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Wellposed Problems of the Calculus of Variations for Nonconvex Integrals

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For Lagrange problems of the calculus of variations we prove wellposedness criteria in Tikhonov's sense and with respect to small perturbations of the boundary data. The integrand depends of the gradient only. No convexity assumption is required in the sufficient conditions for wellposedness.

Keywords : Hadamard and Tikhonov wellposedness, non convex integrals, calculus of variations

1. Introduction

A global minimization problem is called *Tikhonov wellposed* if there exists a unique minimizer to which every minimizing sequence converges, and *Hadamard wellposed* if its unique minimizer depends continuously on problem's data. See [5] for a survey. In [10] a new definition of wellposedness has been introduced, by requiring both Tikhonov and Hadamard wellposedness through suitable embeddings in a parametrized family of minimization problems.
The results of [10] show that one-dimensional Lagrange problems of the classical calculus of variations are well posed with respect to the embedding which takes as a parameter the boundary data, provided the integrand satisfies the classical assumptions of smoothness, coercivity and strict convexity, and the optimal value function is differentiable (see [10, Theorem 3]).
Purpose of this paper is to provide new sufficient or necessary conditions for wellposedness of Lagrange problems involving integral functionals whose integrand is not convex, and not coercive in the one-dimensional case.
Our approach parallels, up to some extent, the corresponding theory of existence of minimizers for non convex integrands (see [8] for a beautiful survey, and also [4, chapter 5]). We consider the relaxed functional obtained by taking the convex envelope of the given integrand. However the results of [10] cannot be applied to it, since the convex envelope

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is not necessarily strictly convex and possibly nonsmooth. A different approach is needed (relying on Olech's lemma, see [9]).
In section 2 we collect the relevant definitions and notations. We prove wellposedness for one-dimensional problems in section 3, and for multiple integrals in section 4.

2. Preliminaries and problems statement.

We consider metric spaces X and P, a fixed point $p^* \in P$ and a given ball L around p^* with positive radius. We are given proper extended real-valued functions

$$J : X \to (-\infty, +\infty];\ F : L \times X \to (-\infty, +\infty] \tag{1}$$

such that

$$J(x) = F(p^*, x),\ x \in X. \tag{2}$$

The corresponding value function is given by

$$V(p) = \inf\{F(p, x) : x \in X\},\ p \in L. \tag{3}$$

The global optimization problem (X, J), of minimizing $J(x)$ as $x \in X$, is called *wellposed* (with respect to the embedding defined by F) iff $V(p) > -\infty$ for all $p \in L$, there exists a unique

$$x^* = \arg\min(X, J)$$

and for every sequences $p_n \to p^*,\ x_n \in X$ such that

$$F(p_n, x_n) - V(p_n) \to 0 \text{ as } n \to +\infty \tag{4}$$

we have $x_n \to x^*$ in X.
Sequences x_n as in (4) will be referred to as asymptotically minimizing (for F) corresponding to the sequence p_n. (See [10] for a discussion of the definition of wellposedness).
Denote by AC (resp. Lip) the usual Banach space of all absolutely (resp. Lipschitz) continuous functions

$$x : [0, 1] \to \mathbb{R}^{\mathbb{N}}.$$

Given a continuous function

$$f : \mathbb{R}^{\mathbb{N}} \to (-\infty, +\infty) \tag{5}$$

and a fixed point $p^* \in \mathbb{R}^{\mathbb{N}}$, we consider, for p close to p^* and $x \in AC$

$$F(p, x) = \begin{cases} \int_0^1 f[\dot{x}(t)]dt & \text{if } f(\dot{x}) \in L^1 = L^1(0, 1) \text{ and } x(0) = 0,\ x(1) = p; \\ +\infty & \text{otherwise;} \end{cases} \tag{6}$$

$$J(x) = F(p^*, x). \tag{7}$$

By *problem* (p) we denote the problem of minimizing $F(p, \cdot)$ on AC, that is of minimizing $\int_0^1 f[\dot{u}(t)]dt$ subject to u absolutely continuous in $[0, 1]$, $u(0) = 0,\ u(1) = p$.

According to the previous definition, *problem* (p^*) *is wellposed* iff there exists a ball L around p^* such that $V(p) > -\infty$ for all $p \in L$, there exists a unique minimizer $u^* \in AC$,

and for every sequence $p_n \to p^*$, every asymptotically minimizing sequence $x_n \in AC$ with $x_n(0) = 0,\ x_n(1) = p_n$ satisfies $\dot{x}_n \to \dot{u}^*$ strongly in L^1.
Of course, convergence of every asymptotically minimizing sequence implies uniqueness of the minimizer.

