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Weak* Sequential Compactness and Bornological Limit Derivatives

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Dedicated to R. T. Rockafellar on his 60th Birthday

In this note we give a self-contained account of the relationship between the sequential and topological constructions of bornological limit derivatives for locally Lipschitzian real-valued functions on Banach spaces.

Keywords : Asplund space, bornological limit derivative, Clarke subgradient, Fréchet derivative, Gateaux derivative, subderivative, weak Hadamard derivative, weakly compactly generated, weak* sequential compactness.

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1. Introduction

In this note we give a self-contained account of the relationship between the sequential and topological constructions of bornological limit derivatives for locally Lipschitzian real-valued functions on Banach spaces. We make some comments on the constructions for lower semicontinuous functions without providing many details.

In the first section we give conditions under which the sequential and topological constructions give essentially the same object. The second section provides examples to show what can happen when those conditions do not hold. The growing interest in infinite dimensional nonsmooth analysis (see for example [3], [4], [5], [14], [15], [16], [19], [20], [25]) seems to warrant careful discussion of these issues.

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2. Positive results.

In this section we prove a result about intersections of sequences of weak* closures of nested bounded sets and deduce some relationships between various bornological limit derivatives. We denote the closed unit ball of E^* by B^* and the weak* closed convex hull of a set A by $\overline{conv}^* A$.
Recall that a Banach space E is *weakly compactly generated (WCG)* provided there is a weakly compact subset K such that $E = \overline{span}\, K$. Clearly reflexive Banach spaces are weakly compactly generated by their balls. Separable Banach spaces are weakly compactly generated and in fact norm compactly generated (take a dense sequence (x_n) in the unit sphere and consider $\{0\} \cup \{x_n/n : n \in \mathbb{N}\}$).
We need the following characterization of WCG spaces.

Proposition 2.1. [8] *A Banach space E is WCG if and only if there is a reflexive Banach space X and an injective continuous linear $T : X \to E$ with dense range.*

We will use Banach spaces which are subspaces of WCG spaces. There are some of these which are not themselves WCG.

Example 2.2. (Rosenthal [24]). A finite measure μ and a non-WCG subspace of the WCG space $L_1(\mu)$.
Let $\mathcal{R} := \{s \in L_1[0,1] : \|s\| = 1, \int_0^1 s = 0\}$ and let μ be the product Lebesgue measure on $[0,1]^{\mathcal{R}}$. For each $r \in \mathcal{R}$ let $f_r(x) := r(x(r))$ for all $x \in [0,1]^{\mathcal{R}}$. Then $X := \overline{span}\{f_r : r \in \mathcal{R}\}$ is not WCG although $L_1(\mu)$ is ([24],[6]). $\square$

If μ is a σ-finite measure then $L_1(\mu)$ is WCG (see [23], page 36.) The converse is not quite true, but if $L_1(\mu)$ is WCG and μ is not σ-finite then μ is rather pathological, for example μ could be any measure which only takes the values 0 and ∞.

Our basic tool is the following result.

Theorem 2.3. *Let E be a Banach space and A_n a sequence of bounded subsets of E^* such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$. Define*

$$\mathcal{L}_t := \bigcap_{n=1}^{\infty} \overline{A_n}^{weak^*} \qquad \text{and} \qquad \mathcal{L}_\sigma := \{\text{weak}^* \lim x_n^* : x_n^* \in A_n \text{ for } n \in \mathbb{N}\}.$$

(a) *If B^* is weak* sequentially compact then $\mathcal{L}_t$ is the weak* closure of $\mathcal{L}_\sigma$.*
(b) *If E is a subspace of a WCG space then $\mathcal{L}_\sigma$ is weak* closed and so $\mathcal{L}_t = \mathcal{L}_\sigma$.*

Proof. (a) Clearly $\mathcal{L}_\sigma$ is contained in the weak* compact set $\mathcal{L}_t$. Let $x^* \in \mathcal{L}_t$ and let W be a weak* closed weak* neighbourhood of x^*. Then we can choose $x_n^* \in W \cap A_n$ for each $n \in \mathbb{N}$. By weak* sequential compactness, there is a subsequence $x_{n_j}^*$ which converges weak* to some $y^* \in W$. Let $y_k^* := x_{n_j}^*$ for $n_{j-1} < k \leq n_j$. Then $y_n^* \in A_n$ and y_n^* converges weak* to y^*. Thus $y^* \in \mathcal{L}_\sigma \cap W$.
(b) If E is a subspace of a WCG space then there is a Banach space Y containing E, a reflexive Banach space X and an injective continuous linear $T : X \to Y$ with dense range, by Proposition 2.1. Let R denote the restriction mapping from Y^* onto E^*. We may and do suppose that $A_1 \subseteq B^*$. Let $H_n := R^{-1}(A_n) \cap B_{Y^*}$ and $K := \bigcap_{n=1}^\infty weak\ cl\, T^*(H_n)$. Suppose $x^* \in \mathcal{L}_t$. Then the sets $V_n := R^{-1}x^* \cap weak^*cl\, H_n$ are weak* compact, nonempty

