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A General Approach to Dual Characterizations of Solvability of Inequality Systems with Applications [1]

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Dedicated to R. T. Rockafellar on his 60th Birthday

A general approach is developed to solvability theorems involving a broad class of functions, here called H-convex functions and inf-H-convex functions. The concept of Minkowski duality is exploited to provide dual characterizations for certain infinite inequality systems. The results not only cover the recently developed solvability results involving DSL functions, concave functions and difference of sublinear and convex functions but also include a new dual characterization for systems with completely difference convex functions. Detailed examples are provided to illustrate the broad nature of the results. Applications to global optimization are also given.

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1. Introduction

Dual characterizations of solvability of nonlinear inequality systems are crucial for the development of dual necessary and sufficient conditions for local (and global) extrema

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of optimization problems. A great deal of attention has been focussed in the recent literature of nonlinear programming and nonsmooth optimization to establishing such characterizations in terms of subdifferentials and approximate subdifferentials (see [11, 12, 20, 21, 14]). More recently, various new forms of solvability characterizations have been given with applications to certain classes of global optimization problems such as convex maximization problems and minimization of the difference of sublinear functions (see [22, 23]).

In this paper, we provide a general approach to the solvability of nonlinear inequality systems involving two broad classes functions, here called H-convex functions and inf-H-convex functions. The notions of H-convexity of sets and functions are given without linearity. This concept of convexity without linearity has been studied extensively by many authors (see, for example, [25, 4, 33, 28, 34]). Recall one of the main results of convex analysis: a function is lower semicontinuous (l.s.c) and convex if and only if it is the pointwise supremum of a subset of the space H of all continuous affine (i.e. linear + constant) functions. Thus 'convexity without linearity' (H-convexity) in our context involves the taking of suprema of subsets of certain classes of functions (not necessarily affine functions). Similarly we can consider H-concavity by taking infima and then inf-H-convexity (i.e. the class of functions expressible as the pointwise infima of a family of H-convex functions). This concept of inf-H-convexity is extremely broad. For example it allows us to consider the important classes of difference convex functions which are easily expressible as the pointwise infimum of a family of convex functions. This concept is also applicable to certain positively homogeneous functions as we will illustrate with extensive examples in this paper.

Two forms of dual characterizations are given for inequality systems involving H-convex functions and inf-H-convex functions with the view to possibly applying, in particular, to convex maximization with convex constraints and general convex minimization problems. This approach covers corresponding recent results, presented for systems involving completely difference sublinear functions [11], difference of sublinear and convex functions [gj2]. More importantly, it yields, a new dual characterization for inequality systems involving functions expressible as the pointwise infima of a family of convex functions (inf-convex functions). The role of a consistency condition required in these results is also clarified and related to a stability property well known in Minkowski duality.

In section 2, we present the notions of H-convex functions and sets and illustrate the broad nature of these concepts by several examples in section 3. The well known dual correspondence between sublinear functions and support sets is now generalized using Minkowski duality (see [25]) to H-convex functions and H-convex sets. In section 4, dual characterizations for the solvability of H-convex inequality systems and inf-H-convex inequality systems are given in terms of H-convex hulls. Section 5 examines ways of characterizing H-convex hulls in terms of convex hulls and cones for certain subclasses of H-convex functions. In section 6, solvability results are given for many important classes of functions using the characterization of H-convex hulls in easily verifiable forms. In section 7, we show how these results can be used to provide dual conditions characterizing global optimum. Finally, we present some open research questions for further development. The appendix contains a new minimax result for H-convex functions.

2. H-convex Functions and Minkowski Duality

In this section we introduce a class of generalized convex functions, here called H-convex functions. The properties of this class of function are discussed within a general concept of Minkowski duality. The approach is a modification of that used in [25]. We begin with some essential definitions and an introduction to the notation to be used throughout.

Definition 2.1. Let X be an arbitrary set and $Z \subseteq X$ a non-empty subset. Let H be a set of functions defined on X and mapping into $\mathbb{R}$. A function $p : Z \to \mathbb{R}_{+\infty}$, where $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$, is called *$H$-convex* on the set Z if there is a set $U \subseteq H$ such that, for all $z \in Z$,

$$p(z) = \sup\{h(z) : h \in U\}.$$

Definition 2.2. Let $f : Z \to \mathbb{R}_{+\infty}$, the set of H-minorants of f on Z is called the *support set* of f and is denoted $\mathrm{s}(f, H, Z)$. Thus

$$\mathrm{s}(f, H, Z) = \{h \in H : (\forall z \in Z)\, h(z) \leq f(z)\}.$$

The function $\mathrm{co}_H^Z f : X \to \mathbb{R}_{+\infty}$ defined, for each $z \in Z$ by

$$(\mathrm{co}_H^Z f)(z) := \sup\{h(z) : h \in \mathrm{s}(f, H, Z)\},$$

is called the *H-convex hull* of the function f.

Clearly, $\mathrm{s}(f, H, Z) = \mathrm{s}(\mathrm{co}_H^Z f, H, Z)$.

It should be noted that Rolewicz [33] has used the expression 'convexity without linearity' in relation to these concepts.

To illustrate the nature of support sets we include the following simple examples. Note that by specifying the set H we generate various special classes of function.

Example 2.3. Let ℓ be a continuous linear function defined on a locally convex Hausdorf topological vector space (l.c.H.t.v.s) X (so $\ell \in X'$, where X' denotes the *continuous dual (or conjugate) space* to X). Then we have the following:

1. Let $H = X'$, $Z = X$. Then $\mathrm{s}(\ell, X', X) = \{\ell\}$.
2. Let $H = X'$, $Z = K$ (a closed convex cone in X) then $\mathrm{s}(\ell, X', K) = \ell - K^*$. Here K^* denotes the *dual cone* to K, i.e. $K^* = \{\ell \in X' : (\forall x \in K)\, \ell(x) \geq 0\}$.
3. Let $H = \{\bar{h} : (\forall x \in X)\ \bar{h}(x) = h(x) - c,\ h \in X',\ c \in \mathbb{R}\}$ (i.e. the set of continuous affine functions defined on X), $Z = X$. Then $\mathrm{s}(\ell, H, X) = \{\ell + c\mathbf{e} : c \leq 0\}$, where $\mathbf{e}$ denotes the function with constant value 1 on X.
4. Let H be the set of all continuous affine functions defined on X, $Z = B$ (the closed unit ball in X (assumed a normed space)). Then $\mathrm{s}(\ell, H, B) = \{(\ell', c) : c \geq \|\ell' - \ell\|\}$. Thus $\mathrm{s}(\ell, H, B)$ is the epigraph of the function $f(\ell') = \|\ell' - \ell\|$.

Clearly $\mathrm{co}_H^Z(\ell) = \{\ell\}$ in examples 1 to 4 above.

Now we consider the function $\mathbf{e}$ defined above.

1. Let $H = X'$, $Z = K$ a closed convex cone. Then $\mathrm{s}(\mathbf{e}, X', K) = -K^*$. $\mathrm{co}_H^Z(\mathbf{e}) = \{0\}$.
2. Let $H = X'$, $Z = B$ (X a normed space). Then $\mathrm{s}(\mathbf{e}, X', B) = \{\ell : (\forall x \in B)\ \ell(x) \leq 1\} = B^*$ (the dual unit ball in X'). $\mathrm{co}_H^Z(\mathbf{e})(x) = \|x\|$.
3. Let $H = X'$, Z an arbitrary set. Then $\mathrm{s}(\mathbf{e}, X', Z) = Z^\circ$ (the polar set of Z).

Now we consider a l.s.c sublinear function $p : X \to \mathbb{R}_{+\infty}$ where X is a l.c.H.t.v.s. Let $H = X'$ and $Z = X$ then

$$s(p, X', X) = \partial p(0),$$

the *subdifferential* of p at zero, i.e. $\partial p(0) = \{h \in X' : (\forall x \in X)\ p(x) \geq h(x)\}$.
Similarly if $Z = K$, a closed convex cone in X, then

$$s(p, X', K) = \mathrm{cl}\,(\partial p(0) - K^*).$$

Here we are taking closure in the weak* topology of X'.
Finally, consider a l.s.c convex function $f : X \to \mathbb{R}_{+\infty}$ where X is a l.c.H.t.v.s. Let $H = \{\bar{h} : (\forall x \in X)\ \bar{h}(x) = h(x) - c,\ h \in X',\ c \in \mathbb{R}\}$ and $Z = X$ then

$$s(f, H, X) = \mathrm{epi}\ f^*.$$

Here f^* denotes the *Fenchel conjugate* of f (see [31, 32] and Definition 2.8 to follow).

Definition 2.4. A set $U \subseteq H$ is called *H-convex* on Z if there is a function f such that $U = s(f, H, Z)$. Clearly U is H-convex if, and only if, $U = s(p_U, H, Z)$ where, for each $z \in Z$,

$$p_U(z) = \sup\{h(z) : h \in U\} \tag{2.1}$$

is a H-convex function.
For an arbitrary subset $U \subseteq H$ the set $s(p_U, H, Z)$ (where p_U is defined by (2.1)) is called the *H-convex hull* of the set U and will be denoted $\mathrm{co}_H^Z\, U$.
The following proposition is immediate from the definitions above. Note that this simple result illustrates the separation property for H-convex sets.

Proposition 2.5. *Let $U \subseteq H$ then U is H-convex on Z if and only if for any $h' \in H$, $h' \notin U$, there is a $x \in Z$ such that $h'(x) > \sup_{h \in U} h(x)$.*

By definition the function $-\infty : z \mapsto -\infty$ (for all $z \in Z$) is H-convex and the empty set is clearly a H-convex set. Note that $\emptyset = s(-\infty, H, Z)$.
Throughout we shall denote the set of all H-convex functions defined on $Z \subseteq X$ by $\overline{\mathcal{F}}(H, Z)$. Similarly the set of all H-convex sets will be denoted $\overline{\mathcal{S}}(H, Z)$. We shall introduce order relations on these sets as follows:

$$p_1,\ p_2 \in \overline{\mathcal{F}}(H, Z),\ p_1 \geq p_2 \iff (\forall z \in Z)\ p_1(z) \geq p_2(z).$$

$$U_1,\ U_2 \in \overline{\mathcal{S}}(H, Z),\ U_1 \geq U_2 \iff U_1 \supseteq U_2.$$

Consider the mapping $\Phi : \overline{\mathcal{F}}(H, Z) \to \overline{\mathcal{S}}(H, Z)$ defined by $\Phi(p) = s(p, H, Z)$. This mapping is known as a *Minkowski duality* (see [25]). It is not difficult to show that Φ is a one-to-one mapping and, moreover, it is an isomorphism between the ordered sets. Minkowski duality extends the well known dual relationship between sublinear functions and support sets to H-convex functions and H-convex sets.
It is easy to show that the mapping $U \mapsto \mathrm{co}_H^Z\, U$ defined on the set 2^H is a closure in the sense of Moore (see [5]). In particular we have the following:

$$(\forall U \subseteq H)\ \ U \subseteq \mathrm{co}_H^Z\, U$$

$$(\forall U \subseteq H)\ \ \mathrm{co}_H^Z\,(\mathrm{co}_H^Z\, U) = \mathrm{co}_H^Z\, U$$

$$U_1,\ U_2 \subseteq H,\ \ U_1 \subseteq U_2 \implies \mathrm{co}_H^Z\, U_1 \subseteq \mathrm{co}_H^Z\, U_2$$

It follows from Moore's theorem ([5]) that the ordered set $\overline{\mathcal{S}}(H,Z)$ is a complete lattice where, for an arbitrary family (U_α) (with $U_\alpha \in \overline{\mathcal{S}}(H,Z)$ for each α), we have

$$\sup_\alpha U_\alpha = \mathrm{co}_H^Z(\cup_\alpha U_\alpha), \quad \inf_\alpha U_\alpha = \cap_\alpha U_\alpha.$$

Since Minkowski duality is an isomorphism between the ordered sets $\overline{\mathcal{F}}(H,Z)$ and $\overline{\mathcal{S}}(H,Z)$ it follows that $\overline{\mathcal{F}}(H,Z)$ is also a complete lattice. In addition the following is valid for any family (p_α) with, for each α, $p_\alpha \in \overline{\mathcal{F}}(H,Z)$:

$$(\sup_\alpha p_\alpha)(\cdot) = \sup_\alpha p_\alpha(\cdot)$$
$$(\inf_\alpha p_\alpha)(\cdot) = \mathrm{co}_H^Z(\inf_\alpha p_\alpha(\cdot))$$

Here $(\sup_\alpha p_\alpha)(\cdot)$ and $(\inf_\alpha p_\alpha)(\cdot)$ are boundaries in the lattice $\overline{\mathcal{F}}(H,Z)$ and $\sup_\alpha p_\alpha(\cdot)$ and $\inf_\alpha p_\alpha(\cdot)$ are pointwise boundaries.
Similarly we have:

$$\mathrm{s}(\sup_\alpha p_\alpha, H, Z) = \mathrm{co}_H^Z(\cup_\alpha \mathrm{s}(p_\alpha, H, Z))$$
$$\mathrm{s}(\inf_\alpha p_\alpha, H, Z) = \cap_\alpha \mathrm{s}(p_\alpha, H, Z)$$

where $\sup_\alpha$ and $\inf_\alpha$ are boundaries in the lattice $\overline{\mathcal{F}}(H,Z)$.

