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Existence of Zeros via Disconnectedness

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Dedicated to R. T. Rockafellar on his 60th Birthday

The purpose of this short paper is very modest if compared with the greatness of the mathematician to whom it is dedicated. Here, indeed, we wish only to signalize a new result on the existence of zeros for operators taking their values in the dual of a real topological vector space. We do hope, however, that the novelty together with the simplicity of its statement can draw the attention of other mathematicians, in order to discover a possible variety of intermediate conditions making easier its applicability to concrete situations.

The reader should think of our result as the extension to the general case of the following elementary observation: if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that the set $\{(x, y) \in \mathbb{R}^2 : f(x)y = 1\}$ is disconnected, then f does vanish at some point. Indeed, if f never vanished, that set would coincide with the graph of the continuous function $1/f$ which is clearly connected.

Our result is as follows:

Theorem 1.1. *Let X be a connected topological space and let E be a real topological vector space, with topological dual E^*. Then, any operator $A : X \to E^*$ for which the set*

$$\{y \in E : x \to \langle A(x), y\rangle \text{ is continuous}\}$$

is dense in E and the set

$$\{(x, y) \in X \times E : \langle A(x), y\rangle = 1\}$$

is disconnected, does vanish at some point of X.

Proof. Denote by p_X the projection from $X \times E$ onto X. Moreover, for any $C \subseteq X \times E$, $x \in X$, put

$$C_x = \{y \in E : (x, y) \in C\}.$$

Arguing by contradiction, assume that $A(x) \neq 0$ for all $x \in X$. Denote by Γ the set

$$\{(x, y) \in X \times E : \langle A(x), y\rangle = 1\}.$$

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Since Γ is disconnected, there are two open sets $\Omega_1, \Omega_2 \subseteq X \times E$ such that

$$\Omega_1 \cap \Gamma \neq \emptyset, \quad \Omega_2 \cap \Gamma \neq \emptyset, \quad \Omega_1 \cap \Omega_2 \cap \Gamma = \emptyset, \quad \Gamma \subseteq \Omega_1 \cup \Omega_2.$$

We now prove that $p_X(\Omega_1 \cap \Gamma)$ is open in X. So, let $(x_0, y_0) \in \Omega_1 \cap \Gamma$. Since E is locally connected ([4], p.35), there are a neighbourhood U_0 of x_0 in X and an open connected neighbourhood V_0 of y_0 in E such that $U_0 \times V_0 \subseteq \Omega_1$. Since $\langle A(x_0), \cdot \rangle$ is a non-null continuous linear functional, it has no local extrema. Consequently, since $\langle A(x_0), y_0 \rangle = 1$, the sets

$$\{u \in V_0 : \langle A(x_0), u \rangle < 1\},$$
$$\{u \in V_0 : \langle A(x_0), u \rangle > 1\}$$

are both non-empty and open. Then, thanks to our density assumption, there are $u_1, u_2 \in V_0$ such that the set

$$\{x \in U_0 : \langle A(x), u_1 \rangle < 1 < \langle A(x), u_2 \rangle\}$$

is a neighbourhood of x_0. Then, if x is in this set, due to the connectedness of V_0, there is some $y \in V_0$ such that $\langle A(x), y \rangle = 1$, and so, x actually lies in $p_X(\Omega_1 \cap \Gamma)$, as desired. Likewise, it is seen that $p_X(\Omega_2 \cap \Gamma)$ is open. Now, observe that, for any $x \in X$, the set $\{x\} \times \Gamma_x$ is non-empty and connected, and so it is contained either in Ω_1 or in Ω_2. Summarizing, we then have that the sets $p_X(\Omega_1 \cap \Gamma)$ and $p_X(\Omega_2 \cap \Gamma)$ are non-empty, open, disjoint and cover X. Hence, X would be disconnected, a contradiction. □

Once Theorem 1.1 has been obtained, we can state the following formally more complete result:

Theorem 1.2. *Let X be a topological space, let E be a real topological vector space, and let $A : X \to E^*$ be such that the set*

$$\{y \in E : x \to \langle A(x), y \rangle \text{ is continuous}\}$$

is dense in E.

Then, the following assertions are equivalent:

(i) *The set*

$$\{(x, y) \in X \times E : \langle A(x), y \rangle = 1\}$$

is disconnected.