and nested so there is $y^* \in \bigcap_{n=1}^{\infty} V_n$. Choose $y^*_{n,j} \in H_n$ such that $T^* y^*_{n,j}$ converges weakly to $T^* y^*$ as $j \to \infty$, for each n, using weak compactness of K (see [13], page 148). Then $\{T^* y^*, T^* y^*_{n,j} : j, n = 1, 2, \ldots\}$ is weakly metrizable and so we can find j_n such that $T^* y^*_{n,j_n}$ converges weakly to $T^* y^*$. Since T^* is a weak* to weak homeomorphism on B_{Y^*} it follows that y^*_{n,j_n} converges weak* to y^* and so $R y^*_{n,j_n}$ converges weak* to $R y^* = x^*$. Since $R y^*_{n,j_n} \in A_n$ that shows $x^* \in \mathcal{L}_\sigma$. □

Remark 2.4. The converse to (a) is also true, in the sense that for a bounded sequence $x^*_n \in E^*$ with no weak* convergent subsequence we can take $A_n := \{x^*_n, x^*_{n+1}, \ldots\}$ and note that $\mathcal{L}_\sigma = \emptyset \neq \mathcal{L}_t$. A topological space T is *angelic* provided for each relatively countably compact subset A of T, (i) A is relatively compact, and (ii) every $t \in \bar{A}$ is the limit of a sequence from A. Thus a compact space is angelic if (ii) holds. The proof of (b) above uses the fact that weakly compact subsets are weak angelic. We do not know if the converse to (b) holds, nor whether $\mathcal{L}_t = \mathcal{L}_\sigma$ follows from B^* being weak* angelic. It is clear that if $\mathcal{L}_t$ always equals $\mathcal{L}_\sigma$ then B^* is weak* angelic. Dr. Jon Vanderwerff (personal communication) has shown that if E has an M-basis and B^* is weak* angelic then $\mathcal{L}_t = \mathcal{L}_\sigma$, so those of our following results which assume that E is a subspace of a WCG space also hold under the weaker assumption that E has an M-basis and B^* is weak* angelic. We give the definition of an M-basis later. If B^* is weak* angelic and the weak* closure of any countable subset of B^* is metrizable then (b) holds, so it will work for the so-called WLD spaces. (In the context of non-smooth analysis one will rarely if ever encounter WLD spaces that are not WCG subspaces.)

Definition 2.5. Let E be a Banach space. Then a *bornology* on E is a collection β of bounded subsets of E such that $E = \bigcup \beta$ and $-A \in \beta$ if $A \in \beta$. In particular, we have the *Gateaux bornology* G consisting of all finite sets, the *Hadamard bornology* H consisting of all relatively compact sets, the *weak Hadamard bornology* WH consisting of all relatively weakly compact sets and the *Fréchet bornology* F consisting of all bounded sets.
We say $\nabla_\beta f(x) \in E^*$ is the *β-derivative* of f at x provided

$$\lim_{t \to 0+} \sup_{y \in A} |(f(x+ty) - f(x))/t - \langle \nabla_\beta f(x), y \rangle| = 0$$

for each $A \in \beta$.
The *β-subderivative* is

$$\partial_\beta f(x) := \{x^* \in \partial_\beta f(x) : \liminf_{t \to 0+} \sup_{y \in A} (f(x+ty) - f(x))/t - \langle x^*, y \rangle \geq 0, x^* \in E^*\}$$

for each $A \in \beta$.
The *Clarke subgradient* is defined by

$$\partial f(x) := \{x^* \in E^* : \langle x^*, y \rangle \leq f^\circ(x; y) \text{ for all } y\}$$

where

$$f^\circ(x; y) := \limsup_{\substack{t \to 0+ \\ z \to x}} \frac{(f(z+ty) - f(z))}{t}.$$

Next we define some limit derivatives by

$$D^\beta_\sigma f(x) := \{weak^* \lim \nabla_\beta f(x_n) : x_n \to x\}$$

(the *sequential* β *derivative*),

$$D^\beta_t f(x) := \bigcap_{n=1}^{\infty} weak^*\, cl\,\{\nabla_\beta f(y) : \|x-y\| < 1/n\}$$