Definition 2.6. A set A is called a *semilinear* space if there is a binary operation $+$ and the operation of multiplication by a positive scalar defined on the set A such that, for a, a_1, a_2, $a_3 \in A$ and λ, $\mu > 0$,

$$\begin{array}{cc} a_1+(a_2+a_3) = (a_1+a_2)+a_3 & a_1+a_2 = a_2+a_1 \\ \lambda(a_1+a_2) = \lambda a_1 + \lambda a_2 & (\lambda+\mu)a = \lambda a + \mu a \\ (\lambda\mu)a = \lambda(\mu)a & 1\cdot a = a \end{array}$$

It is easy to check that if the cancellation law holds in a semilinear space A, i.e a, b, $c \in A$, $a+c = b+c \implies a = b$, then there is an isomorphism between A and a convex cone in a vector space (namely A as the convex cone embedded in the vector space $A-A$). Convex sets are defined in a semilinear space in the usual way. Note that the set of all functions $f : X \to \mathbb{R}_{+\infty}$, where X is an arbitrary nonempty set, is a semilinear space (given the usual pointwise definition of function addition and positive scalar multiplication). The set of all l.s.c convex functions $f : X \to \mathbb{R}_{+\infty}$ defined on a l.c.H.t.v.s X is a semilinear space.
Let A be both a semilinear space and a lattice. This set will called a *semilinear lattice* if

(a) $x \geq y \implies (\forall z \in A)\ x+z \geq y+z$
(b) $x \geq y \implies (\forall \lambda > 0)\ \lambda x \geq \lambda y$
(c) For arbitrary $U \subseteq A$ and $z \in A$, $\sup(z+U) = z + \sup U$.

Let H be a semilinear space with respect to the usual operations of pointwise addition and positive scalar multiplication. It follows in this case that $\overline{\mathcal{F}}(H,Z)$ is a semilinear lattice. It should be noted that $\overline{\mathcal{F}}(H,Z)$ cannot be represented as a cone in a vector space since if p, p_1, $p_2 \in \overline{\mathcal{F}}(H, Z)$, $p_1 + p = p_2 + p$ and there is a subset $Z_o \subseteq Z$ such that $p(z) = +\infty$ (for $z \in Z_o$) then $p_1 = p_2$ does not necessarily hold.

Now let us consider the lattice $\overline{\mathcal{S}}(H, Z)$. It is straightforward to verify that a H-convex set is convex with respect to the algebraic operations on H. We define on $\overline{\mathcal{S}}(H, Z)$ a binary operation $\oplus$ and positive scalar multiplication as follows (for U_1, U_2, $U \in \overline{\mathcal{S}}(H, Z)$, $\lambda > 0$):

$$U_1 \oplus U_2 = \mathrm{co}_H^Z (U_1 + U_2) \quad \lambda U = \{\lambda u : u \in U\}.$$

It is easy to verify that Minkowski duality in this case, i.e the mapping $f \mapsto \mathrm{s}(f, H, Z)$, is an algebraic isomorphism between $\overline{\mathcal{F}}(H, Z)$ and $\overline{\mathcal{S}}(H, Z)$ and therefore $\overline{\mathcal{S}}(H, Z)$ is a semilinear lattice.

We can define, in a symmetric manner, concepts of H-concave functions and sets. A function $q : Z \to \mathbb{R} \cup \{-\infty\}$ is said to be *H-concave* if there is a set $U \subseteq H$ such that, for each $z \in Z$,

$$q(z) = \inf\{h(z) : h \in U\}.$$

A set $U \subseteq H$ is called *H-concave* if $U = \{h \in H : (\forall z \in Z)\, h(z) \geq \inf_{h' \in U} h'(z)\}$. We shall denote the set of H-concave functions defined on a set $Z \subseteq X$ by $\underline{\mathcal{F}}(H, Z)$ and the set of all H-concave sets by $\underline{\mathcal{S}}(H, Z)$.

We shall be particularly interested in the sequel in the class of functions expressible as the pointwise infimum of a family of H-convex functions, in the notation above we are refering to the class $\underline{\mathcal{F}}(\overline{\mathcal{F}}(H, Z), Z)$. Thus we require the following definition.

Definition 2.7. Let X be an arbitrary set and $Z \subseteq X$ a non-empty subset. Let H be a set of functions defined on X and mapping into $\mathbb{R}$. A function $q : Z \to \mathbb{R}_{+\infty}$ is called *inf-H-convex* on Z if there is a set $\Delta \subseteq \overline{\mathcal{F}}(H, Z)$ such that, for all $z \in Z$,

$$q(z) = \inf\{h(z) : h \in \Delta\}$$

If X is a locally convex topological vector space and H is the set of continuous affine functions defined on X then we shall denote inf-H-convex functions as merely inf-convex functions. Note that we can define the class of min-H-convex functions in an analogous fashion simply replacing infimum by minimum.
This definition extends the concept of inf-convexity defined in [13] and inf-sublinearity used in, for example, [10].
In addition to a semilinear space we shall require the weaker idea of a *conic set*, defined as follows.

Definition 2.8. A set A is called a *conic set* if there is an operation of positive scalar multiplication defined on A such that the following hold, for each $a \in A$, λ, $\mu > 0$:

$$1 \cdot a = a, \quad (\lambda\mu)a = \lambda(\mu)a.$$

Let H be a set of functions defined on the set $Z \subseteq X$ and assume that H is a conic set with respect to the usual positive scalar multiplication. Clearly if p is a H-convex function in this case then λp is a H-convex function for all $\lambda > 0$. Thus $\overline{\mathcal{F}}(H, Z)$ is a conic set. Since H is a conic set it also immediately follows (defining positive scalar multiplication in the usual way) that $\overline{\mathcal{S}}(H,Z)$ is also a conic set. It easily follows that these conic sets, $\overline{\mathcal{F}}(H,Z)$ and $\overline{\mathcal{S}}(H,Z)$, are isomorphic where we identify support sets as follows, for $p \in \overline{\mathcal{F}}(H,Z)$ and $\lambda > 0$:

$$\mathrm{s}(\lambda p, H, Z) = \lambda\, \mathrm{s}(p, H, Z).$$

We now recall the concept of a H-conjugate function (see, for example, [4, 25, 29, 34]) and develop connections between these functions and the concept of an ϵ-subdifferential. In the following we shall assume that $X = Z$ and that H is a set of real valued functions defined on X. Let $\bar{H}$ denote the set of all functions $\bar{h}$ which have the form $\bar{h}(x) = h(x) - c$ for some $h \in H$ and $c \in \mathbb{R}$ and every $x \in X$.

Definition 2.9. Let $f : X \to \mathbb{R}$, then the *H-conjugate* of f, denoted f^*, is defined by

$$f^*(h) = \sup_{x \in X}(h(x) - f(x))$$

for each $h \in H$.

The concept of a generalized conjugate has been studied extensively in the literature as an extension of the classical Fenchel-Moreau conjugate of a convex function; see, for example [28, 34, 25, 4].
The *epigraph* of a function f is the set epi $f = \{(x,\alpha) : x \in \mathrm{dom} f,\ \alpha \geq f(x)\}$.

Proposition 2.10. *For a function $f : X \to \mathbb{R}$ the epigraph of its H-conjugate f^* coincides with the support set of f with respect to $\bar{H}$, i.e*

$$\mathrm{epi}\ f^* = \mathrm{s}(f, \bar{H}, X).$$

Proof. We have the following:

$$(h,\lambda) \in \mathrm{epi}\ f^* \iff \lambda \geq f^*(h) \iff (\forall x \in X)\, \lambda \geq h(x) - f(x)$$

$$\iff (\forall x \in X)\, f(x) \geq h(x) - \lambda \iff (h,\lambda) \in \mathrm{s}(f, \bar{H}, X).$$

□

Definition 2.11. Let p be a $\bar{H}$-convex function and $\epsilon \geq 0$. A function $h \in H$ is called an *ϵ-subgradient* of the function f at the point x_o if, for all $x \in X$,

$$h(x) - h(x_o) \leq p(x) - p(x_o) + \epsilon.$$

The set $\partial_{H,\epsilon} p(x_o)$ of all ϵ-subgradients of p at x_o is called the *ϵ-subdifferential* of p at x_o with respect to H. Since p is $\bar{H}$-convex, for each $\epsilon > 0$ there is a function $h \in H$ and a $\lambda \in \mathbb{R}$ such that, for all $x \in X$:

$$h(x) - \lambda \leq p(x) \ \text{ and } \ h(x_o) - \lambda > p(x_o) - \epsilon.$$

Thus we have $h(x) - h(x_o) \leq p(x) - p(x_o) + \epsilon$. In particular $\partial_{H,\epsilon} p(x_o) \neq \emptyset$ for all $\epsilon > 0$. However it may be possible that $\partial_{H,0} p(x_o)$ is empty.
The ϵ-subdifferential, $\partial_{H,\epsilon} p(x_o)$, is studied systematically in [34].

Let p be a $\bar{H}$-convex function. Then for each $\bar{h} \in \mathrm{s}(p, \bar{H}, X)$ there are $h \in H$ and $\lambda \in \mathbb{R}$ such that $\bar{h}(\cdot) = h(\cdot) - \lambda$. Thus, for simplicity, we identify $\bar{h}$ and the pair (h, λ) in a natural way.

Proposition 2.12. *Let p be a $\bar{H}$-convex function and let $x_o \in \operatorname{dom} p = \{x \in X : p(x) < +\infty\}$. Then the following holds:*

$$\mathrm{s}(p, \bar{H}, X) = \cup_{\epsilon \geq 0}\{(h, \lambda) : h \in \partial_{H,\epsilon} p(x_o),\ \lambda = -(p(x_o) - h(x_o)) + \epsilon\}. \tag{2.2}$$

Proof. Let V denote the set on the righthand side of (2.2). If $(h, \lambda) \in \mathrm{s}(p, \bar{H}, X)$ then, for all $x \in X$, $p(x) \geq h(x) - \lambda$. In particular $p(x_o) \geq h(x_o) - \lambda$. Let $\epsilon = (p(x_o) - h(x_o)) + \lambda$. Then $\epsilon \geq 0$ and $p(x) - p(x_o) \geq h(x) - h(x_o) - \epsilon$, in particular $h \in \partial_{H,\epsilon} p(x_o)$ and $(h, \lambda) \in V$. Conversely, if $(h, \lambda) \in V$ then there is an $\epsilon \geq 0$ such that $h \in \partial_{H,\epsilon} p(x_o)$ and $\lambda = -(p(x_o) - h(x_o)) + \epsilon$. Thus, for all $x \in X$,

$$p(x) \geq p(x_o) + h(x) - h(x_o) - \epsilon \quad \text{and} \quad -\lambda = [p(x_o) - h(x_o)] - \epsilon.$$

Thus $(h, \lambda) \in \mathrm{s}(p, \bar{H}, X)$ as required. □

Proposition 2.13. *Let p be a $\bar{H}$-convex function and $\epsilon \geq 0$. Let $x_o \in X$, then*

$$h \in \partial_{H,\epsilon}(x_o) \iff p^*(h) + p(x_o) - h(x_o) \leq \epsilon.$$

Proof. Follows easily from the Definition of the H-conjugate and the ϵ-subgradient. □

3. Examples of H-convex and inf-H-convex Functions

The following examples are included to illustrate the broad nature of the classes of H-convex and inf-H-convex functions. We begin with examples of H-convex functions for various sets H.

Example 3.1. Let X be a locally convex topological vector space and let $H = X'$ the set of continuous linear functions defined on X. Furthermore let K be a closed convex cone in X. It is well known that a function p defined on K is H-convex if and only if p is l.s.c and sublinear.

Example 3.2. Let X be as in example 3.1 and suppose that Z is an arbitrary nonempty subset of X. Let $H = X'$ and so $\bar{H} = \{(\ell, c) : \ell \in X',\ c \in \mathbb{R}\}$ denotes the set of all continuous affine functions defined on X. It is easy to check that a function f defined on Z is $\bar{H}$-convex if and only if there is a l.s.c convex function $\bar{f}$ defined on the closed convex hull of Z (cl co Z) such that, for all $z \in Z$, $f(z) = \bar{f}(z)$. In particular if Z is a closed convex set then a function f is $\bar{H}$-convex if and only if it is a l.s.c convex function.

Example 3.3. Let X be as above with $K \subseteq X$ a closed convex cone. Let H be the cone K^* of all continuous linear functions nonnegative on K (i.e. the dual cone to K). Let $Z \subseteq K$ be a closed convex cone. It can be shown (see [25, 35]) that $p \in \overline{\mathcal{F}}(K^*, Z)$ if and only if there is a sublinear function $\tilde{p} : X \to \mathbb{R}_{+\infty}$ such that

(i) for all $z \in Z$, $\tilde{p}(z) = p(z)$, and
(ii) $\tilde{p}$ is K-increasing; that is if $x_1 \geq x_2$ (i.e $x_1 \in x_2 + K$) then $\tilde{p}(x_1) \geq \tilde{p}(x_2)$.

Example 3.4. Let Z be subset of $\mathbb{R}^n$ and let H be the set of all quadratic functions, h, defined on $\mathbb{R}^n$, of the form:

$$h(x) = a\|x\|^2 + [\ell, x] + c$$

where $a \leq 0$, $\ell \in \mathbb{R}^n$ ($[\cdot,\cdot]$ denotes the usual inner product), $c \in \mathbb{R}$. In this case it can be shown (see [25]) that $\overline{\mathcal{F}}(H, Z)$ is the set of all l.s.c functions, p, defined on Z such that there is a quadratic function h with the property $h(z) \leq p(z)$ for all $z \in Z$. In particular if Z is a compact set then $\overline{\mathcal{F}}(H, Z)$ is the set of all l.s.c functions defined on Z.
In Examples 3.1 to 3.4 the set H (or $\bar{H}$) is a semilinear space.

Example 3.5. Let $X = Z = \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the nonnegative orthant. Let H be the set of all functions h defined on $\mathbb{R}^n_+$ generated by a vector $(h_1, \ldots, h_n) \in \mathbb{R}^n_+$ where

$$h(x) = \min_{i \in \mathcal{T}(h)} h_i x_i,$$

where $\mathcal{T}(h) = \{i : h_i > 0\}$. We assume that the minimum over the empty set is zero for convenience. It can be shown (see [1, 3]) that $\overline{\mathcal{F}}(H, \mathbb{R}^n_+)$ is the set of all increasing positively homogeneous functions defined on $\mathbb{R}^n_+$.