For a given function $f : \mathbb{R}^{\mathbb{N}} \to (-\infty, +\infty)$ we denote by f^{**} its convex envelope and by epi$f$ its epigraph. For set A, extr$A$ denotes the set of its extremal points, cl$A$ the closure and co$A$ the convex hull. Du denotes the gradient of u.

3. Wellposedness criteria for one-dimensional integrals.

We start with the abstract setting and obtain a wellposedness criterion mimicking the approach to existence in the calculus of variations without lower semicontinuity via the relaxed problem (see [8, section 3]).

Given F, J, V as in (1), (2), (3) let

$$E : L \times X \to (-\infty, +\infty]$$

be such that

$$V(p) = \inf\{E(p, x) : x \in X\},\ \ p \in L, \tag{8}$$

$$E(p, x) \le F(p, x) \text{ for all } p, x. \tag{9}$$

Proposition 3.1. *Suppose that E fulfils* (8) *and* (9). *If $[X, E(p^*, \cdot)]$ is wellposed with solution u^* and $F(p^*, u^*) = E(p^*, u^*)$, then $[X, F(p^*, \cdot)]$ is wellposed.*

Proof. Let $p_n \to p^*$ and $u_n \in X$ be such that

$$F(p_n, u_n) - V(p_n) \to 0.$$

By (8), (9) and wellposedness we get $u_n \to u^*$.
Since $F(p^*, \cdot)$ and $E(p^*, \cdot)$ agree on u^* we see that

$$u^* \in \arg\min[X, F(p^*, \cdot)].$$

Thus every asymptotically minimizing sequence for $[X, F(p^*, \cdot)]$ converges to some minimizer, whence wellposedness. □

Let us consider now wellposedness of one-dimensional problems of the calculus of variations.
Given f as in (5) and the point $p^* \in \mathbb{R}^{\mathbb{N}}$ we posit

$$\begin{gathered} f \text{ is continuous in } \mathbb{R}^{\mathbb{N}} \text{ and there exist } a \in \mathbb{R}^{\mathbb{N}},\ b \in \mathbb{R} \text{ such that} \\ f(x) \ge a \cdot x + b \text{ for all } x. \end{gathered} \tag{10}$$

If (10) is satisfied then

$$f^{**} : \mathbb{R}^{\mathbb{N}} \to (-\infty, +\infty)$$

is continuous everywhere. For any $x \in AC$ and $p \in \mathbb{R}^{\mathbb{N}}$ consider

$$E(p,x) = \begin{cases} \int_0^1 f^{**}[\dot{x}(t)]dt & \text{if } \ f(\dot{x}) \in L^1 \ \text{ and } \ x(0)=0, x(1)=p; \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

Write

$$T(p) = \{x \in AC : x(0) = 0, x(1) = p\}.$$

Lemma 3.2. *If* (10) *is satisfied, for every p we have*

$$V(p) = \ inf \ \{E(p,x) : x \in T(p)\} = f^{**}(p).$$

Proof. By (10), $F(p,\cdot)$ and $E(p,\cdot)$ are bounded from below for every p. For any fixed p write

$$T = T(p).$$

Then by [1, theorem 2.4 and remark 2.8] we have

$$\begin{aligned} \inf F(p,T) &= \inf F(p, T \cap \mathrm{Lip}), \\ \inf E(p,T) &= \inf E(p, T \cap \mathrm{Lip}). \end{aligned} \tag{12}$$

Given $\epsilon > 0$ let $u \in T \cap \mathrm{Lip}$ be such that

$$E(p,u) < \inf E(p, T \cap \mathrm{Lip}) + \epsilon.$$

By Bogoljubov's theorem [6, theorem 5 p.383] there exists a sequence $x_n \in C^1([0,1])$ such that $x_n(0) = 0, \ x_n(1) = p$, and

$$\liminf F(p, x_n) \le E(p,u).$$

Hence for infinitely many n

$$V(p) \le F(p,x_n) \le \inf E(p, T \cap \mathrm{Lip}) + \epsilon.$$

It follows that

$$V(p) \le \inf E(p, T \cap \mathrm{Lip}). \tag{13}$$

The opposite inequality is trivial since $f^{**} \le f$ and (12) holds. Hence by (12) and (13)

$$\begin{aligned} V(p) &= \inf E(p,T) = \inf F(p,T) = \\ &= \inf E(p, T \cap \mathrm{Lip}) = \inf F(p, T \cap \mathrm{Lip}). \end{aligned} \tag{14}$$

This proves the first equality. Now let $x^*(t) = pt, \ 0 \le t \le 1$. Then by Jensen's inequality, for every $x \in T$

$$\int_0^1 f^{**}[\dot{x}(t)]dt \ge f^{**}(\int_0^1 \dot{x}(t)dt) = f^{**}(p) = \int_0^1 f^{**}[\dot{x}^*(t)]dt$$

hence the conclusion by (14). □

Remark 3.3. In [8, section 3] we find the conclusion of Lemma 3.2 under the assumption of coercivity of f.