(the *topological* β *derivative*),

$$\partial^\beta_\sigma f(x) := \{weak^* \lim x_n^* \in \partial_\beta f(x_n) : x_n \to x\}$$

(the *sequential* β *subderivative*) and

$$\partial^\beta_t f(x) := \bigcap_{n=1}^{\infty} weak^*\, cl\,\{\partial_\beta f(y) : \|x-y\| < 1/n\}$$

(the *topological* β *subderivative*).
A Banach space E is a *Gateaux differentiability space (GDS)* if every continuous convex function on E is Gateaux differentiable at a dense set of points, and E is an *Asplund space* if every continuous convex function on E is Fréchet differentiable at a dense set of points. We recommend [28] as an informative introduction to Asplund space theory rich with examples.

Every Banach space with separable dual and every reflexive space is Asplund; more generally every space with an equivalent Fréchet smooth renorm is Asplund. There are however Asplund spaces which fail to have even a Gateaux smooth renorm [11]. Asplund spaces are characterized as those spaces all of whose separable subspaces have separable duals. We refer to [23] and [9] for details.

We say E is a *β-Preiss space* provided every locally Lipschitzian $f : E \to \mathbb{R}$ is densely β-differentiable and $\partial f(x) = \overline{conv}^* D^\beta_t f(x)$. Preiss [22] showed that every Banach space with a β-smooth norm is a β-Preiss space and that Asplund spaces are Fréchet-Preiss spaces. Every subspace of a WCG Banach space has a smooth renorming (see [9]) so is a Gateaux-Preiss space. If μ is a σ-finite measure then $L_1(\mu)$ is WCG and has an equivalent WH-smooth norm [1], so each subspace of $L_1(\mu)$ is a WH-Preiss space. If E is a Gateaux differentiability space then B^* is weak* sequentially compact [18]. Since every β-Preiss space is a Gateaux differentiability space, β-Preiss spaces have weak* sequentially compact dual balls.

Theorem 2.6. *Let f be a locally Lipschitzian real-valued function on a Banach space E and β a bornology on E.*

(a) *If B^* is weak* sequentially compact then $D^\beta_t f(x) = weak^* cl D^\beta_\sigma f(x)$.*

(a′) *If B^* is weak* sequentially compact then $\partial^\beta_t f(x) = weak^* cl \partial^\beta_\sigma f(x)$.*

(b) *If E is a subspace of a WCG space then $D^\beta_t f(x) = D^\beta_\sigma f(x)$.*

(b′) *If E is a subspace of a WCG space then $\partial^\beta_t f(x) = \partial^\beta_\sigma f(x)$.*

Proof. For (a) and (b) use Theorem 2.3(a) and (b) respectively putting

$$A_n := \bigcup \{\nabla_\beta\ f(y) : \|x-y\| < 1/n\}.$$

To get (a′) and (b′) put $A_n := \{\partial_\beta f(y) : \|x-y\| < 1/n\}$ instead. □

Corollary 2.7. *If E is a Banach space with B^* weak* sequentially compact and f is a locally Lipschitzian real-valued function on E and $x \in E$ then:*

(i) *If $\partial f(x) = \overline{conv}^* D_t^\beta f(x)$ then $\partial f(x) = \overline{conv}^* D_\sigma^\beta f(x)$.*

(ii) *If $\partial f(x) = \overline{conv}^* \partial_t^\beta f(x)$ then $\partial f(x) = \overline{conv}^* \partial_\sigma^\beta f(x)$.*

Remark 2.8. If f is a convex or concave function or more generally if ∂f is a minimal weak* cusco (see [2]) then one can deduce $\partial f(x) = \overline{conv}^* D_t^\beta f(x)$ as soon as one knows $D_t^\beta f(x) \neq \emptyset$ for all x.

Corollary 2.9. *If E is a Banach space and f is a locally Lipschitzian real-valued function on E then*

(a) *If E is a subspace of a WCG space then $D_t^H f(x) = D_t^G f(x) = D_\sigma^G f(x) = D_\sigma^H f(x)$ (which is weak* closed) and $\partial f(x) = \overline{conv}^* D_\sigma^G f(x)$.*

(b) *If E is an Asplund space then $D_t^F f(x) = weak^* cl D_\sigma^F f(x)$ and $\partial f(x) = \overline{conv}^* D_\sigma^F f(x)$.*

(c) *If μ is a σ-finite measure and E is a subspace of $L_1(\mu)$ then $D_t^{WH} f(x) = D_\sigma^{WH} f(x)$ and $\partial f(x) = \overline{conv}^* D_\sigma^{WH} f(x)$.*