Example 3.6. where H is the set defined in example 3.4 above. Thus $\bar{H}$ is the set of all functions $\bar{h}$ which have the form $\bar{h}(x) = h(x) - c$, $h \in H$, $c \in \mathbb{R}$, for every $x \in \mathbb{R}^n_+$. It can be shown (see [2, 3]) that $\overline{\mathcal{F}}(\bar{H}, \mathbb{R}^n_+)$ is the set of all increasing convex-along-rays functions defined on $\mathbb{R}^n_+$. A function f defined on $\mathbb{R}^n_+$ is said to be convex-along-rays if, for each $y \in \mathbb{R}^n_+$, the function $f_y(\lambda) = f(\lambda y)$ is convex (for $\lambda > 0$).

Example 3.7. Let $h \in \mathbb{R}^n_+$ and $c \in (0, +\infty)$. We consider the function $\tilde{h}$ defined on $\mathbb{R}^n_+$ by the formula:

$$\tilde{h}(x) = \min_{i \in \mathcal{T}(h)} (h_i x_i, c), \quad x \in \mathbb{R}^n_+$$

Let $\tilde{H}$ be the set of all functions of this form for $h \in \mathbb{R}^n_+$ and $c > 0$. It can be shown (see [3]) that a function f defined on $\mathbb{R}^n_+$ is $\tilde{H}$-convex on a set $Z \subseteq \mathbb{R}^n_+$ if and only if f is increasing and quasihomogeneous (i.e $(\forall x \in \mathbb{R}^n_+)(\forall \alpha \in [0,1])\ f(\alpha x) \geq \alpha f(x)$).

Example 3.8. Here we extend the collection of examples which are connected with the operation of taking *minima* on the cone $\mathbb{R}^n_+$. Let $X = Z = \mathbb{R}^n_+$ and let h be a function which is generated by a vector $(h_1, \ldots, h_n) \in \operatorname{int} \mathbb{R}^n_+$ as follows:

$$h(x) = \min_{i=1,\ldots,n} h_i x_i, \quad x \in \mathbb{R}^n_+ \tag{3.1}$$

Let $\tilde{H}$ be the union of zero (as a function defined on $\mathbb{R}^n_+$) and all functions of the form (3.1) with $(h_1, \ldots, h_n) \in \operatorname{int} \mathbb{R}^n_+$. Clearly $\tilde{H}$ is a subset of the set H considered in example 3.5.

Proposition 3.9. *A function p defined on $\mathbb{R}^n_+$ is $\tilde{H}$-convex on the cone $Z = \mathbb{R}^n_+$ if and only if p is increasing and positively homogeneous of degree one with $p(x) = 0$ for all x lying on the boundary of the cone $\mathbb{R}^n_+$ (i.e $p(x) = 0$ if $\min_i x_i = 0$).*

Proof. It is easy to check that a $\tilde{H}$-convex function p is increasing and positively homogeneous of degree one with $p(x) = 0$ whenever $\min_i x_i = 0$. Conversely, suppose p is a function with the properties mentioned above. Let $\bar{x} \in \operatorname{int} \mathbb{R}^n_+$ and $h^{\bar{x}} = (h^{\bar{x}}_1, \dots, h^{\bar{x}}_n)$ where $h^{\bar{x}}_i = p(\bar{x})/\bar{x}_i$, $i = 1, \dots, n$. We then define the function h_x on $\mathbb{R}^n_+$ by (3.1). Thus we have

$$h_{\bar{x}}(\bar{x}) = \min_i h^{\bar{x}}_i \bar{x}_i = p(\bar{x}).$$

On the other hand if $x \in \mathbb{R}^n_+$ and $\lambda = \min_i(x_i/\bar{x}_i)$ then $x \geq \lambda \bar{x}$. Therefore

$$p(x) \geq p(\lambda \bar{x}) = \lambda p(\bar{x}) = \min_i \frac{p(\bar{x}) x_i}{\bar{x}_i} = \min_i h^{\bar{x}}_i x_i = h_{\bar{x}}(x).$$

Let $U = \{h_x \in \tilde{H} : x \in \operatorname{int} \mathbb{R}^n_+\}$. We have, for $x \in \operatorname{int} \mathbb{R}^n_+$,

$$p(x) = \max_{h \in U} h(x).$$

Clearly $\max_{h \in U} h(x) = 0$ if $\min_i x_i = 0$. Thus $p(x) = \max_{h \in U} h(x)$ for all x and consequently p is a H-convex function. □

Remark 3.10. The main notion underlying the H-convex functions discussed in examples 3.5 and 3.8 is that of a *normal set*. A subset U of the cone $\mathbb{R}^n_+$ is called *normal* if the relations $x \in U$, $x' \leq x$, $x' \in \mathbb{R}^n_+$ imply $x' \in U$. It can be shown (see [35]) that a function p defined on $\mathbb{R}^n_+$ is positively homogeneous of degree one and increasing if and only if there is a closed normal set U such that p is the Minkowski gauge of U, i.e. $p(x) = \inf\{\lambda > 0 : x \in \lambda U\}$. Normal sets play an important role in the study of various models in Mathematical Economics (see [35, 26]). We shall meet various modifications of these sets when studying H-convex sets.

Example 3.11. Let $X = \mathbb{R}^n$. For $\ell \in X'$ let us denote $\ell^+(x) = \max\{\ell(x), 0\}$. Let us fix a natural number N and consider the set H_N of all functions defined on $\mathbb{R}^n$ which have the form

$$h(x) = \min_{i=1,\dots,N} \ell^+_i(x)$$

with $\ell_i \in X'$ $(i = 1, \dots, N)$.

Clearly a H_N-convex function is nonnegative and positively homogeneous. Applying results by Shveidel [37] it is not difficult to show that there is a number N which depends only on n such that the set of H_N-convex functions coincides with the set of all positively homogeneous nonnegative l.s.c functions.

Example 3.12. Let X be a locally convex topological vector space and X' its conjugate space. For $\ell \in X'$ and $c \geq 0$ let us consider the functions $h_{\ell,c}$ which have the form

$$h_{\ell,c}(x) = \begin{cases} c & [\ell, x] > 1 \\ 0 & [\ell, x] \leq 1 \end{cases}$$

Let H be the set of all functions $h_{\ell,c}$ for $\ell \in X'$ and $c \geq 0$. Then it follows (see [36]) that a function p defined on X is H-convex if and only if p is a l.s.c nonnegative quasiconvex function with $p(0) = 0$.

Example 3.13. Let us extend example 3.12 by considering the set H of all functions h of the form

$$h(x) = \begin{cases} c & [\ell, x - x_o] > 1 \\ c' & [\ell, x - x_o] \leq 1 \end{cases}$$

where $\ell \in X'$, $x_o \in X$, $c, c' \in \mathbb{R}_{+\infty}$ and $c \geq c'$. It follows easily from [36] that a function p defined on X is H-convex if and only if p is l.s.c quasiconvex and bounded from below on X.

It is worth noting that H (or $\bar{H}$) in examples 3.5 to 3.13 are conic sets.

We now consider examples of inf-H-convex functions.

Example 3.14. From the definition it is clear that every H-concave function is inf-H-convex.

Example 3.15. Let H be a linear space. Then the difference, $f_1 - f_2$, of two H-convex functions is inf-H-convex. If $f_1, f_2 \in \overline{\mathcal{F}}(H, Z)$ then, for each $x \in Z$,

$$\begin{aligned} f_1(x) - f_2(x) &= f_1(x) - \sup_{h \in \mathrm{s}(f_2, H, Z)} h(x) \\ &= f_1(x) + \inf_{h \in \mathrm{s}(f_2, H, Z)} (-h)(x) \\ &= \inf_{h \in \mathrm{s}(f_2, H, Z)} (f_1 - h)(x) \end{aligned}$$

Since $f_1 - h$ is H-convex for each $h \in H$ it follows that $f_1 - f_2$ is inf-H-convex.
In particular the difference of two l.s.c sublinear functions is inf-H-convex when $H = X'$; the difference of two l.s.c convex functions is inf-H-convex when $H = X' \times \mathbb{R}$.

Example 3.16. Let f be a positively homogeneous (of degree one) continuous and quasiconvex function defined on $X = \mathbb{R}^n$. It can be shown (see [6]) that there are two l.s.c sublinear functions f_1, f_2 such that, for each $x \in \mathbb{R}^n$,

$$f(x) = \min(f_1(x), f_2(x)).$$

Thus f is inf-H-convex where $H = X'$ (in fact min-H-convex).

Example 3.17. Let Z be a convex subset of the space X and $f : Z \to \mathbb{R}$ be an u.s.c function such that for some continuous convex function h, $h(x) > f(x)$ for all $x \in Z$. Then (see [13]) f is inf-convex (i.e. f is inf-H-convex with H the set of continuous affine functions defined on X). In particular if Z is a compact convex set then every u.s.c function defined on Z is inf-convex.

Example 3.18. Let Z be a compact convex set in $\mathbb{R}^n$ and let G be an open bounded set with $Z \subseteq G$. We shall denote the set of all twice continuously differentiable functions defined on the set G by $C^2(G)$. We require the following lemma:

Lemma 3.19. *Let $f \in C^2(G)$ and $z \in Z$. Then there is a number k_z such that the quadratic function p_z defined as follows:*

$$p_z(x) = k_z\|x - z\|^2 + [\nabla f(z), x - z] + f(z) \tag{3.2}$$

possesses the following properties:

$$p_z(z) = f(z) \ \text{ and } \ (\forall x \in Z)\ p_z(x) \geq f(x).$$

Proof. Clearly $p_z(z) = f(z)$ is valid for any k_z. Now let us consider the quadratic (3.2) with an arbitrary $k_z \geq \frac{1}{2}\|\nabla^2 f(z)\| + m$ where $m > 0$. Let $F(x) = p_z(x) - f(x)$. We have

$$\nabla F(x) = \nabla p_z(x) - \nabla f(x) = 2k_z(x - z) + \nabla f(z) - \nabla f(x).$$

Thus $\nabla^2 F(x) = 2k_z \,\mathrm{Id} - \nabla^2 f(x)$, where Id denotes the identity mapping. In particular $\nabla F(z) = 0$ and

$$\begin{aligned} x^T \nabla^2 F(z) x &= 2k_z\|x\|^2 - x^T \nabla^2 f(z) x \\ &\geq 2k_z\|x\|^2 - \|\nabla^2 f(z)\|\|x\|^2 \\ &> m\|x\|^2, \ \text{ for all } x \end{aligned}$$

Therefore for an arbitrary $k_z > \frac{1}{2}\|\nabla^2 f(z)\|$ the function $F(x)$ achieves a local maximum at the point z, i.e. there is a neighbourhood V_z of the point z such that $f(x) < p_z(x)$ for all $x \in V_z$ $(x \neq z)$. On the other hand since the set $Z \backslash V_z$ is compact it is easy to check that for large k_z, $p_z(x) > f(x)$ for all $x \in Z \backslash V_z$. □

Corollary 3.20. *If $f \in C^2(G)$ then*

$$f(x) = \min_{z \in Z}(k_z\|x - z\|^2 + [\nabla f(z), x - z] + f(z))$$

Thus Corollary 3.20 shows that every C^2 function defined on a compact set G is inf-H-convex (in fact min-H-convex) where H is the semilinear space of all convex quadratic functions defined on $\mathbb{R}^n$.

4. General Solvability Theorems

In this section we develop solvability theorems for infinite systems of inequalities involving H-convex functions. Furthermore we extend the recent results in [13] by establishing very general solvability theorems for systems of inequalities involving functions expressible as the pointwise infimum of H-convex functions, that is functions within the class $\underline{\mathcal{F}}(\overline{\mathcal{F}}(H, Z), Z)$.

In the following cone V represents the conic hull of a set V, i.e.

$$\operatorname{cone} V = \bigcup_{\lambda > 0} \lambda V.$$

This operation is well defined for a conic set.

Theorem 4.1. *Let Z be a subset of set X, let H be a conic set of functions defined on X and let I be an arbitrary index set. Furthermore let f and, for each $i \in I$, g_i be H-convex functions defined on Z. Then the following statements are equivalent:*

(i) $(\forall i \in I)\; g_i(x) \leq 0 \implies f(x) \leq 0$

(ii) $\mathrm{s}(f, H, Z) \subseteq \mathrm{co}_H^Z \,\mathrm{cone} \bigcup_i \mathrm{s}(g_i, H, Z)$

Proof. For each $z \in Z$ let $g(z) = \sup_{i \in I} g_i(z)$. The function g is H-convex. Define the level sets $S_g = \{z : g(z) \leq 0\}$ and $S_f = \{z : f(z) \leq 0\}$. Clearly statement (i) can be written as follows:
(iii) $S_g \subseteq S_f$
Let δ denote the indicator function of the set S_g:

$$\delta(z) = \begin{cases} +\infty & z \notin S_g \\ 0 & z \in S_g \end{cases}$$

It is easy to check that inclusion (iii) is equivalent to the inequality $f \leq \delta$. In particular if $f \leq \delta$ then $g(x) \leq 0$ implies $f(x) \leq 0$. On the other hand if $S_g \subseteq S_f$ then $f(x) \leq 0$ whenever $x \in S_g$ i.e. $f \leq \delta$.
Clearly $\delta = \sup_{\lambda > 0} \lambda g$. Since H is a conic set we have that λg is a H-convex function since g is H-convex. Therefore δ is a H-convex function as the supremum of H-convex functions. Now let us compute the support set $\mathrm{s}(\delta, H, Z)$ of the function δ. Since $\delta = \sup_{\lambda>0} \sup_i g_i = \sup_{\lambda>0,\, i} \lambda g_i$ we have

$$\begin{aligned} \mathrm{s}(\delta, H, Z) &= \mathrm{co}_H^Z \bigcup_{\lambda>0,\, i} \lambda \mathrm{s}(g_i, H, Z) \\ &= \mathrm{co}_H^Z \bigcup_{\lambda>0} \bigcup_{i \in I} \mathrm{s}(g_i, H, Z) \\ &= \mathrm{co}_H^Z \,\mathrm{cone} \bigcup_{i \in I} \mathrm{s}(g_i, H, Z) \end{aligned}$$

Minkowski duality shows that the inequality $f \leq \delta$ is equivalent to the inclusion

$$\mathrm{s}(f, H, Z) \subseteq \mathrm{s}(\delta, H, Z) = \mathrm{co}_H^Z \,\mathrm{cone} \bigcup_{i \in I} \mathrm{s}(g_i, H, Z).$$

Hence the result follows. □

We shall discuss special cases of this solvability result (and Theorem 4.3 to follow) in Section 6. It suffices to note that Theorem 4.1 is a very general nonlinear extension of the classical Farkas lemma for finite systems of linear inequalities which is applicable to systems involving sublinear, convex and certain quasiconvex functions for example. The application of Theorem 4.1 to systems of quasiconvex functions raises several open questions (see the Conclusion for a detailed discussion) related to the description of the H-convex hull in such examples.

