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On the Measurability of the Conjugate and the Subdifferential of a Normal Integrand

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Dedicated to R. T. Rockafellar on his 60th Birthday

Let $(T, \mathcal{T})$ be an arbitrary measurable space, X a Banach space whose dual X^* is strongly separable and f an integrand defined on $T \times X$. If f is normal in the sense of Rockafellar [18], then the conjugate integrand f^* is normal. Moreover the subdifferential multifunction is Effros (or weakly) measurable. These results extend those of Rockafellar [18] and of Hess [6], and provide an alternate approach to that of Beer [2].

1. Introduction

The notion of normal integrand, introduced and studied extensively by R. T. Rockafellar in several papers (for example, [15–18]), is known to be well suited for dealing with minimization problems arising in many fields of applied mathematics such as, optimal control, calculus of variation and mathematical economics.

The normality of an *integrand* f, that is, an extended real valued function defined on a product space $T \times X$, is generally defined by considering the *epigraphical multifunction* F associated with it, namely $F(t) := \text{epi} f(t, .)$, for t in T, where "epi" stands for the *epigraph*. In this context $(T, \mathcal{T})$ is a measurable space and X a Banach space, or sometimes a more general topological vector space. The integrand f is said to be *normal* if it satisfies the two following conditions :

a) for any $t \in T$, $f(t, .)$ is lower semicontinuous
b) multifunction F is *Effros-measurable*, which means that, for any open subset W of the product space $X \times R$, the subset $F^{-}W := \{t \in T | F(t) \cap W \neq \emptyset\}$ is a member of $\mathcal{T}$.

So, the normality of an integrand is closely related to the measurability of a multifunction. The name of Effros refers to the *Effros σ-field* on the set of all closed subsets of X, whose definition will be recalled in section 2.

It is often necessary to consider various operations on integrands, which give rise to func-

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tions, to other integrands or to multifunctions. Then, the preservation of measurability, normality or Effros-measurability under these operations is of evident interest. In the present paper, we shall be concerned by two of them which are most important in the framework of convex analysis : the conjugacy and the subdifferential operations. So, the problems we address are :

(1) the normality of the integrand f^* associated with f
(2) the Effros measurability of the corresponding subdifferential multifunction $t \to \partial f(t, u(t))$, where u denotes a given measurable function from T into X.

In [18] a positive answer was provided to problems (1) and (2) when X is finite dimensional and $(T, \mathcal{T})$ is an arbitrary measurable space. The case where X is a reflexive separable Banach space (possibly infinite dimensional) was treated later in [6]. Recently, by an indirect method involving the slice topology and assuming that X is a normed space with strongly separable dual, G. Beer showed (theorem 5. 12 in [2]) that if Γ is an Effros-measurable multifunction with non empty closed values in X then the multifunction $t \to \Gamma(t)^\circ$, the polar of $\Gamma(t)$, is Effros-measurable. Clearly, this is close to questions (1) and (2) (see also [3] for the detailed study of the slice topology).

In an other direction, considering a complete σ-finite measure space $(T, \mathcal{T}, \mu)$, but only assuming that X and X^* are Suslin locally convex vector spaces, Castaing and Valadier [4] (chapter VII) had proved the normality of the conjugate integrand. But their definition of a normal integrand is not the same as above : condition b) is replaced by the measurability of f with respect to the product σ-field $\mathcal{T} \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ stands for the Borel σ-field of X. It is known that, due to the possibility of appealing to Auman-Von Neumann-Sainte-Beuve's projection theorem ([4], [20]), the completeness assumption on $\mathcal{T}$ allows for easier proofs. However, as pointed out by several authors, for example, Rockafellar [18] or Nougès-Sainte-Beuve [14], the completeness hypothesis on $\mathcal{T}$ is difficult to handle in some situations, especially when product measure spaces are involved.

This is why, in our approach, we shall consider an arbitrary measurable space $(T, \mathcal{T})$ (i. e. we shall not assume that the σ-field $\mathcal{T}$ is complete with respect to any measure). Further, as in [2], we shall assume that X is an infinite dimensional Banach space whose dual X^* is strongly separable. But our arguments resembles those employed by Rockafellar more, and one of our goals is precisely to show that this type of arguments, of elementary (but often astute) nature, are also quite tractable when X is infinite dimensional, provided suitable adaptations are made.