(d) *If E is a β-Preiss space then $D_t^\beta f(x) = weak^* cl D_\sigma^\beta f(x)$ and $\partial f(x) = \overline{conv}^* D_\sigma^\beta f(x)$.*

Proof. (a) If E is a subspace of a WCG space then f is densely Gateaux differentiable and Hadamard differentiability is equivalent to Gateaux differentiability for locally Lipschitzian functions so Theorem 2.3(b) applies.
(b) If E is an Asplund space then B^* is weak* sequentially compact and f is densely Fréchet differentiable so Theorem 2.3(a) applies.
(c) In this case E is a subspace of a WCG space and f is densely WH differentiable so Theorem 2.3(b) applies.
(d) Since E is a β-Preiss space it is a GDS and so B^* is weak* sequentially compact and Theorem 2.3(a) applies. □

Remark 2.10. Corresponding results hold for the subdifferentials of Lipschitzian functions but to apply Theorem 2.3 to subdifferentials of lower semicontinuous functions we would need to deal with unbounded sets A_n and the conclusions of Theorem 2.3 need not follow. The appropriate thing to do seems to be to intersect the A_n with a large ball before closing when constructing $\mathcal{L}_t$. However the resulting $\mathcal{L}_t$ can fail to be weak* closed.

Theorem 2.11. *Let E be a Banach space and A_n a sequence of subsets of E^* such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$. Define*

$$\mathcal{L}_t := \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A_n \cap mB^*}^{weak^*} \qquad \text{and} \qquad \mathcal{L}_\sigma := \{\text{weak}^* \lim x_n^* : x_n^* \in A_n \text{ for } n \in \mathbb{N}\}.$$

(a) *If B^* is weak* sequentially compact then $\mathcal{L}_t$ has the same weak* closure as $\mathcal{L}_\sigma$.*
(b) *If E is a subspace of a WCG space then $\mathcal{L}_t = \mathcal{L}_\sigma$.*

Proof. This follows from Theorem 2.3 because any weak* convergent sequence in E^* is bounded. □

One can be quite precise about when an Asplund space is WCG. Recall that an *M-basis* for E is a system $(x_i, x_i^*)_{i\in I}$ for an arbitrary index set I, such that $\langle x_i^*, x_j\rangle$ is 1 if $i = j$ and 0 otherwise, and furthermore $E = \overline{span}\{x_i : i \in I\}$ and $E^* = \overline{span}^*\{x_i^* : i \in I\}$.

Proposition 2.12. (Valdivia [26], Corollary 3). *If E is an Asplund space then E is WCG if and only if E has an M-basis and B^* is weak* angelic.*

There are Asplund spaces with weak* angelic dual balls which are not WCG. An example is the space JT^* (see [27]).

We say a Banach space E has the *Schur property* provided every weakly convergent sequence in E is norm convergent. A Banach space E has the *Dunford-Pettis property* if given weakly null sequences (x_n) and (x_n^*) in E and E^* respectively then $\lim_n\langle x_n^*, x_n\rangle = 0$. If either E or E^* has the Schur property then E enjoys the Dunford-Pettis property. A compact Hausdorff space is *scattered* or *dispersed* provided it contains no nonempty perfect subsets. We may now gather up a striking set of equivalences for continuous function spaces.

Theorem 2.13. *Let K be a compact Hausdorff topological space. The following are equivalent:*
(a) *K is scattered.*
(b) *$C(K)$ is Asplund.*
(c) *$C(K)$ contains no isomorphic copy of ℓ_1.*
(d) *$C(K)$ contains no isometric copy of $C[0,1]$.*
(e) *$C(K)^*$ has the Schur property.*

Proof. By [21], (a) implies (b). Also [21] show that subspaces of Asplund spaces are Asplund spaces so (b) implies (c) as ℓ_1 is not an Asplund space. Now [7] (page 212) shows that E^* has the Schur property if and only if E has the Dunford-Pettis property and E contains no copy of ℓ_1. Since each $C(K)$ has the Dunford-Pettis property (a consequence of Egoroff's theorem also discussed in [7]) it follows that (c) implies (e) and (e) implies (d), as $C[0,1]$ contains a copy of ℓ_1. Now if K is not scattered, [17] (page 29) shows there is a continuous surjection $s : K \to [0,1]$. Then $F(f)(r) := f(s(r))$ defines an isometry of $C[0,1]$ into $C[K]$ so (d) implies (a). □

Remark 2.14. This provides a large selection of Schur spaces and shows that $C[0,1]$ is universally present in non-Asplund $C(K)$ spaces. While Haydon's examples (see [9]) have shown how much can be illustrated in $C(K)$ these equivalences emphasize how special $C(K)$ is among Banach spaces in that four usually distinct properties coincide there.