Extensions of Farkas' lemma have been used extensively in applications to nonsmooth, and most recently, global optimization. This type of solvability theorem provides a dual characterization of inconsistency for a specific inequality system which arises when considering a suitable first order approximation to the programming problem under consideration (see [9, 11, 12, 20, 21]).

Proposition 4.2. *Let I, and for each $i \in I$, g_i be as in* Theorem 4.1*, then the system*

$$(\forall i \in I)\ g_i(z) \leq 0,\ z \in Z \tag{4.1}$$

is inconsistent on the set Z if and only if

$$\mathrm{co}_H^Z \ \mathrm{cone} \bigcup_{i \in I} \mathrm{s}(g_i, H, Z) = H.$$

Proof. Let $g(z) = \sup_i g_i(z)$ and $\delta(z) = \sup_{\lambda > 0} \lambda g(z)$. Clearly the system (4.1) is inconsistent on the set Z if and only if $g(z) > 0$ for all $z \in Z$. Equivalently, $\delta(z) = +\infty$ for all $z \in Z$. Thus we have

$$\mathrm{s}(\delta, H, Z) = \mathrm{co}_H^Z \ \mathrm{cone} \bigcup_{i \in I} \mathrm{s}(g_i, H, Z)$$

$$\mathrm{s}(+\infty, H, Z) = H.$$

Thus the result follows. □

We shall now study characterizations of solvability theorems involving functions expressible as the pointwise infimum of H-convex functions.
In the following let $Z \subseteq X$ and let H be a semilinear space of functions defined on X. Let I be an arbitrary index set with f and, for each $i \in I$, g_i inf-H-convex. Thus for each $i \in I$ there is a set Δ_i and a family $\{p_{\alpha_i} : \alpha_i \in \Delta_i\}$ with $p_{\alpha_i} \in \overline{\mathcal{F}}(H, Z)$ such that, for each $z \in Z$,

$$g_i(z) = \inf_{\alpha_i \in \Delta_i} p_{\alpha_i}(z)$$

In addition there is a set Δ and a family $\{p_\alpha : \alpha \in \Delta\} \subseteq \overline{\mathcal{F}}(H, Z)$ such that, for each $z \in Z$,

$$f(z) = \inf_{\alpha \in \Delta} p_\alpha(z).$$

Let $(A(t))_{t \in T}$ be a family of nonempty sets, then a selection function a is a function of the form

$$a = (a_t)_{t \in T} \in \prod_{t \in T} A(t).$$

Thus in particular $a_t \in A(t)$ for each $t \in T$. We will use the notation

$$(a_t) \in \prod_{t \in T} A(t).$$

Theorem 4.3. *Let $Z \subseteq X$ and let H be a semilinear space of functions defined on X. Let I be an arbitrary index set with f and, for each $i \in I$, g_i $\inf -H$-convex. Then the following statements are equivalent:*

(i) $(\forall i \in I)\, g_i(z) \leq 0 \implies f(z) \geq 0$

(ii) *For each $\alpha \in \Delta$ and $(\alpha_i) \in \prod_{i \in I} \Delta_i$*

$$0 \in \mathrm{co}_H^Z \left(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \,\mathrm{cone} \bigcup_{i \in I} \mathrm{s}(p_{\alpha_i}, H, Z)\right)$$

Note that Theorem 4.3 extends Theorem 3.2 in [13] which was established for systems of inf-convex functions and involved an additional boundedness assumption on the domain of the functions involved.

We require the following lemmas (see [13] for Lemma 4.4).

Lemma 4.4. *Consider a set I and a family of sets $(\Delta_i)_{i \in I}$. For convenience let $\mathcal{S}$ denote the set of all selection functions a defined on I with, for each $i \in I$, $a(i) = a_i \in \Delta_i$. Assume that for each $i \in I$ there is a function $t_i : \Delta_i \to \mathbb{R}$ such that $\inf_{\alpha_i \in \Delta_i} t_i(\alpha_i) > -\infty$. Then for the function $t : \mathcal{S} \times I \to \mathbb{R}$, $t(a, i) = t_i(a(i))$ we have*

$$\inf_{a \in \mathcal{S}} \sup_{i \in I} t(a, i) = \sup_{i \in I} \inf_{a \in S} t(a, i).$$

Proof. Since $\mathcal{S}$ is the set of all selection functions we have, for all $i \in I$,

$$\inf_{a \in \mathcal{S}} t(a, i) = \inf_{a \in \mathcal{S}} t_i(a(i)) = \inf_{\alpha_i \in \Delta_i} t_i(\alpha_i).$$

Let $\epsilon > 0$ and, for each $i \in I$, let $\bar{\alpha}_i \in \Delta_i$ be such that

$$t_i(\bar{\alpha}_i) \leq \inf_{\alpha_i \in \Delta_i} t_i(\alpha_i) + \epsilon.$$

Define the selection function $\bar{a} \in \mathcal{S}$ by $\bar{a}(i) = \bar{\alpha}_i$ for all $i \in I$. Then

$$\begin{aligned} \sup_{i \in I} t(\bar{a}, i) &= \sup_{i \in I} t_i(\bar{\alpha}_i) \\ &\leq \sup_{i \in I} \big(\inf_{\alpha_i \in \Delta_i} t_i(\alpha_i) + \epsilon \big) \\ &= \sup_{i \in I} \inf_{a \in \mathcal{S}} t(a, i) + \epsilon. \end{aligned}$$

Consequently,

$$\inf_{a \in \mathcal{S}} \sup_{i \in I} t(a, i) \leq \sup_{i \in I} t(\bar{a}, i) \leq \sup_{i \in I} \inf_{a \in \mathcal{S}} t(a, i) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that inf sup $\leq$ sup inf. The reverse inequality is always true and so the result follows. $\square$

Lemma 4.5. *Let X, Z, H, I, g_i and f be as above. Then the following statements are equivalent:*

(i) $(\forall \alpha \in \Delta)(\forall (\alpha_i) \in \prod_{i \in I} \Delta_i)\ 0 \in \mathrm{co}_H^Z\,(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \bigcup_i \mathrm{s}(p_{\alpha_i}, H, Z))$

(ii) $(\forall z \in Z)\ f(z) + \sup_i g_i(z) \geq 0.$

Proof. Let $\alpha \in \Delta$ and $a = (\alpha_i) \in \prod_i \Delta_i$. For convenience let

$$V_{\alpha, a} = \mathrm{co}_H^Z\,(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \bigcup_i \mathrm{s}(p_{\alpha_i}, H, Z)),$$

and consider the function

$$p_{\alpha, a} = p_\alpha + \sup_i p_{\alpha_i}.$$

Since H is a semilinear space it follows that $\overline{\mathcal{F}}(H, Z)$ is a semilinear lattice and therefore $p_{\alpha, a}$ is a H-convex function. Minkowski duality shows that $V_{\alpha, a}$ is the support set of the function $p_{\alpha, a}$.

Since $V_{\alpha, a} = \{h \in H : h \leq p_{\alpha, a}\}$ and $0 \in H$ we have $0 \in V_{\alpha, a}$ if and only if $p_{\alpha, a}(z) \geq 0$ for all $z \in Z$. Therefore condition (i) is equivalent to

$$\inf_{\alpha \in \Delta} \inf_{a \in \mathcal{S}} p_{\alpha, a}(z) \geq 0, \quad z \in Z$$

(Here $\mathcal{S}$ is the set of all selections of the indexed family $(\Delta_i)_{i \in I}$). To simplify this expression we note the following for each $z \in Z$:

$$\begin{aligned} \inf_{\alpha \in \Delta} \inf_{a \in \mathcal{S}} p_{\alpha, a}(z) &= \inf_{\alpha \in \Delta} \inf_{a \in \mathcal{S}} (p_\alpha(z) + \sup_i p_{\alpha_i}(z)) \\ &= \inf_{\alpha \in \Delta} p_\alpha(z) + \inf_{a \in \mathcal{S}} \sup_i p_{\alpha_i}(z) \\ &= f(z) + \inf_{a \in \mathcal{S}} \sup_i p_{\alpha_i}(z). \end{aligned}$$

Using Lemma 4.4 with $t_i(\alpha_i) = p_{\alpha_i}(z)$ (where $\alpha_i = a(i)$) we have

$$\inf_{a \in \mathcal{S}} \sup_{i \in I} p_{\alpha_i}(z) = \sup_{i \in I} \inf_{a \in \mathcal{S}} p_{\alpha_i}(z) = \sup_{i \in I} g_i(z).$$

Hence

$$\inf_{\alpha \in \Delta} \inf_{a \mathcal{S}} p_{\alpha, a}(z) = f(z) + \sup_{i \in I} g_i(z).$$

So that condition (i) is equivalent to the following:

$$(\forall z \in Z)\ f(z) + \sup_i g_i(z) \geq 0 \tag{4.2}$$

as required. □

Proof of Theorem 4.3: In order to prove that (ii) implies (i) let us consider the set $\mathcal{I} = I \times (0, +\infty)$. Define the function $\tilde{g}_j$ for each $j = (i, \mu) \in \mathcal{I}$ as follows $\tilde{g}_j(z) = \mu g_i(z)$. Clearly statement (i) is equivalent to the following:

(i′) $(\forall j \in \mathcal{I})\ \tilde{g}_j(z) \leq 0 \implies f(z) \geq 0.$

Clearly each function $\tilde{g}_j$, $j \in \mathcal{I}$, is inf-H-convex. Let $\bar{p}_j = \mu p_{\alpha_i}$ and with $\bar{\Delta}_j = \Delta_i$ for each $j = (i, \mu) \in \mathcal{I}$. We have, for each $z \in Z$,

$$\tilde{g}_j(z) = \mu g_i(z) = \inf_{\alpha_i \in \Delta_i} \mu p_{\alpha_i}(z) = \inf_{\bar{\alpha}_j \in \bar{\Delta}_j} \bar{p}_{\bar{\alpha}_j}(z).$$

Clearly $\mathrm{s}(\bar{p}_{\bar{\alpha}_j}, H, Z) = \mu\, \mathrm{s}(p_{\alpha_i}, H, Z)$. Let $\alpha \in \Delta$ and let $\bar{\alpha} = (\bar{\alpha}_j)_{j \in \mathcal{I}}$ be a selection from the set $\prod_{j \in \mathcal{I}} \bar{\Delta}_j$. Let us consider the function $p_{\alpha, \bar{\alpha}} = p_\alpha + \sup_j \bar{p}_{\bar{\alpha}_j}$ and the support set $\mathrm{s}(p_{\alpha,\bar{\alpha}}, H, Z)$ of this function. We have

$$\begin{aligned}
\mathrm{s}(p_{\alpha,\bar{\alpha}}, H, Z) &= \mathrm{co}_H^Z \left(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \bigcup_{j \in \mathcal{I}} \mathrm{s}(\bar{p}_{\bar{\alpha}_j}, H, Z)\right) \\
&= \mathrm{co}_H^Z \left(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \bigcup_{i \in I, \mu > 0} \mu\, \mathrm{s}(p_{\alpha_i}, H, Z)\right) \\
&= \mathrm{co}_H^Z \left(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \,\mathrm{cone} \bigcup_{i \in I} \mathrm{s}(p_{\alpha_i}, H, Z)\right)
\end{aligned}$$

Therefore the condition $0 \in \mathrm{s}(p_{\alpha,\bar{\alpha}}, H, Z)$ is equivalent to:

$$0 \in \mathrm{co}_H^Z \left(\mathrm{s}(p_\alpha, H, Z) + \mathrm{co}_H^Z \,\mathrm{cone} \bigcup_{i \in I} \mathrm{s}(p_{\alpha_i}, H, Z)\right)$$

Thus, by Lemma 4.5, (ii) is equivalent to the following:

$$(\forall z \in Z)\ f(z) + \sup_{j \in \mathcal{I}} \tilde{g}_j(z) \geq 0 \tag{4.3}$$

However,

$$\sup_j \tilde{g}_j(z) = \begin{cases} 0 & \text{if } g_i(z) \leq 0, \text{ for all } i \in I \\ +\infty & \text{if there is an } i \in I \text{ such that } g_i(z) > 0 \end{cases}$$

We now check that (4.3) implies (i). Suppose that (4.3) is true and that $(\forall i \in I)\ g_i(z) \leq 0$. Then $\sup_{j \in \mathcal{I}} \tilde{g}_j(z) = 0$ and consequently by (4.3) $f(z) \geq 0$. Thus (i) is true. Hence since (4.3) is equivalent to (ii) it follows that (ii) implies (i) as required. Similarly we can show that (i$'$) implies (4.3). If (4.3) is not true then there is a $z \in Z$ such that $f(z) + \sup_j \tilde{g}_j(z) < 0$. In particular $\sup_j \tilde{g}_j(z) < +\infty$ and so $\sup_j \tilde{g}_j(z) = 0$. Hence $f(z) < 0$ and so (i$'$) is not true. Thus since (i) and (i$'$) are equivalent the result follows.□

It is important to note that Theorem 4.1 requires only that H is a conic set (and thus would be applicable, for example, to the class of quasiconvex functions) whereas Theorem 4.3 requires that H be a semilinear space. It is worth noting that if H is a semilinear space then Theorem 4.1 is a straightforward corollary of Theorem 4.3.