The paper is organized as follows : in Section 2 we set our notations and give some preliminaries. Section 3 is devoted to a short study of the measurability of multifunctions with w^*-closed values in a separable dual and provides several equivalent formulations. Section 4 contains the statements and the proofs our main results.

2. Notations and preliminaries

Let $(T, \mathcal{T})$ be an abstract measurable space and X a separable Banach space with dual space X^*. Denote by s the strong topology on X and by s^* (resp. w^*) the strong (resp. the weak-star) topology of X^*. Further, denote by 2^X the set of all subsets of X and by $\mathcal{C}(X)$ (resp. $\mathcal{C}^*(X^*)$) the set of all *closed subsets* of X (resp. *w^*-closed subsets* of X^*). Moreover, the subscript "c" will indicate that the subsets are convex. Often, if no ambiguity may occur, we shall simply write $\mathcal{C}$, $\mathcal{C}^*$, $\mathcal{C}^*_c$, ...

In X (or X^*) the closed ball of radius r and centered at x is denoted by $B(x,r)$. For any subset C of X the *distance function* $d(.,C)$ is defined on X by

$$d(x,C) := \inf\{||x-y|| \mid y \in X\} \quad x \in X$$

where $||.||$ denotes the norm of X ; the *indicator function* χ_C is defined by

$$\chi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C. \end{cases}$$

The *Effros σ-field* on 2^X , denoted by $\mathcal{E}_s$ (or simply $\mathcal{E}$), is the σ-field generated by the subsets

$$U^- := \{F \in 2^X \mid F \cap U \neq \emptyset\}$$

where U ranges over the s-open subsets of X (see [5]). A map F from T into 2^X is called a *multifunction*. It is said to be *Effros-measurable* if we have $F^{-1}(\mathcal{E}) \subset \mathcal{T}$. Clearly, this inclusion holds if and only if, for every s-open subset U in X,

$$F^{-1}(U^-) = \{t \in T \mid F(t) \cap U \neq \emptyset\} \in \mathcal{T}.$$

$F^{-1}(U^-)$ is also denoted by F^-U. If multifunction F is Effros-measurable then the *domain* of F, that is, the subset

$$\mathrm{dom}F := \{t \in T \mid F(t) \neq \emptyset\},$$

is a member of $\mathcal{T}$, because $\mathrm{dom}F = F^-X$. Originally the Effros σ-field was defined on $\mathcal{C}(X)$ [Chr, p. 53], so that the above definition is a little more general than the usual one. But the relation between these two definitions is well-known and obvious [11, proposition 2. 6] : a multifonction $F : T \to 2^X$ is Effros-measurable if and only if the multifunction $G :=$ s-cl$F$ is measurable with respect to the trace σ-field $\mathcal{E} \cap \mathcal{C}(X)$ (where 's-cl' denotes the s-closure operation).

It will be also necessary to consider multifunctions whose values lie in the dual space X^*, i. e. maps from T into 2^{X^*} which will be endowed with one of the two following σ-fields:

1) the Effros σ-field $\mathcal{E}_{s^*}$ relative to topology s^*. $\mathcal{E}_{s^*}$ is generated by the subsets

$$V^- := \{C \in 2^{X^*} \mid C \cap V \neq \emptyset\}$$

when V ranges over the set of s^*-open subsets of X^*.

2) the Effros σ-field $\mathcal{E}_{w^*}$ relative to topology w^*. $\mathcal{E}_{w^*}$ is generated by the subsets

$$W^- := \{C \in 2^{X^*} \mid C \cap W \neq \emptyset\}$$

when W ranges over the set of w^*-open subsets of X^*.

But, in fact, we shall only have to deal with multifunctions with values in $\mathcal{C}^*(X^*)$, the set of w^*-closed subsets of X^*, so that we shall only consider the traces (or restrictions) of the σ-fields $\mathcal{E}_{s^*}$ and $\mathcal{E}_{w^*}$ to $\mathcal{C}^*(X^*)$. These trace σ-fields will be respectively denoted by $\mathcal{E}_{s^*} \cap \mathcal{C}^*$ and $\mathcal{E}_{w^*} \cap \mathcal{C}^*$.