The following condition

$$[p^*, f^{**}(p^*)] \in \text{extr epi} f^{**} \tag{15}$$

will play a basic role in the next results.

Lemma 3.4. *Let* (10) *and* (15) *hold. Suppose that* $p_n \to p^*$ *and* $u_n \in AC$ *is asymptotically minimizing for* E *corresponding to* p_n. *Then* u_n *converges strongly in* AC.

Proof. We have

$$(\dot{u}_n(t), f^{**}[\dot{u}_n(t)]) \in \text{epi} f^{**}$$

for all n and a.e.t. By Lemma 3.2

$$\int_0^1 f^{**}[\dot{u}_n(t)]dt \to f^{**}(p^*)$$

since $f^{**} = V$ is continuous at p^*. It follows that

$$(\int_0^1 \dot{u}_n(t)dt, \int_0^1 f^{**}[\dot{u}_n(t)]dt) \to [p^*, f^{**}(p^*)]$$

and such a limit point is extremal for

$$\text{clco} \int_0^1 \text{epi } f^{**}dt = \text{ clco } (\text{clco epi } f^{**}) = \text{ epi } f^{**}.$$

By Olech's Lemma [9, Lemma 1 p. 88] it follows that $[\dot{u}_n, f^{**}(\dot{u}_n)]$ is a Cauchy sequence in L^1, hence u_n converges strongly in AC (since $u_n(0) = 0$). □

Theorem 3.5. *Suppose that* (10) *and* (15) *hold. Then problem* (p^*) *is wellposed.*

Proof. By Lemma 3.2, E given by (11) fulfils properties (8) and (9). Let $p_n \to p^*$ and u_n be asymptotically minimizing for E corresponding to p_n. By Lemma 3.4 there exists $u^* \in T(p^*)$ such that $u_n \to u^*$ in AC, hence for some subsequence

$$\dot{u}_n(t) \to \dot{u}^*(t) \text{ a. e. in } (0,1).$$

Fatou's lemma and continuity of V (Lemma 3.2) yield

$$V(p^*) = \liminf E(p_n, u_n) \geq \int_0^1 f^{**}(\dot{u}^*)dt$$

hence u^* minimizes $E(p^*, \cdot)$. It follows that $[AC, E(p^*, \cdot)]$ is wellposed. Hence $u^*(t) = p^*t$ by Lemma 3.2. Let v_n be a minimizing sequence for $[AC, F(p^*, \cdot)]$. By (9), v_n is asymptotically minimizing for E as well, hence $v_n \to u^*$ in AC. By Fatou's lemma, u^* minimizes $F(p^*, \cdot)$ and $V(p^*) = f(p^*) = f^{**}(p^*)$, hence

$$E(p^*, u^*) = F(p^*, u^*).$$

The conclusion follows from proposition 3.1. □

Remark 3.6. A direct proof of theorem 3.5 can be given, as is obvious from the above proof, avoiding proposition 3.1. We prefer here to emphasize a general approach instead of ad hoc arguments.

Theorem 3.7. *Let f be convex. If problem (p^*) is wellposed, then* $[p^*, f(p^*)] \in \text{extr epi}f$.

Proof. By uniqueness of the minimizer, the solution of problem (p^*) is given by

$$u^*(t) = p^*t, \quad 0 \leq t \leq 1$$

by Lemma 3.2. Let $0 < \alpha < 1$, p and q in $\mathbb{R}^{\mathbb{N}}$, y and z in $\mathbb{R}$ be such that (p, y), $(q, z) \in$ epi$f$ and $\alpha(p, y) + (1-\alpha)(q, z) = [p^*, f(p^*)]$. Thus

$$\begin{aligned} y \geq f(p), \quad z \geq f(q), \quad \alpha p + (1-\alpha)q &= p^* \\ \alpha y + (1-\alpha)z &= f(p^*). \end{aligned} \tag{16}$$

Consider $w \in AC$, $w(0) = 0$,

$$\dot{w}(t) = \begin{cases} p & \text{if } \ 0 < t < \alpha, \\ q & \text{if } \ \alpha < t < 1. \end{cases}$$