There is also a version of the equivalence of (a) through (d) for C^* algebras [10].

3. Limiting examples

These examples are continuous concave functions. This gives examples for the derivative case and the subderivative case simultaneously because for a continuous concave function

f there is $x^* \in \partial_\beta f(x)$ if and only if $x^* = \nabla_\beta f(x)$. The first two examples show that the sets of Theorem 2.6 can be empty, while the third example shows that equality can fail even when both limit derivatives are nonempty.

Example 3.1. A continuous concave function on ℓ_∞ which is nowhere Gateaux subdifferentiable.
The function $f(x) := -\limsup x_n$ is nowhere Gateaux differentiable: see [23], page 13.

Example 3.2. A continuous concave function on ℓ_1 which is nowhere Fréchet subdifferentiable.
If $f(x) := -\|x\|_1$ then f is nowhere Fréchet differentiable: see [23], page 8.

Example 3.3. A continuous concave function f on ℓ_∞ which is densely Fréchet differentiable but has points where $D_t^\beta f(x) \neq weak^* cl D_\sigma^\beta f(x) \neq \emptyset$ and $\partial f(x) \neq \overline{conv}^* D_\sigma^\beta f(x)$, for each bornology β with $G \subseteq \beta \subseteq F$.
Define $f(x) := -\|x\|_\infty$; by [9] (page 5) the β-derivative of the norm at y exists if and only if $|y_n| > \sup_{m \neq n} |y_m|$ for some n, and then the derivative is $sgn(y_n)e_n \in \ell_1$.

Now at $x := (1, 1/2, 2/3, 3/4, \ldots)$ we have $D_\sigma^\beta f(x) = \{-e_1\}$ and $D_t^\beta f(x) = \{-e_1\} \cup Z$ for a nonempty set Z not containing e_1. In fact the derivatives at nearby points can only be $-e_n$ for large n which gives $Z = \bigcap_{n=1}^\infty weak^*\, cl\{-e_n, -e_{n+1}, \ldots\}$. Also $\partial f(x) = \overline{conv}^* D_t^\beta f(x) \neq \overline{conv}^* D_\sigma^\beta f(x)$. $\square$

This shows the Corollary 2.7 needs the weak* sequential compactness hypothesis; also see Remark 2.8. Note that ℓ_∞^* is far from having a weak* sequentially compact unit ball: a sequence in ℓ_∞^* converges weak* if and only if it converges weakly ([7], page 103). Contrast this with Theorem 2.13(e). Of course by the Josefson-Nissenzweig Theorem (see [7], page 219) the norm and weak* sequential convergences can never coincide in an infinite dimensional Banach space.

Our final example shows the need for of our WCG hypothesis in Theorem 2.6, even in an Asplund space.

Example 3.4. A compact Hausdorff scattered space K and a continuous concave function f on $C(K)$ such that $\partial f(x) \neq D_t^F f(x) \neq D_\sigma^F f(x)$ for some x.
Let ω_1 be the first uncountable ordinal and let K be the compact topological space $[0, \omega_1]$. Define $f(x) := -\|x\|$ for $x \in C(K)$. Let μ_ω be the point mass at $\omega \in K$. By [9] (page 5) the norm is Fréchet differentiable at $x \in C(K)$ if and only if there is ω, an isolated point of K (that is, not a limit ordinal) such that $|x(\omega)| > |x(t)|$ for $t \neq \omega$. In that case the derivative is μ_ω. By considering for ω any non-limit ordinal

$$y_\omega(t) := \begin{cases} 1+\epsilon & , t = \omega \\ 1 & , \text{otherwise} \end{cases}$$

at $x \equiv 1$ we get

$$D_t^F f(x) = \{-\mu_\omega : \omega \in K\} \neq D_\sigma^F f(x) = \{-\mu_\omega : \omega < \omega_1\}$$

because ω_1 is not the limit of a sequence of countable ordinals, while other $w \in K$ are limits of sequences of non-limit ordinals. On the other hand $\partial f(x) = \{\mu : \mu \leq 0, \mu(K) = -1\}$.

Remark 3.5. This $C(K)$ is Asplund but not WCG, it has an M-basis and B^* is not weak* angelic. By Theorem 2.13, $C([0, \omega_1])^*$ has the Schur property. Also $C([0, \omega_1])$ has a C^∞ renorm [12] but nonetheless Theorem 2.3(b) and Corollary 2.9(b) have failed in this setting.

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