We now complete this section with a potentially interesting application of Theorems 4.1 and 4.3.

Let us introduce the notion of an extension of a H-convex function and show that the solvability theorems developed above are equivalent both for H-convex systems and the corresponding systems of H-convex extensions.

Consider two subsets Z and Z_1 of the set X with $Z \subseteq Z_1$. Let H be a set of functions defined on X. Let $p \in \overline{\mathcal{F}}(H, Z)$ with $U = \mathrm{s}(p, H, Z)$ the support set of p. Now consider the function $\bar{p}$ defined on Z_1 by the following for each $z \in Z_1$,

$$\bar{p}(z) = \sup_{h \in U} h(z)$$

Clearly the support set $U_l = \mathrm{s}(\bar{p}, H, Z_1)$ of the function $\bar{p}$ contains the set U. On the other hand if $h \in U_l$ then

$$(\forall z \in Z)\ h(z) \leq \bar{p}(z) = p(z)$$

and therefore $h \in U$. So $U = U_l$. We shall say that $\bar{p}$ is a *H-extension* of the function p on the set Z_1.

We now assume that the conditions of Theorem 4.1 are satisfied. Let $Z \subseteq Z_1$ and let $\bar{f}$ and $\bar{g}_i$ be H-extensions of f and g_i respectively to the set Z_1.

Theorem 4.6. *The following statements are equivalent:*

(i) $z \in Z,\ (\forall i \in I)\ g_i(z) \leq 0 \implies f(z) \leq 0$

(ii) $z \in Z_1,\ (\forall i \in I)\ \bar{g}_i(z) \leq 0 \implies \bar{f}(z) \leq 0$

Proof. Theorem 4.1 shows that the characterization of statement (i) depends only on the connections between the support sets $\mathrm{s}(f, H, Z)$ and $\mathrm{s}(g_i, H, Z)$ $(i \in I)$. Consequently since the support sets of the given functions coincide with those of their extensions the result follows. □

Example 4.7. Let Z be the unit sphere in $\mathbb{R}^n$ and Z_1 the unit ball. Let H be the set of all continuous affine functions defined on $\mathbb{R}^n$. It is well-known (see, for example [25]) that an arbitrary l.s.c function p defined on Z is H-convex. Let p be l.s.c on the sphere Z and define the function $\bar{p}$ on the ball Z_1 as follows:

$$\bar{p}(z) = \inf \{\sum_{i=1}^{m} \alpha_i p(x_i) : \sum_{i=1}^{m} \alpha_i x_i = z,\ x_i \in Z,\ \alpha_i \geq 0,\ \sum_{i=1}^{m} \alpha_i = 1,\ m = 1, 2, \ldots\}. \quad (4.4)$$

Since each point of the sphere is an extreme point of the ball we have $p(x) = \bar{p}(x)$ for all $x \in Z$. It is well known that $\bar{p}$ is a l.s.c convex function and therefore $\bar{p}$ is a H-convex function on the ball Z_1. Let $h \in \mathrm{s}(p, H, Z)$. If $z \in Z,\ \alpha_i \geq 0,\ \sum_i \alpha_i = 1,\ \sum_i \alpha_i x_i = z$ then $h(z) = \sum_i \alpha_i h(x_i) \leq \sum_i \alpha_i p(x_i)$ and therefore $h(z) \leq \bar{p}(z)$, i.e. $h \in \mathrm{s}(\bar{p}, H, Z_1)$. Since $p(x) = \bar{p}(x)$ on the sphere we have $\mathrm{s}(\bar{p}, H, Z_1) \subseteq \mathrm{s}(p, H, Z)$. Thus $\mathrm{s}(\bar{p}, H, Z_1) = \mathrm{s}(p, H, Z)$. Hence $\bar{p}$ is a H-extension of the function p.

Now we consider l.s.c functions f and g_i $(i \in I)$ defined on the sphere Z and functions $\bar{f}$, $\bar{g}_i$ $(i \in I)$ defined on the ball Z_1 by means of the approach illustrated in (4.4). We can now apply Theorem 4.6.

Now let q be an inf-H-convex function defined on the set Z and $\{p_\alpha : \alpha \in \Delta\}$ is a family of H-convex functions such that $q(z) = \inf_{\alpha \in \Delta} p_\alpha(z)$ for all $z \in Z$. Let $\bar{p}_\alpha$ be a H-extension of the function p_α on the set $Z_1 \supseteq Z$ and $\bar{q}(z) = \inf_{\alpha \in \Delta} \bar{p}_\alpha(z)$, $(z \in Z_1)$. We say that $\bar{q}$ is a H-extension of the function q with respect to the family $(p_\alpha)_{\alpha \in \Delta}$.

Theorem 4.8. *Let X, Z, H, I, g_i and f be the same as in* Theorem 4.3 *and let $\bar{g}_i$ be a H-extension of g_i $(i \in I)$ with respect to the family $(\bar{p}_{\alpha_i})_{\alpha_i \in \Delta_i}$ and $\bar{f}$ be a H-extension of the function f with respect to the family $(p_\alpha)_{\alpha \in \Delta}$. Then the following are equivalent:*

(i) $z \in Z\ (\forall i \in I)\ g_i(z) \leq 0 \implies f(z) \leq 0$

(ii) $z \in Z_1\ (\forall i \in I)\ \bar{g}_i(z) \leq 0 \implies \bar{f}(z) \leq 0$

5. Characterizations of the H-Convex Hull

In order to apply the solvability theorems developed in the preceding section it is necessary to describe the H-convex hull, co_H^Z in verifiable terms (i.e. involving standard notions of convex hull or convex cone). This is, in general, a very difficult problem, particularly for nonconvex systems. However there are a number of cases in which co_H^Z has a relatively simple characterization. In the following we provide a detailed analysis for several important special cases.

Example 5.1. Consider the closed convex cone K in a locally convex Hausdorff topological vector space X. Let H be the set of all continuous linear functions defined on X. If $p \in \overline{\mathcal{F}}(H, K)$ then p is a l.s.c sublinear function on K. In order to adequately describe the H-convex sets we require the following definition.

Definition 5.2. Let L be a cone in a vector space X. A set $\Omega \subseteq X$ is called *L-stable* if $\Omega + L = \Omega$.

It is well known (see [25, 35]) that a nonempty set U is H-convex in this case if and only if U is closed convex and $-K^*$-stable.

It should be noted that the H-extension of a H-convex function p (i.e. p is a l.s.c sublinear function defined on K) to the entire space X coincides with the function $\tilde{p}$ defined on X by the formula:

$$\tilde{p}(x) = \begin{cases} p(x) & x \in K \\ +\infty & x \notin K \end{cases}$$

and the support set of p is the subdifferential of the l.s.c function $\tilde{p}$, defined on all of X with the property $\mathrm{dom}\tilde{p} \subseteq K$.

Now let us consider an arbitrary nonempty set $U \subseteq H = X'$. Let

$$U_\ell = \mathrm{cl}\,(\mathrm{co}\, U - K^*)$$

where $\mathrm{co}\, U$ denotes the convex hull of U. Clearly the set U_ℓ is closed convex and $-K^*$-stable. Therefore this set is H-convex. In addition

$$\sup_{h \in U_\ell} h(x) = \sup_{h \in U} h(x) \quad x \in K$$

Therefore U_ℓ is the H-convex hull of the set U. Thus we have the following:

Proposition 5.3. *Let $H = X'$ be the set of all continuous linear functions defined on X and let K be a closed convex cone in X. Then*

$$\mathrm{co}_H^K\, U = \mathrm{cl}\,(\mathrm{co}\, U - K^*)$$

Remark 5.4. If U is a weak* compact convex set then $\mathrm{co}_H^K = U - K^*$. If moreover $K = X$ then U is a H-convex set and $\mathrm{co}_H^K U = U$.
More generally in this section we have developed characterizations of H-convex hulls that involve the closure of sums (or differences) of convex sets (as in Proposition 5.3). The question of when such sums are actually closed is difficult, for example in Proposition 5.3 $(\mathrm{co}\,(U) - K^*)$ may not be closed in general. Here are some conditions that guarantee closedness. These conditions were given in [24]:

(i) If K and L are closed convex cones, then $K \cap (-L) = \{0\}$, K locally compact implies $K + L$ closed.
(ii) Suppose that X is a Hilbert space, K and L are closed convex cones, and the angle between K and $-L$ is positive, i.e.

$$\sup\{[k, l] : \|k\| = \|l\|,\, k \in K,\, l \in L\} < 1,$$

then $K + L$ is closed.
(iii) Suppose that C is a closed subspace and D is a finite dimensional subspace then $C + D$ is closed.
(iv) Suppose K and L are closed convex sets, $0 \in K$, $L \cap \mathrm{cone}\,(K - L)$ a closed subspace. Then $K^* + L^*$ is closed.

Example 5.5. Now we consider closed convex cones K and Z in the l.c.H.t.v.s X such that $Z \subseteq K$. Let H be the set K^* of all continuous linear functions defined on X and nonnegative on K. We require the following definition.

Definition 5.6. Let L_1 and L_2 be cones in X with $L_1 \subseteq L_2$. Then a subset U of the cone L_1 is called (L_1, L_2)-*normal* if $U = \mathrm{cl}\,(U - L_2) \cap L_1$. If U is an arbitrary subset of the cone L_1 then the set

$$\mathcal{N}\,(U) = \mathrm{cl}\,(U - L_2) \cap L_1$$

is called the (L_1, L_2)-*normal hull* of the set U.

Normal sets play a very important role in Mathematical Economics. These sets and normal hulls are studied in detail in [26, 35].

It can be shown ([35]) that a subset U of the cone K^* is H-convex with respect to Z (i.e. $U \in \overline{\mathcal{S}}(H, Z)$) if and only if this set is closed convex and (K^*, Z^*)-normal. The H-convex hull $\mathrm{co}_H^Z U$ of an arbitrary set U in K^* has the form

$$\mathrm{co}_H^Z U = \mathcal{N}\,(\mathrm{co}\, U).$$

Example 5.7. Now we let H be the set of all continuous affine functions defined on X. Let Z be a closed convex subset of X. Define the set

$$K_Z = \{(\ell, c) \in H : (\forall x \in Z)\, \ell(x) - c \le 0\}$$

Clearly K_Z is a closed convex cone. The following proposition follows:

Proposition 5.8. *Let f be a l.s.c convex function defined on X. Then*

$$\mathrm{dom} f \subseteq Z \iff \mathrm{s}(f, H, X) \text{ is a } K_Z\text{-stable set.}$$

Proof. The following holds:

$$\mathrm{dom} f \subseteq Z \iff f + \delta_Z = f \iff (f+\delta)^* = f^* \iff \mathrm{epi}\ (f+\delta_Z)^* = \mathrm{epi}\ f^* \tag{5.1}$$

Here δ_Z denotes the indicator function of the set Z, i.e.

$$\delta_Z(x) = \begin{cases} 0 & x \in Z \\ +\infty & x \notin Z \end{cases}$$

If g is a convex function then g^* is a conjugate function (see [31]).
Note that $\mathrm{dom}\delta_Z = Z \supseteq \mathrm{dom} f$. Using a well known result (see for example Rockafellar [31]) we have

$$(f+\delta_Z)^* = \mathrm{cl}\,(f^* \oplus \delta_Z^*)$$

where $\oplus$ denotes inf-convolution. Since the epigraph of the inf-convolution of two functions is equal to the sum of the epigraphs of the functions it follows that

$$\mathrm{epi}\ (f+\delta_Z)^* = \mathrm{epi}\ (\mathrm{cl}\,(f^* \oplus \delta_Z^*)) = \mathrm{cl}\,(\mathrm{epi}\ (f^* \oplus \delta_Z^*)) = \mathrm{cl}\,(\mathrm{epi}\ f^* + \mathrm{epi}\ \delta_Z^*) \tag{5.2}$$

We now calculate epi δ_Z^*:

$$\begin{aligned} \mathrm{epi}\ \delta_Z^* &= \{(\ell, c) : c \geq \delta_Z^*(\ell)\} \\ &= \{(\ell, c) : c \geq \sup_{z \in Z} \ell(z)\} \\ &= \{(\ell, c) : (\forall z \in Z)\ c \geq \ell(z)\} \\ &= K_Z \end{aligned} \tag{5.3}$$

Let $\mathrm{dom} f \subseteq Z$. Then using (5.1), (5.2) and (5.3) we have

$$\mathrm{cl}\,(\mathrm{epi}\ f^* + K_Z) = \mathrm{epi}\ f^*$$

and therefore $\mathrm{epi}\ f^* + K_Z \subseteq \mathrm{epi}\ f^*$. On the other hand, since $0 \in K_Z$ we have $\mathrm{epi}\ f^* + K_Z \supseteq \mathrm{epi}\ f^*$. Therefore $\mathrm{epi}\ f^* + K_Z = \mathrm{epi}\ f^*$.
Conversely if $\mathrm{epi}\ f^* + K_Z = \mathrm{epi}\ f^*$ then $\mathrm{cl}\,(\mathrm{epi}\ f^* + K_Z) = \mathrm{epi}\ f^* + K_Z = \mathrm{epi}\ f^*$ and therefore $\mathrm{epi}\ (f+\delta_Z^*)^* = \mathrm{epi}\ f^*$ and $\mathrm{dom} f \subseteq Z$. □

We also require the following result.

