepsilon^{-1} z \leq z' \leq a_\varepsilon z$ and

$$V(x,s,z') \leq a_\varepsilon \tilde{V}(x,s,z). \tag{4.17}$$

Now, consider a sequence $(s_h, z_h) \to (s,z)$ for which $\tilde{V}(x,s_h,z_h) \to \tilde{W}(x,s,z)$ and let us denote by z'_h the associated points for the first term of (4.17). We can suppose, by passing to subsequences, that $z'_h \to z''$. Then we obtain $a_\varepsilon^{-1} z \leq z'' \leq a_\varepsilon z$ and

$$\tilde{W}(x,s,z) \geq a_\varepsilon^{-1} V(x,s,z'').$$

By Theorem 2.1 and (4.15) we obtain finally

$$\begin{aligned} \tilde{L}(u) &\geq \limsup_{x\to 0} \tilde{W}\left(x, u(x), 1 - \int_x^1 \tilde{g}(t,u)\,dt\right) \geq \\ &\geq \liminf_{x\to 0} a_\varepsilon^{-1} \left[V\left(x, u(x), 1 - \int_x^1 g(t,u)\,dt\right) - \omega(\varepsilon)k(x,u(x)) \right] \geq \\ &\geq a_\varepsilon^{-1}[L(u) - K\omega(\varepsilon)]. \end{aligned}$$

The conclusion now follows by taking $\varepsilon \to 0$. □

As an application consider the following generalized version of the Lezenina and Sobolevskii problem on the unit ball B of $\mathbb{R}^n$, where $p > 1$ and

$$F(u) = \int_B |Du|^p \, dx, \qquad G(u) = \int_B (1-|x|)^{-p} |u|^p \, dx.$$

This leads in the radially symmetric case to

$$F(u) = \int_0^1 r^{n-1} |u'|^p \, dr, \qquad G(u) = \int_0^1 \frac{r^{n-1}}{(1-r)^p} |u|^p \, dr.$$

On putting $t = 1 - r$ and considering u as a function of t we are led to

$$F(u) = \int_0^1 (1-t)^{n-1} |u'|^p \, dt, \qquad G(u) = \int_0^1 (1-t)^{n-1} t^{-p} |u|^p \, dt.$$

Here the integrands

$$\tilde{f}(t, s, \zeta) = (1-t)^{n-1} |\zeta|^p, \qquad \tilde{g}(t, s) = (1-t)^{n-1} t^{-p} |s|^p$$

are asymptotically equivalent to the integrands

$$f(t, s, \zeta) = |\zeta|^p, \qquad g(t, s) = t^{-p} |s|^p$$

corresponding to the case $n = 1$ of Corollary 4.4. Consequently, we deduce by Theorem 4.6

$$\tilde{L}(0) \geq L(0) = \left(1 - \frac{1}{p}\right)^p.$$

In the present case one can actually show that equality holds. Namely, we must estimate $\tilde{V}(x, 0, 1)$; to do so we modify a sequence (u_h) of optimizers for $V(x, s_h, 1)$ where $s_h \to 0$. By Corollary 4.4 (with $n = 1$) such a sequence is given by

$$u_h(t) = s_h (t/x)^{(|s_h|^p x^{1-p} + p - 1)/p}.$$

The modified sequence v_h is defined as follows:

$$v_h(t) = \begin{cases} \lambda_h u_h(t) & \text{if } 0 \leq t \leq x/2 \\ 2\lambda_h u_h(x/2) \left(1 - \dfrac{t}{x}\right) & \text{if } x/2 \leq t \leq x. \end{cases}$$

Note that for $\lambda_h = 1$ we have $v_h \leq u_h$. Moreover, by direct calculation

$$\tilde{G}_x(v_h) = \lambda_h^p \left[\int_0^{x/2} (1-t)^{n-1} t^{-p} |u_h(t)|^p \, dt + 2^p \int_{x/2}^x (1-t)^{n-1} t^{-p} |u_h(x/2)|^p \left(1 - \frac{t}{x}\right)^p dt \right]$$

The first integral has the value $(1-\xi)^{n-1} 2^{-|s_h|^p x^{1-p}}$ with $\xi \in (0, x/2)$, whereas the second integral tends to zero as $s_h \to 0$. It then follows that there is a choice of $\lambda_h^p \leq 2^{|s_h|^p x^{1-p}} (1 - x/2)^{1-n}$ such that $\tilde{G}_x(v_h) = 1$. Moreover, by direct computation

$$\tilde{F}_x(v_h) \leq \left(1 - \frac{x}{2}\right)^{1-n} \left(\frac{|s_h|^p x^{1-p} + p - 1}{p}\right)^p + o(1) \qquad \text{as } h \to +\infty$$

and, passing to the limit first as $h \to +\infty$ and then as $x \to 0$, the right hand side goes to $\left(\frac{p-1}{p}\right)^p = L(0)$.

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