Then by (16) $w(1) = p^*$ and

$$\int_0^1 f(\dot{w})dt = \alpha f(p) + (1-\alpha) f(q) \leq f(p^*).$$

Hence, by Lemma 3.2, w solves problem (p^*). By uniqueness of the minimizer, $\dot{w} = \dot{u}^*$ a. e., whence $p = q = p^*$. If $y > f(p)$ then by (16) $f(p^*) > \alpha f(p) + (1-\alpha) f(q) = f(p^*)$, similarly if $z > f(q)$ we get the same contradiction. This yields $y = z = f(p^*)$, whence extremality. □

Remark 3.8. If f is (finite and) convex, it follows by theorem 3.5 and 3.7 that for problem (p^*), uniqueness of the minimizer, wellposedness and (15) are equivalent properties. Similar conclusions, involving a weaker notion of wellposedness (see the last remark of the present paper) are known under coercivity assumptions, see [3]. (The equivalence between extremality and uniqueness is obtained in [2].)

4. Wellposedness for multiple integrals.

We are given a bounded open set $\Omega \subset \mathbb{R}^{\mathbb{N}}$ with Lipschitz boundary, a point $p^* \in \mathbb{R}^{\mathbb{N}}$, a positive constant k such that $|p^*| \leq k$ and a real-valued continuous function f as in (5), such that

$$a|u|^\alpha + b \leq f(u) \leq m|u|^\alpha + n \tag{17}$$

for all u, some costants $a > 0$, b, m, n and $\alpha > 1$. Write

$$P = \{\varphi \in W^{1,\infty}(\Omega) : |\ \mathrm{D}\ \varphi(x)| \leq k \text{ a.e. in } \Omega\}.$$

Given $\varphi \in P$ and $u \in W^{1,\alpha}(\Omega)$ consider

$$F(\varphi, u) = \begin{cases} \int_\Omega f(\mathrm{D}u)dx & \text{if } u - \varphi \in W_0^{1,\alpha}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

$$\varphi^*(x) = p^* \cdot x, \ \ x \in \Omega,$$

and

$$T(\varphi) = \{u \in W^{1,\alpha}(\Omega) : u - \varphi \in W_0^{1,\alpha}(\Omega)\}.$$

We consider $T(\varphi)$ equipped with the strong convergence of $W^{1,\alpha}(\Omega)$. Wellposedness of

$$\text{problem}(\varphi^*) = [T(\varphi^*), F(\varphi^*, \cdot)]$$

will be considered taking into account small perturbations of the boundary datum φ^*, according to the general definition of section 2, as follows. *Problem* (φ^*) *is* called *wellposed* iff

a) the optimal value function

$$V(\varphi) = \inf\{F(\varphi, u) : u \in T(\varphi)\}$$

is finite for all $\varphi \in P$ such that φ is sufficiently close to φ^* in the norm of $W^{1,1}(\Omega)$;

b) there exists a unique minimizer

$$u^* = \text{argmin}(\varphi^*);$$

c) for every sequence $\varphi_n \in P$ such that $\varphi_n \to \varphi^*$ in $W^{1,1}(\Omega)$, every sequence $u_n \in T(\varphi_n)$ such that

$$\int_\Omega f(\mathrm{D}u_n)dx - V(\varphi_n) \to 0$$

converges strongly to u^* in $W^{1,1}(\Omega)$.

Theorem 4.1. *Problem* (φ^*) *is wellposed provided* f *is continuous and conditions* (15), (17) *are fulfilled.*

Proof. For $\varphi \in P$ and $u \in T(\varphi)$ write

$$E(\varphi, u) = \int_\Omega f^{**}(\mathrm{D}u)dx.$$

Step 1: $V(\varphi^*) = \text{meas }(\Omega) f^{**}(p^*)$.

Let $u \in T(\varphi^*)$. Then $u = v + \varphi^*$ with $v \in W_0^{1,\infty}(\Omega)$. By Jensen's inequality

$$\begin{aligned} \int_\Omega f^{**}(\mathrm{D}u)dx &\geq \text{meas }(\Omega) f^{**}(\text{meas }(\Omega)^{-1} \int_\Omega \mathrm{D}u dx) = \\ &= \text{meas }(\Omega) f^{**}(\text{meas }(\Omega)^{-1} \int_\Omega \mathrm{D}\varphi^* dx) = \\ &= \text{meas }(\Omega) f^{**}(p^*) \end{aligned}$$

since $\int_\Omega \mathrm{D}v dx = 0$. Hence $\inf E[T(\varphi^*)] = \text{meas}\ (\Omega) f^{**}(p^*)$. By the relaxation theorem (see [4, section 5.2])

$$\inf F[\varphi, T(\varphi)] = \inf E[\varphi, T(\varphi)],\ \varphi \in P, \tag{18}$$

hence step 1 is proved. By (17) we have

$$V(\varphi) > -\infty \text{ for every } \varphi \in P.$$

Step 2: if $\varphi_n, \varphi \in P$ and $\varphi_n \to \varphi^*$ in $W^{1,1}(\Omega)$ then $V(\varphi_n) \to V(\varphi^*)$.