Proposition 5.9. (see [25]) *Let $H = \{h : h(x) = \ell(x) - c,\ \ell \in X',\ c \in \mathbb{R},\ x \in X\}$ be the set of all continuous affine functions defined on X. Then a nonempty set $U \neq H$ is H-convex (with respect to the entire space X) if and only if U is closed convex $(\{0\} \times \mathbb{R}_+)$-stable and not $(\{0\} \times \mathbb{R})$-stable.*

Clearly H and $\emptyset$ are H-convex sets.

The following holds:

Proposition 5.10. *A nonempty set $U \neq H$ is H-convex (on Z) if and only if U is closed convex K_Z-stable and not $(\{0\} \times \mathbb{R})$-stable.*

Proof. It is easy to check that a H-convex set (on Z) is closed convex and K_Z-stable. Now let U be closed convex K_Z-stable and not $(\{0\} \times \mathbb{R})$-stable. Since $\{0\} \times \mathbb{R}_+ \subseteq K_Z$,

U is a $(\{0\} \times \mathbb{R}_+)$-stable set. Proposition 5.9 shows that there is a l.s.c convex function f defined on X such that $U = \mathrm{s}(f, H, X)$. Applying Proposition 5.8 we have $\mathrm{dom} f \subseteq Z$. Therefore U is a H-convex set (on Z). □

Corollary 5.11. *Let H be the set of all continuous affine functions defined on X and let U be an arbitrary nonempty subset of H which is not $(\{0\} \times \mathbb{R})$-stable. Then*

$$\mathrm{co}_H^Z U = \mathrm{cl}\,(\mathrm{co}\, U + K_Z). \tag{5.4}$$

Note that the set $\mathrm{cl}\,(\mathrm{co}\, U + K_Z)$ is closed convex K_Z-stable and therefore this set is H-convex (on Z). Since, for $z \in Z$,

$$\sup_{h \in U} h(z) = \sup_{h \in \mathrm{cl}\,(\mathrm{co}\, U + K_Z)} h(z)$$

it follows that $\mathrm{cl}\,(\mathrm{co}\, U + K_Z)$ is the support set of the function $z \mapsto \sup_{h \in U} h(z)$. Hence the formula (5.4) follows.

Remark 5.12. If U is a $(\{0\} \times \mathbb{R})$-stable set then, for all $x \in X$,

$$\sup_{(\ell, c) \in U} (\ell(x) - c) = +\infty$$

and therefore $\mathrm{co}_H^Z U = H$.

Proposition 5.13. *Let I be an arbitrary index set and, for each $i \in I$, let g_i be a l.s.c convex function defined on X. If the following system has a solution*

$$(\forall i \in I)\; g_i(x) \le 0$$

then the set U is not $\{0\} \times \mathbb{R}$-stable, where

$$U = \mathrm{cone} \bigcup_{i \in I} \mathrm{epi}\; g_i^*.$$

Proof. Let x_o be a solution to the system and take any $i \in I$. Then, for each $\ell \in \mathrm{dom} g_i^*$, we have

$$g_i^*(\ell) = \sup_{x \in X} \{\ell(x) - g_i(x)\} \ge \ell(x_o) - g_i(x_o) \ge \ell(x_o).$$

Thus if $(\ell, \lambda) \in \mathrm{epi}\; g_i^*$ then $\lambda \ge g_i^*(\ell) \ge \ell(x_o)$. Define the evaluation function x_o on X' by $x_o(\ell) = \ell(x_o)$. By the above if $(\ell, \lambda) \in \mathrm{epi}\; g_i^*$ then $(\ell, \lambda) \in \mathrm{epi}\;\; x_o$. Thus, for all i, $\mathrm{epi}\; g_i^* \subseteq \mathrm{epi}\; x_o$. Hence

$$\bigcup_{i \in I} \mathrm{epi}\; g_i^* \subseteq \mathrm{epi}\; x_o.$$

Since $\mathrm{epi}\; x_o$ is a halfspace in $X' \times \mathbb{R}$ it follows immediately that U cannot be $\{0\} \times \mathbb{R}$-stable as required. □

Example 5.14. Consider the set $\tilde{H}$ of all functions h defined on the cone $\mathbb{R}^n_+$ which are generated by a vector $h = (h_1, \dots, h_n) \in \operatorname{int} \mathbb{R}^n_+ \cup \{0\}$ and have the form

$$h(x) = \min_{i=1,\dots,n} h_i x_i.$$

Clearly $\tilde{H}$ is a conic set. We have two alternative approaches to introducing an order relation in $\tilde{H}$. Let $h_1 = (h^1_1, \dots, h^1_n)$, $h_2 = (h^2_1, \dots, h^2_n)$, then

(i) $h_1 \ge h_2$ if $h_1(x) \ge h_2(x)$, i.e. $\min_i h^1_i x_i \ge \min_i h^2_i x_i$.

(ii) $h_1 \ge h_2$ if $h^1_i \ge h^2_i$, for all i.

It is straightforward to check that these relations coincide.

We identify $\tilde{H}$ with the cone $\operatorname{int} \mathbb{R}^n_+ \cup \{0\}$. However we consider this cone as an ordered conic set without summation since $\tilde{H}$ is not a semilinear space. Recall that a function p defined on $\mathbb{R}^n_+$ is $\tilde{H}$-convex if and only if p is increasing, positively homogeneous of degree one and $p(x) = 0$ if $\min_i x_i = 0$.

The subset U of the cone $\operatorname{int} \mathbb{R}^n_+ \cup \{0\}$ is called *normal* if $h \in U$, $h' \in \tilde{H}$, $h' \le h$ implies $h' \in U$. Compare this definition with the definitions of normal subsets of the cone $\mathbb{R}^n_+$ and (L_1, L_2)-normal sets.

Proposition 5.15. *A subset U of the cone $\tilde{H}$ is $\tilde{H}$-convex if and only if U is closed (in the topological space int $\mathbb{R}^n_+ \cup \{0\}$) and a normal set.*

Proof. It is easy to check that a $\tilde{H}$-convex set is closed and normal. Now let U be a closed and normal subset of the cone $\tilde{H}$. We have to show that the inequality

$$(\forall x \in \mathbb{R}^n_+) \;\; h(x) \le \sup_{h' \in U} h'(x)$$

implies the inclusion $h \in U$. Equivalently we need to show that if $h \in \tilde{H}$ and $h \notin U$ then there is a $x \in \mathbb{R}^n_+$ such that $h(x) > \sup_{h' \in U} h'(x)$. Let us consider such a vector $h \notin U$. Clearly $h \ne 0$ and therefore $h \in \operatorname{int} \mathbb{R}^n_+$. Since U is closed there is an $\epsilon > 0$ such that $(1-\epsilon)h \notin U$. Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ be the vector such that, for each i

$$\bar{x}_i = \frac{1}{(1-\epsilon)h_i}.$$

We have $h(\bar{x}) = \min_i h_i \bar{x}_i = 1/(1-\epsilon) > 1$. Now let $h' \in U$. Since U is normal the inequality $h' \ge (1-\epsilon)h$ is not true, therefore there is an index i_o such that $h'_{i_o} < (1-\epsilon)h_{i_o}$. Hence

$$h'(\bar{x}) = \min_i h'_i \bar{x}_i \le h'_{i_o} \bar{x}_{i_o} < (1-\epsilon) h_{i_o} \bar{x}_{i_o} = 1.$$

Thus we have constructed a vector $\bar{x}$ with the property

$$h(\bar{x}) > 1 \ge \sup_{h' \in U} h'(\bar{x}).$$

□

Let U be an arbitrary subset of the cone $\tilde{H}$. The normal hull, $\mathcal{N}(U)$, of the set U is defined as follows:

$$\mathcal{N}(U) = \{h \in \tilde{H} : (\exists h' \in U)\ h \leq h'\}.$$

It is easy to check that closure of a normal set (in the topological space $\operatorname{int} \mathbb{R}^n_+ \cup \{0\}$) is a normal set too. Therefore $\operatorname{cl}(\mathcal{N}(U))$ is a closed normal set. Thus we have the following:

Corollary 5.16. *If U is an arbitrary subset of the cone $\mathbb{R}^n_+$ then*

$$\operatorname{co}_{\tilde{H}} U = \operatorname{cl}(\mathcal{N}(U)).$$

Note that $\operatorname{cl}(\mathcal{N}(U))$ is a $\tilde{H}$-convex set. In addition, for each $x \in \mathbb{R}^n_+$,

$$\sup_{h \in U} h(x) = \sup_{h \in \operatorname{cl}(\mathcal{N}(U))} h(x).$$

6. Solvability Theorems in Special Cases

In this section we consider the general solvability theorems developed in Section 4 along with the characterizations of H-convex hull outlined in Section 5 in a number of special cases. Namely we transform the dual conditions in Theorems 4.1 and Theorem 4.3 using the description of the H-convex hull developed above.

6.1. Sublinear inequality systems

We begin by considering solvability theorems involving l.s.c sublinear functions f and g_i $(i \in I)$ defined on a closed convex cone K. Thus we wish to characterize Theorem 4.1 (i). The set H coincides in this case with the space X' of all continuous linear functions defined on X. Proposition 5.3 shows that in this case:

$$\begin{aligned} \operatorname{co}^K_H \operatorname{cone} \bigcup_i \partial g_i(0) &= \operatorname{cl}(\operatorname{co}\operatorname{cone} \bigcup_i \partial g_i(0) - K^*) \\ &= \operatorname{cl}(\operatorname{cone}\operatorname{co} \bigcup_i \partial g_i(0) - K^*) \end{aligned} \tag{6.1}$$

If $\operatorname{dom} g_i \subseteq K$ (for each i) then the support set (subdifferential) of the sublinear function g_i is $-K^*$-stable and therefore $\partial g_i(0) - K^* = \partial g_i(0)$. The formula (6.1) has a particularly simple form in this case

$$\operatorname{co}^K_H \operatorname{cone} \bigcup_i \partial g_i(0) = \operatorname{cl}\operatorname{cone}\operatorname{co} \bigcup_i \partial g_i(0) \tag{6.2}$$

In particular (6.2) follows if $K = X$. Thus statement (ii) of Theorem 4.1 can be written in the following form in this case:

$$\partial f(0) \subseteq \operatorname{cl}(\operatorname{cone}\operatorname{co} \bigcup_i \partial g_i(0))$$

In relation to Theorem 4.3 suppose that $(p_{\alpha_i})_{i\in\Delta_i}$ is a family of l.s.c sublinear functions and p_α is a l.s.c sublinear function then

$$\begin{aligned}&\mathrm{co}_H^K\,(\partial p_\alpha(0)+\mathrm{co}_H^K\,\mathrm{cone}\bigcup_i\partial p_{\alpha_i}(0))\\&=\mathrm{cl}\,(\partial p_\alpha(0)+\mathrm{cl}\,[\mathrm{co\,cone}\bigcup_i\partial p_{\alpha_i}(0)-K^*]-K^*)\end{aligned}$$

To simplify this formula we note that, in general $\mathrm{cl}\,(C+\mathrm{cl}\,D)=\mathrm{cl}\,(C+D)$, and the set

$$\mathrm{cl}\,(\mathrm{co\,cone}\bigcup_i\partial p_{\alpha_i}(0)-K^*)$$

is $-K^*$-stable. Thus we have

$$\begin{aligned}&\mathrm{co}_H^K\,(\partial p_\alpha(0)+\mathrm{co}_H^K\,\mathrm{cone}\bigcup_i\partial p_{\alpha_i}(0))\\&=\mathrm{cl}\,(\partial p_\alpha(0)+\mathrm{cone\,co}\bigcup_i\partial p_{\alpha_i}(0)-K^*)\end{aligned}\tag{6.3}$$

Once again we can remove $-K^*$ if $\mathrm{dom}p_\alpha\subseteq K$ and, for each i, $\mathrm{dom}p_{\alpha_i}\subseteq K$. Thus statement (ii) of Theorem 4.3 can be written as follows in thus case:

$$0\in\mathrm{cl}\,(\partial p_\alpha(0)+\mathrm{cone\,co}\bigcup_i\partial p_{\alpha_i}(0)).$$

For a comparison of this result with known sublinear versions of Farkas' lemma, see [8, 10].

6.2. *Convex inequality systems*

We now consider systems involving proper l.s.c convex functions, f and g_i $(i\in I)$ and assume that the set Z is closed and convex. The set H is now the set of all continuous affine functions defined on the space X. We will throughout assume that the system

$$i\in I,\ g_i(x)\le 0$$

is consistent. Thus, by Proposition 5.13, the set

$$\mathrm{cone}\bigcup_{i\in I}\mathrm{epi}\ g_i^*$$

is not $\{0\}\times\mathbb{R}$-stable. Then using Corollary 5.11 we have

$$\mathrm{co}_H^Z\,\mathrm{cone}\bigcup_i\mathrm{epi}\ g_i^*=\mathrm{cl}\,(\mathrm{co\,cone}\bigcup_i\mathrm{epi}\ g_i^*+K_Z)$$

where the cone K_Z is defined by the formula

$$K_Z=\{h=(\ell,c)\in H:(\forall z\in Z)\ \ell(z)-c\le 0\}.$$

If, for each i, dom$g_i \subseteq Z$ then, by applying Proposition 5.8, we can remove K_Z. Therefore statement (ii) of Theorem 4.1 has the following form:

$$\text{epi } f^* \subseteq \text{cl}\,(\text{co cone} \bigcup_i \text{epi } g_i^*).$$

Now let p_α and p_{α_i} be l.s.c proper convex functions. We have

$$\begin{aligned} &\text{co}_H^Z\,(\text{epi } p_\alpha^* + \text{co}_H^Z \text{ cone} \bigcup_i \text{epi } p_{\alpha_i}^* \\ &= \text{cl}\,\{(\text{epi } p_\alpha^* + \text{cl}\,[\text{cone co} \bigcup_i \text{epi } p_{\alpha_i}^* + K_Z]) + K_Z\} \\ &= \text{cl}\,(\text{epi } p_\alpha^* + (\text{cone co} \bigcup_i \text{epi } p_{\alpha_i}^*) + K_Z) \end{aligned}$$

and consequently statement (ii) of Theorem 4.3 has the following form:

$$0 \in \text{cl}\,(\text{epi } p_\alpha^* + (\text{co cone} \bigcup_i \text{epi } p_{\alpha_i}^*) + K_Z).$$

We can remove K_Z in this case if, for all i, dom$g_i \subseteq Z$. In this case we have:

$$0 \in \text{cl}\,(\text{epi } p_\alpha^* + (\text{co cone} \bigcup_i \text{epi } p_{\alpha_i}^*)).$$

6.3. Systems involving K-increasing sublinear functions

Let K be a closed convex cone in X. A sublinear function p defined on the space X is called K-increasing if, for all $x_1, x_2 \in X$ with $x_1 - x_2 \in K$ we have $p(x_1) \geq p(x_2)$. In addition assume that $Z \subseteq K$ is a closed convex cone in X. We now consider $H = K^*$, the set of all continuous linear functions which are nonnegative on the cone K.