(ii) *The set $X \setminus A^{-1}(0)$ is disconnected.*

Proof. Let (i) hold. Since

$$\{(x, y) \in X \times E : \langle A(x), y \rangle = 1\} = \{(x, y) \in (X \setminus A^{-1}(0)) \times E : \langle A(x), y \rangle = 1\},$$

if $X \setminus A^{-1}(0)$ were connected, we could apply Theorem 1.1 to $A_{|(X \setminus A^{-1}(0))}$, and so A would vanish at some point of $X \setminus A^{-1}(0)$, which is absurd.
Conversely, if (ii) holds, then (i) follows at once observing that, with the notations of the proof of Theorem 1.1, one has $X \setminus A^{-1}(0) = p_X(\Gamma)$. □

Remark 1.3. It is worth pointing out that one can also prove Theorem 1.1 by applying some known results of set-valued analysis. Namely, assuming, as before, that $A(x) \neq 0$ for all $x \in X$, we directly infer from Theorem 2.2 of [3] that the multifunction $H : X \to 2^E$ defined by $H(x) = \Gamma_x$ $(x \in X)$, is lower semicontinuous. Then, since X is connected and each $H(x)$ is non-empty and connected, Theorem 3.2 of [1] ensures that the graph of H is connected too. But such graph is nothing else than Γ itself, and so we get a contradiction.

Remark 1.4. When X is a connected topological space, E is an infinite-dimensional real vector space (with algebraic dual E'), and $A : X \to E'$ is a $\sigma(E', E)$-continuous operator, one could try to apply Theorem 1.1 endowing E with the strongest vector topology ([2], p.53).

Remark 1.5. In Theorem 1.1, the role of the constant 1 can actually be assumed by any continuous real function on X. Precisely, we have the following

Proposition 1.6. *Let X be a topological space, let E be a real topological vector space, and let $A : X \to E'$. Assume that, for some continuous function $\alpha : X \to \mathbb{R}$, the set*

$$\Lambda := \{(x, y) \in X \times E : \langle A(x), y\rangle = \alpha(x)\}$$

is disconnected.
Then, either $A(x) = 0$ for some $x \in X$, or the set

$$\Gamma := \{(x, y) \in X \times E : \langle A(x), y\rangle = 1\}$$

is disconnected.

Proof. Assume that $A^{-1}(0) = \emptyset$. So, $p_X(\Gamma) = X$. Consider the function $f : X \times E \to X \times E$ defined by putting $f(x, y) = (x, \alpha(x)y)$ for all $(x, y) \in X \times E$. Of course, f is continuous. Arguing by contradiction, assume that Γ is connected. Then, $f(\Gamma)$ is connected too. Now, observe that

$$\Lambda = \bigcup_{x\in\alpha^{-1}(0)} (f(\Gamma) \cup (\{x\} \times \Lambda_x)).$$

Furthermore, note that, if $x \in \alpha^{-1}(0)$, then $(x, 0) \in f(\Gamma) \cap (\{x\} \times \Lambda_x)$, and so $f(\Gamma) \cup (\{x\} \times \Lambda_x)$ is connected. In turn, the sets $f(\Gamma) \cup (\{x\} \times \Lambda_x)$ $(x \in \alpha^{-1}(0))$ are clearly pairwise non-disjoint, and hence Λ is connected, a contradiction. □

Remark 1.7. A further reasonable remark about possible uses of Theorem 1.1 arises from Theorem 1.2. In fact, in dealing with the existence of zeros for a given $\sigma(E^*, E)$-continuous operator $A : X \to E^*$, if we were trying to apply Theorem 1.1 directly to A, we would have not only that $A^{-1}(0) \neq \emptyset$, but also that $X \setminus A^{-1}(0)$ is disconnected, which is a quite special property. In other words, such a direct application could imply rather severe assumptions on A. So, it would be reasonable to find another connected topological space T and a continuous function $f : T \to X$ in such a way to prove more easily the disconnectedness of the set $\{(t, y) \in T \times E : \langle A(f(t)), y\rangle = 1\}$. So doing, we would still have the requested information $A^{-1}(0) \neq \emptyset$, the set $X \setminus A^{-1}(0)$ being, however, not necessarily disconnected.

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