By step 1, $V(\varphi^*) = \int_\Omega f^{**}(\mathrm{D}\varphi^*)dx$. Moreover $V(\varphi_n) \le \int_\Omega f^{**}(\mathrm{D}\varphi_n)dx$ for every n by (18). By (17), f^{**} is Lipschitz continuous on bounded sets. We get

$$V(\varphi_n) - V(\varphi^*) \le \int_\Omega [f^{**}(\mathrm{D}\varphi_n) - f^{**}(\mathrm{D}\varphi^*)]dx \le$$
$$\le (\text{costant}) \int_\Omega |\mathrm{D}\varphi_n - \mathrm{D}\varphi^*| dx$$

hence

$$\limsup V(\varphi_n) \le V(\varphi^*).$$

Now let $u_n \in \text{argmin}[T(\varphi_n), E(\varphi_n, \cdot)]$, whose existence follows by standard results (see [4, theorem 3.4 p. 82]). Then $u_n = \varphi_n + z_n$ with $z_n \in W_0^{1,\alpha}(\Omega)$. By (17) and convergence of the equi - Lipschitz sequence φ_n, some subsequence fulfils

$$u_n \rightharpoonup w \text{ in } W^{1,\alpha}(\Omega),\ w = \varphi^* \text{ on } \partial\Omega.$$

By lower semicontinuity (see [4, theorem 3.4 p. 74]), for the corresponding subsequence we have

$$\liminf V(\varphi_n) = \liminf \int_\Omega f^{**}(\mathrm{D}u_n)dx \ge \int_\Omega f^{**}(\mathrm{D}w)dx \ge V(\varphi^*)$$

since $w \in T(\varphi^*)$. This proves step 2 (for the original sequence of course).
Step 3: $[T(\varphi^*), E(\varphi^*, \cdot)]$ is wellposed.

Let $\varphi_n \in P$ and $\varphi_n \to \varphi^*$ in $W^{1,1}(\Omega)$, let $u_n \in T(\varphi_n)$ be asymptotically minimizing for E corresponding to φ_n. By step 2 it follows that

$$E(\varphi_n, u_n) \to V(\varphi^*). \tag{19}$$

Moreover $[\mathrm{D}u_n(x), f^{**}[\mathrm{D}u_n(x)] \in \text{epi} f^{**}$ for every n and a. e . x, and

$$(\int_\Omega \mathrm{D}u_n dx, \int_\Omega f^{**}(\mathrm{D}u_n)dx) = (\int_\Omega \mathrm{D}\varphi_n dx, \int_\Omega f^{**}(\mathrm{D}u_n)dx) \to \text{meas}\ \Omega\ \ [p^*, f^{**}(p^*)]$$

by (19) and step 1. Then (15) and Olech's Lemma [9, Lemma 1 p. 88] yield strong convergence of $\mathrm{D}u_n$ in $L^1(\Omega)$. Since $u_n = \varphi_n$ on $\partial\Omega$ and φ_n converges strongly, it follows that $u_n \to w$ in $W^{1,1}(\Omega)$. Since f^{**} fulfils (17), standard reasoning shows that

$$V(\varphi^*) = \liminf \int_\Omega f^{**}(\mathrm{D}u_n)dx \ge \int_\Omega f^{**}(\mathrm{D}w)dx,$$

and $w \in T(\varphi^*)$. Hence $w \in \operatorname{argmin}[T(\varphi^*), E(\varphi^*, \cdot)]$. Every asymptotically minimizing sequence converges to some minimizer, and this proves step 3. By step 1, $\varphi^* \in \operatorname{argmin}[T(\varphi^*), E(\varphi^*, \cdot)]$ and $f(p^*) = f^{**}(p^*)$ by (15) and [6, proposition 1.5.4 (ii) p. 51]. An application of proposition 1 ends the proof. □

Remark 4.2. In [3] wellposedness is proved in a weaker sense than here and for a broader class of integrands: the perturbed boundary data are there linear and the convergence of minimizers only is considered. In [3] it is also shown that there are illposed problems with a unique minimizer.

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