Theorem 6.1. *Let K and Z be closed convex cones in X with $Z \subseteq K$ and suppose that f and g_i $(i \in I)$ are K-increasing sublinear functions defined on X (here I is an arbitrary index set). Then the following statements are equivalent:*

(i) $x \in Z$, $(\forall i \in I)\; g_i(x) = 0 \implies f(x) = 0$.

(ii) $\partial f(0) \subseteq \mathcal{N}_{K^*, Z^*}(co \bigcup_i \partial g_i(0))$ *where $\mathcal{N}_{K^*, Z^*}$ denotes the (K^*, Z^*)-normal hull (see* Example 5.5*).*

Proof. We begin by noting that f and g_i $(i \in I)$ are H-convex functions (see [25, 35]) (to be precise this applies to the restriction of the functions to the cone Z). Thus we can use Theorem 4.1 in this case. Following Example 5.5 we have that statement (ii) of Theorem 4.3 can be written in the form of statement (ii) above.
Note that a K-increasing function p is nonegative on K. Since $Z \subseteq K$ the inequality $p(z) \leq 0$ $(z \in Z)$ is equivalent to $p(z) = 0$. Therefore statement (i) of Theorem 4.1 is equivalent to statement (i) above. ☐

6.4. Positively homogeneous systems

Let us consider a conic set H of positively homogeneous functions defined on a conic set K. Let $\bar{H} = \{\bar{h} = (h, c) : \bar{h} = h(x) - c,\ h \in H,\ c \in \mathbb{R}\}$. In the following we consider inequalities involving functions of the form $\bar{p}$ where $\bar{p}(x) = p(x) - 1$ with p a H-convex function defined on a conic set $Z \subseteq K$. We begin with the following lemma.

Lemma 6.2. *Let p be a H-convex function and $\bar{p}(x) = p(x) - 1$. Then*

$$\mathrm{s}(\bar{p}, \bar{H}, Z) = \mathrm{s}(p, H, Z) \times [1, +\infty).$$

Proof. Clearly if $h(x) \le p(x)$ for each $x \in Z$ and $c \ge 1$ then

$$(\forall x \in Z)\ \ h(x) - c \le p(x) - 1 \tag{6.4}$$

and therefore $(h, c) \in \mathrm{s}(\bar{p}, \bar{H}, Z)$. Now let $(h, c) \in \mathrm{s}(\bar{p}, \bar{H}, Z)$, i.e. the formula (6.4) holds for all $x \in Z$. If $x = 0$ then (6.4) gives $c \ge 1$. If $x \in Z$, $x \ne 0$ then $\mu x \in Z$ with $\mu > 0$ and we have

$$\mu h(x) - c \le \lambda p(x) - 1$$

or equivalently

$$(\forall x \in Z)\ h(x) - \frac{c}{\mu} \le p(x) - \frac{1}{\mu}.$$

Letting $\mu \to +\infty$ we have $h(x) \le p(x)$ for all $x \in Z$, i.e. $h \in \mathrm{s}(p, H, Z)$. □

Proposition 6.3. *Let g be a H-convex and nonnegative function defined on the conic set Z with* $\mathrm{dom} g \subseteq Z$. *We define the function δ on Z as follows:*

$$\delta(x) = \begin{cases} 0 & g(x) \le 1 \\ +\infty & g(x) > 1 \end{cases}$$

Then

$$\mathrm{s}(\delta, \bar{H}, Z) = \mathrm{cone}\,(\mathrm{s}(g, H, Z) \times [1, +\infty)) \cup V$$

where $V = \{(h, 0) \in \bar{H} : (\forall x \in \mathrm{dom} g)\ h(x) \le 0\}$.

Proof. We have by the definition of δ that

$$\mathrm{s}(\delta, \bar{H}, Z) = \{(h, c) \in \bar{H} : h(x) \le 1 \text{ if } g(x) \le 1\}.$$

Let $(h, c) \in \mathrm{s}(\delta, \bar{H}, Z)$. Since g is a positively homogeneous function we have $g(0) = 0 < 1$. Therefore $-c = h(0) - c \le 0$ and $c \ge 0$. If $c = 0$ then $h(x) \le 0$ for all x such that $g(x) \le 1$. Let $x \in \mathrm{dom} g$ and $g(x) > 0$. Then $g(x/g(x)) = 1$ and therefore $h(x) = g(x) h(x/g(x)) \le 0$. If $g(x) = 0$ then $h(x) \le 0$ also. So $h(x) \le 0$ for all $x \in \mathrm{dom} g$ and so $h \in V$.

Now let $c > 0$. We have $h(x) \le c$ if $g(x) \le 1$. In particular $\frac{1}{c} h(x) \le 1 = g(x)$ if $g(x) = 1$. Since g is nonnegative and positively homogeneous we have $\frac{1}{c} h(x) \le g(x)$ for all $x \in \mathrm{dom} g$ such that $g(x) > 0$. If $g(x) = 0$ then $g(\lambda x) = 0$ for all $\lambda > 0$ and therefore $\frac{1}{c} h(\lambda x) \le 1$

for all $\lambda > 0$. We have $\frac{1}{c}h(x) = 0 = g(x)$ in this case so that $\frac{1}{c}h(x) \le g(x)$ for all $x \in Z$ and therefore $\frac{1}{c}h \in \mathrm{s}(g, H, Z)$. Thus we have

$$(h, c) \in c\,[\mathrm{s}(g, H, Z) \times [1, +\infty)] \subseteq \mathrm{cone}\,[\mathrm{s}(g, H, Z) \times [1, +\infty)].$$

It follows that we have proved that

$$\mathrm{s}(\delta, \bar{H}, Z) \subseteq \mathrm{cone}\,[\mathrm{s}(g, H, Z) \times [1, +\infty)] \cup V.$$

Now we establish the reverse inclusion. By definition if $(h, 0) \in V$ then $h(x) \le 0$ for all $x \in \mathrm{dom} g$ and so $(h, 0) \in \mathrm{s}(\delta, \bar{H}, Z)$. If $(h, c) \in \mathrm{cone}\,[\mathrm{s}(g, H, Z) \times [1, +\infty)]$ then there is a $\lambda \ge 0$ such that

$$(h, c) \in \lambda\, \mathrm{s}(g, H, Z) \times [\lambda, +\infty).$$

We have $c \ge \lambda$ and $h \in \lambda\, \mathrm{s}(g, H, Z) = \mathrm{s}(\lambda g, H, Z)$. Therefore

$$h(x) \le \lambda g(x) \le c g(x).$$

If $g(x) \le 1$ then $h(x) - c \le 0$, i.e. $(h, c) \in \mathrm{s}(\delta, \bar{H}, Z)$. □

Theorem 6.4. *Let f and g_i $(i \in I)$ be nonnegative H-convex functions defined on the conic set Z. Then the following statements are equivalent:*
(i) $(\forall i \in I)\; g_i(x) \le 1 \implies f(x) \le 1$
(ii) $\mathrm{s}(f, H, Z) \times [1, +\infty) \subseteq \mathrm{cone}\,(\mathrm{co}_H^Z \bigcup_i \mathrm{s}(g_i, H, Z) \times [1, +\infty))$

Remark 6.5. Note that the statement
(i′) $(\forall i \in I)\; \tilde{g}_i(x) \le c_i \implies \tilde{f}(x) \le c$
where c_i, $c > 0$, can be written in the form of statement (i) above with $g_i = \tilde{g}_i/c_i$ and $f = \tilde{f}/c$.

Proof. We apply Theorem 4.1 to show that statement (i) is equivalent to the following statement
(ii′) $\mathrm{s}(\bar{f}, \bar{H}, Z) \subseteq \mathrm{s}(\delta, \bar{H}, Z)$
where $\bar{f} = f - 1$ and

$$\delta(x) = \begin{cases} 0 & \text{if} \quad \sup_i g_i(x) \le 1 \\ +\infty & \text{if} \quad \sup_i g_i(x) > 1 \end{cases}$$

In particular if we let $g = \sup_i g_i$ then

$$\delta(x) = \begin{cases} 0 & \text{if } g(x) \le 1 \\ +\infty & \text{if } g(x) > 1 \end{cases}$$

Applying Proposition 6.3 we have

$$\mathrm{s}(\delta, \bar{H}, Z) = \mathrm{cone}\,(\mathrm{s}(g, H, Z) \times [1, +\infty)) \cup V$$

where $V = \{(h, 0) \in \bar{H} : (\forall x \in \mathrm{dom} g)\; h(x) \le 0\}$. However, by Lemma 6.2, we have

$$\mathrm{s}(\bar{f}, \bar{H}, Z) = \mathrm{s}(f, H, Z) \times [1, +\infty).$$

If $(h,\lambda) \in \mathrm{s}(\bar{f}, \bar{H}, Z)$ then $\lambda \geq 1$; if $(h,\lambda) \in V$ then $\lambda \leq 0$. Therefore we can say that statement (ii′) is equivalent to the following inclusion:

$$\mathrm{s}(f, H, Z) \times [1, +\infty) \subseteq \mathrm{cone}\,(\mathrm{s}(g, H, Z) \times [1, +\infty)).$$

Since $\mathrm{s}(g, H, Z) = \mathrm{s}(\sup_i g_i, H, Z) = \mathrm{co}_H^Z \bigcup_i \mathrm{s}(g_i, H, Z)$ we have that (ii′) is equivalent to (ii). □

We consider the following example:

Example 6.6. Let K and Z be closed convex cones in a l.c.H.t.v.s X. Let f and g_i $(i \in I)$ be nonnegative l.s.c sublinear functions defined on X. Then the following are equivalent:

(i) $x \in Z,\ (\forall i \in I)\ g_i(x) \leq 1 \implies f(x) \leq 1$

(ii) $\partial f(0) \times [1, +\infty) \subseteq \mathrm{cone}\,([\mathrm{cl}\,\mathrm{co} \bigcup_i \partial g_i(0) - Z^*] \times [1, +\infty)).$

6.5. Systems involving C^2 functions

In this section we apply Theorem 4.3 to obtain a solvability result for systems involving twice continuously differentiable functions. As in Example 3.18 let Z be a compact convex set in $\mathbb{R}^n$ and let G be an open bounded set with $Z \subseteq G$.

Now let us consider a semilinear space P of all quadratic functions p defined on $\mathbb{R}^n$ which have the form

$$p(x) = a\|x\|^2 + [\ell, x] + c$$

where $a \geq 0$, $\ell \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Let I be an arbitrary index set with functions f, g_i $(i \in I)$ such that $f \in C^2(G)$, $g_i \in C^2(G)$ $(i \in I)$. There are numbers $k_z > 0$ and $k_z^i > 0$ $(z \in Z,\ i \in I)$ such that

$$f(x) = \min_{z \in Z} p_z(x), \quad g_i(x) = \min_{z \in Z} p_z^i(x), \ x \in Z$$

where

$$p_z(x) = k_z\|x - z\|^2 + [\nabla f(z), x - z] + f(z) \in P \tag{6.5}$$

$$p_z^i(x) = k_z^i\|x - z\|^2 + [\nabla g_i(z), x - z] + g_i(z) \in P \tag{6.6}$$

Now let H be the set of all continuous affine functions defined on $\mathbb{R}^n$. Since $k_z > 0$ and $k_z^i > 0$ the functions p_z and p_z^i are convex and therefore H-convex.
Let $p \in P$, $p(x) = a\|x - z\|^2 + [\ell, x - z] + c$. It is straightforward to compute affine functions $h(x) = [k, x - z] + b$ such that $h(x) \leq p(x)$ for all $x \in \mathbb{R}^n$. We have

$$\begin{aligned}\mathrm{s}(p, H, \mathbb{R}^n) =& \{(k, b) : [k, x - z] + b \leq a\|x - z\|^2 + [\ell, x - z] + c\} \\ =& \{(k, b) : c - b \geq \frac{1}{4a}\|k - \ell\|^2\}\end{aligned}$$

Corollary 5.11 shows that

$$\mathrm{s}(p, H, Z) = \mathrm{cl}\,(\mathrm{s}(p, H, \mathbb{R}^n) + K_Z)$$

where $K_Z = \{h = (\ell, c) \in H : (\forall x \in Z)\ \ell(x) - c \leq 0\}$. We can now apply Theorem 4.3 to yield necessary and sufficient conditions for the following statement:

$$(\forall i \in I)\ g_i(x) \leq 0 \implies f(x) \geq 0$$

6.6. Systems involving inf-convex functions

We now provide the extension of Theorem 3.2 in [13], a solvability theorem for systems of inf-convex functions. In particular we remove a boundedness assumption on the domain Z required in [13]. The result follows directly by Theorem 4.3. We assume now that X is a locally convex Hausdorff topological vector space.

In the following we shall assume that f and g_i (for $i \in I$) are inf-convex functions. Thus, as for Theorem 4.3, there are families $(p_\alpha)_{\alpha \in \Delta}$ and, for each $i \in I$, $(p_{\alpha_i})_{\alpha_i \in \Delta_i}$ of l.s.c convex functions such that, for each $z \in Z$,

$$f(z) = \inf_{\alpha \in \Delta} p_\alpha(z), \quad g_i(z) = \inf_{\alpha_i \in \Delta_i} p_{\alpha_i}(z).$$

In order to apply Theorem 4.3 and Corollary 5.11 (to characterize the H-convex hull in this case) we require the following assumption:

Assumption 6.7. For each selection (α_i), where $\alpha_i \in \Delta_i$ for each i, the following set is not $\{0\} \times \mathbb{R}$-stable:

$$\text{cone} \bigcup_{i \in I} \text{epi } p^*_{\alpha_i}.$$

By Proposition 5.13 this assumption is valid if, for each selection (α_i), the following system is consistent:

$$(\forall i \in I)\ p_{\alpha_i}(z) \leq 0, \ z \in Z.$$

Theorem 6.8. *Let $Z \subseteq X$ be a convex set. Let I be an arbitrary index set with $f : Z \to \mathbb{R}_{+\infty}$ and, for each $i \in I$, $g_i : Z \to \mathbb{R}_{+\infty}$ inf-convex. Furthermore assume Assumption 6.7 is valid. Then the following statements are equivalent:*

(i) $(\forall i \in I)\ g_i(z) \leq 0 \implies f(z) \geq 0$

(ii) *For each $\alpha \in \Delta$ and $(\alpha_i) \in \prod_{i \in I} \Delta_i$*

$$0 \in \text{cl}\,(\text{epi } p^*_\alpha + (\text{co cone} \bigcup_{i \in I} \text{epi } p^*_{\alpha_i}) + K_Z)$$

Note that if f and, for each i, g_i are continuous convex functions with $\text{dom} f$ and $\text{dom} g_i$ contained in Z then (ii) becomes

$$0 \in \text{cl}\,(\text{epi } f^* + (\text{co cone} \bigcup_{i \in I} \text{epi } g_i^*)) \tag{6.7}$$

As a special case of Theorem 4.1 let f and, for each $i \in I$, let g_i be a continuous convex functions defined on X. Then the following holds:

Corollary 6.9. *Let f and, for each $i \in I$, g_i be continuous convex functions defined on X and suppose the system*

$$i \in I, \ g_i(x) \leq 0$$

is consistent. Then the following statements are equivalent:

(i) $(\forall i \in I)\, g_i(x) \le 0 \implies f(x) \le 0$

(ii) $\text{epi } f^* \subseteq \text{cl co cone} \bigcup_{i \in I} \text{epi } g_i^*$

We complete this section with the following result, a direct corollary of Corollary 6.9, which extends a result in Gwinner [14] and Ha [15].

Corollary 6.10. *Let I and J be nonempty index sets and let, for each $j \in J$ and $i \in I$, f_j and g_i be continuous convex functions defined on X with the system*

$$i \in I,\ g_i(x) \le 0$$

consistent. Then the following statements are equivalent:

(i) $(\forall i \in I)\, g_i(x) \le 0 \implies (\forall j \in J)\, f_j(x) \le 0$

(ii) $\bigcup_{j \in J} \text{epi } f_j^* \subseteq \text{cl co cone} \bigcup_{i \in I} \text{epi } g_i^*$

Proof. Follows immediately by Corollary 6.9 with $f = \sup_j f_j$. □

It is straightforward, using Example 3.15 and Theorem 4.3, to establish a solvability theorem for systems involving DC (difference convex) functions.

7. Applications to Global Optimization with Convex Constraints

In this section we will consider the direct application of the solvability theorems developed in the preceding sections to convex minimization and convex maximization problems. The latter class of programming problem provide an important class of global optimization problems which have received considerable recent attention in the literature (see, for example, [16, 17, 19]). We are able to obtain characterizations of optimality in both cases involving epigraphs of Fenchel conjugates via the application of Theorem 6.8 and Corollary 6.9 respectively.

We begin by considering the following programming problem

$$\text{(P1)} \quad \min \ f(x) \ \text{ subject to } \ g_i(x) \le 0,\ i \in I.$$

Here f and, for all $i \in I$, g_i are continuous convex functions defined on a Banach space X. The set I is a possibly infinite index set. Thus (P1) is a standard convex minimization problem. Thus $a \in X$ is optimal for (P1) if and only if

$$(\forall i \in I)\, g_i(x) \le 0 \implies f(x) \ge f(a). \tag{7.1}$$

Now, by a direct application of Theorem 6.8 (using (6.7)), we find that (7.1) is equivalent to the following

$$0 \in \text{cl}\,(\text{epi } f^* + (0, f(a)) + (\text{co cone} \bigcup_{i \in I} \text{epi } g_i^*)) \tag{7.2}$$

Note that we are using the fact that for a continuous convex function f and $\alpha \in \mathbb{R}$, $\text{epi } (f - \alpha)^* = \text{epi } f^* + (0, \alpha)$.

To understand the significance of (7.2) let us assume that the set

$$\text{epi } f^* + (0, f(a)) + (\text{co cone} \bigcup_{i \in I} \text{epi } g_i^*)$$

is closed and, for convenience $f(a) = 0$. Thus there exists (finitely many) $\lambda_{i_j} > 0$ $(j = 1, \ldots, n)$ such that

$$0 \in \text{epi } f^* + \sum_j \lambda_{i_j} \text{ epi } g_{i_j}^*. \tag{7.3}$$

Since, for continuous convex functions g, k and h and $\lambda > 0$, we have $\lambda \text{ epi } k^* = \text{epi } (\lambda k)^*$ and $\text{epi } (g+h)^* = \text{epi } (g^* \oplus h^*) = \text{cl} [\text{epi } g^* + \text{epi } h^*]$, (7.3) becomes

$$0 \in \text{epi } (f + \sum_j \lambda_{i_j} g_{i_j})^* \tag{7.4}$$

Thus (7.4) is equivalent to the following:

$$\begin{aligned} & 0 \geq (f + \sum_j \lambda_{i_j} g_{i_j})^*(0) \\ \Longleftrightarrow\ & 0 \geq \sup_{x \in X} (-(f + \sum_j \lambda_{i_j} g_{i_j}(x)) \\ \Longrightarrow\ & (\forall x \in X)\ f(x) + \sum_j \lambda_{i_j} g_{i_j})(x) \geq 0 \geq f(a) + \sum_j \lambda_{i_j} g_{i_j}(a) \end{aligned} \tag{7.5}$$

Thus (7.3), under any condition which guarantees closure in the dual condition, implies the well known Lagrangian saddlepoint condition for a convex minimization problem. Thus the condition (7.3) is a generalization of the Lagrangian saddlepoint characterization of optimality for a convex mimimization problem.

Now consider the convex maximization (P2), where f, g_i and I are as defined above.

$$\text{(P2)} \quad \max \quad f(x) \ \text{ subject to } \ g_i(x) \leq 0,\ i \in I.$$

In this case we consider $-f$ as an inf-convex function, since every concave function is expressible as the pointwise infimum of its affine majorants. Thus by a direct application of Corollary 6.9, a is optimal for (P2) if and only if the following holds:

$$\text{epi } f^* + (0, f(a)) \subseteq \text{cl co cone} \bigcup_{i \in I} \text{epi } g_i^*. \tag{7.6}$$

This condition provides a characterization of global optimality for the convex maximization problem (P2). By a direct application of Proposition 2.12 we can express (7.6) in terms of ϵ-subdifferentials. This approach, using the ϵ-subdifferential, to characterizing global optimality has been used recently in [22, 23, 16, 17]. It should be noted that convex maximization problems have received considerable recent attention (see [19]) in the literature. Such problems arise frequently in applications. It is straightforward to obtain characterizations of global optimality for a range of related problems such as DC (difference convex) programming problems by a suitable application of Theorem 6.8 and Proposition 2.12.

8. Conclusion and Open Questions

In this paper we have given a general framework for studying dual characterizations of solvability of nonlinear inequality systems. We have presented solvability results for a wide range of both convex and nonconvex inequality systems. Moreover, our framework provides a unified approach for developing global optimality conditions for many classes of nonlinear minimization problems as well as maximization problems.

Our approach to dual characterizations of solvability of inequality systems raises a number of potentially interesting open questions. These questions involve the problem of characterizing the H-convex hull, as outlined in Section 5, for various special classes H. Some questions related to this problem are as follows:

1. Let H be the set described in Example 3.12. The set $\overline{\mathcal{F}}(H,X)$ consists of all nonnegative l.s.c quasiconvex functions q such that $q(0)=0$. If, in this case, one could give a verifiable description of H-convex hull with H as in Example 3.12 then it would be possible to extend this description to the set H described in Example 3.13. So, Theorem 4.1 will be applicable to give verifiable dual conditions characterizing the solvability of inequality systems involving l.s.c. quasiconvex functions. Hence, the following question arises: Is it possible, using the construction suggested in [36], to give a verifiable description of the H-convex hull in this case? More generally the H-convexity of l.s.c quasiconvex functions has been discussed in [30, 27] using a different set H. In this case is it possible to apply Theorem 4.1?
2. Let $\bar{H}$ be the set described in Example 3.6. Let $x \in \mathbb{R}^n_+$ and f be $\bar{H}$-convex (increasing convex-along-rays) defined on $\mathbb{R}^n_+$. A function $h \in \mathrm{s}(f,\bar{H},\mathbb{R}^n_+)$ with the property $h(x)=f(x)$ has found application in [1, 3]. (The set $\mathrm{s}_x(f,\bar{H},\mathbb{R}^n_+)=\{h \in \mathrm{s}(f,\bar{H},\mathbb{R}^n_+) : h(x)=f(x)\} \neq \emptyset$ for such functions). We can describe $\mathrm{s}(f,\bar{H},\mathbb{R}^n_+)$ by applying functions $h \in \mathrm{s}_x(f,\bar{H},\mathbb{R}^n_+)$. Is it possible in this case to describe the $\bar{H}$-convex hull?
3. Let H be the set of quadratic functions defined on $\mathbb{R}^n_+$, considered in Example 3.4, and let Z be a compact subset of $\mathbb{R}^n$. Note that H is a very *thin* subset of the space of all continuous functions $C(\mathbb{R}^n_+)$, it is isomorphic to a halfspace of an $n+2$-dimensional space. Is it possible to apply the geometry of $\mathbb{R}^n$ in order to describe the H-convex hull (with respect to the compact set Z). If such a description exists then we can apply Theorem 4.3 to study solvability theorems for systems involving arbitrary l.s.c functions.

9. Appendix - Minimax results for H-convex functions

We conclude this paper with a minimax result for H-convex functions.
Let H be a set of functions defined on X such that the following hold:

1. $0 \in H$
2. If $h \in H$ and $\lambda \in \mathbb{R}$ then $h+\lambda\mathbf{e} \in H$. (Here $\mathbf{e}(x)=1$ for all $x \in X$.)

Theorem 9.1. *Let Z be an arbitrary subset of X and let U be a H-convex set with respect to Z. If*

$$\inf_{z\in Z}\sup_{h\in U} h(z) < +\infty$$

then

$$\inf_{z\in Z}\sup_{h\in U} h(z) = \sup_{h\in U}\inf_{z\in Z} h(z).$$

Proof. The following inequality is always valid:

$$\sup_{h\in U}\inf_{z\in Z} h(z) \le \inf_{z\in Z}\sup_{h\in U} h(z). \tag{9.1}$$

Let $v = \inf_{z\in Z}\sup_{h\in U} h(z)$. Consider the following two cases:

1. $v = 0$. Let $\psi(z) = \sup_{h\in U} h(z)$. We have $\inf_{z\in Z}\psi(z) = 0$. Therefore $\psi(z) \ge 0$ for all $z \in Z$ and $0 \in \mathrm{s}(\psi, H, Z)$. Since U is H-convex we have $\mathrm{s}(\psi, H, Z) = U$. Hence $0 \in U$ and $\sup_{h\in U}\inf_{z\in Z} h(z) \ge 0$.
2. Now assume $v \neq 0$. The result holds trivially if $v = -\infty$ by (9.1). By assumption $v < +\infty$. Thus we have $\inf_{z\in Z}\sup_{h\in U}(h(z) - v) = 0$. Let $h'(z) = h(z) - v = (h - v\mathbf{e})(z)$ and $U' = \{h' : h' = h - v\mathbf{e},\ h \in U\}$. Let $\psi'(z) = \psi(z) - v$.
   Clearly $U' = \mathrm{s}(\psi', H, Z)$ (since $h + \lambda\mathbf{e} \in H$ for each $h \in H$ and $\lambda \in \mathbb{R}$). Note this follows since $h \in U \implies h \le \psi \implies h - v \le \psi'$ and $\tilde{h} \in \mathrm{s}(\psi', H, Z) \implies (\forall z \in Z)\ \tilde{h}(z) \le \psi'(z) \implies \tilde{h} + v\mathbf{e} \le \psi \implies \tilde{h} + v\mathbf{e} \in U \implies \tilde{h} = h - v\mathbf{e} \in U'$.
   Therefore U' is a H-convex set. We have

$$\inf_{z\in Z}\sup_{h'\in U'} h'(z) = 0.$$

Therefore $\sup_{h'\in U'}\inf_{z\in Z} h'(z) = 0$ and so $\sup_{h\in U}\inf_{z\in Z} h(z) = v$ as required. □

Corollary 9.2. *Let Z be an arbitrary set, U be a closed convex subset of the space H of all continuous affine functions defined on X, and let U be K_Z-stable (see* Section 5*) and assume*

$$\inf_{z\in Z}\sup_{h\in U} h(z) < +\infty$$

then

$$\inf_{z\in Z}\sup_{h\in U} h(z) = \sup_{h\in U}\inf_{z\in Z} h(z).$$

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