Concerning the terminology, it is worthwhile to mention that Effros-measurable multifunctions have also been called 'weakly measurable' multifunctions (e.g. [11], [12]).

A function f from T into X is called a *selector* of the multifunction F if, for any $t \in \mathrm{dom}F$, one has $f(t) \in F(t)$. A *Castaing representation* of F is a sequence $(f_n)_{n \geq 1}$ of measurable selectors of F such that

$$\text{s-cl}F(t) = \text{s-cl}\{f_n(t) \mid n \geq 1\} \quad \forall t \in \mathrm{dom}F$$

where 's-cl' denotes the closure with respect to topology s. It is known (theorem III. 9 of [4]) that a multifunction F, with closed values in X, is Effros-measurable if and only if $\mathrm{dom}F \in \mathcal{T}$ and if F, restricted to $\mathrm{dom}F$, has a Castaing representation.

Besides the above point of view which consists in regarding a multifunction as a map from T into 2^X , many authors also consider the *graph* of F which is denoted $\mathrm{Gr}F$ and defined by

$$\mathrm{Gr}F := \{(t,x) \in T \times X \mid x \in F(t)\}.$$

In this definition, F is regarded as a relation on $T \times X$ rather than a map from T into 2^X.

Given an extended-real valued function ϕ defined on X, the *epigraph* of ϕ is the subset of $X \times \mathbb{R}$ defined by

$$\mathrm{epi}\phi \ := \{(x,\lambda) \mid \phi(x) \leq \lambda\}.$$

Function ϕ is said to be *proper* if it does not take the value $-\infty$ and is not identically $+\infty$. The *effective domain* of ϕ is denoted $\mathrm{dom}\phi$ and defined by

$$\mathrm{dom}\phi := \{x \in X \mid \phi(x) < +\infty\}.$$

As usual in convex analysis, we shall adopt the convention $+\infty - (+\infty) = +\infty$. The *conjugate* of ϕ is the function ϕ^* defined on X^* by

$$\phi^*(y) := \sup\{\langle y,x\rangle - \phi(x) \mid x \in X\} \quad y \in X^*.$$

Given $x_0 \in X$ such that $\phi(x_0)$ is finite, the *subdifferential* of ϕ at x_0 is the w^*-closed subset of X^* denoted $\partial\phi(x_0)$ and defined by

$$\partial\phi(x_0) := \{y \in X^* \mid \phi(x) \geq \phi(x_0) + \langle y, x - x_0\rangle, \forall x \in X\}.$$

An extended-real valued function f defined on $T \times X$ will be called an *integrand* ; we shall say that f is proper (resp. convex, ...) if, for any $t \in T, f(t,.)$ is proper (resp. convex, ...). The following definition, already mentioned in the introduction, is essential for our purpose.

Definition 2.1. Consider an integrand f which satisfies the two following conditions :

a) for any $t \in T$, $f(t,.)$ is s-lower semicontinuous on X.

b) the multifunction $L : T \to 2^{X \times \mathbb{R}}$ defined by

$$L(t) := \mathrm{epi}f(t,.) \quad t \in T$$

is Effros-measurable. Such an f is called a *normal integrand* on $T \times X$ and multifunction L is called the *epigraphical multifunction* associated with f.

Clearly integrand f satisfies a) if and only if multifunction L is s-closed valued. On the other hand an integrand satisfying only condition b) will be called an *Effros-measurable integrand.*

Remark 2.2. As already mentioned, the notion of normal integrand, and variants close to it, were introduced and studied by Rockafellar in a series of papers (for instance [15–18]). It can be observed that, unlike the joint measurability hypothesis also encountered in the literature, this notion is of one-sided nature ; it is well-suited for dealing with minimization problems and was introduced precisely for this purpose. The corresponding notion for maximization problems is obtained by replacing conditions a) and b) above by the following ones :

a') for any $t \in T, f(t,.)$ is s-upper semicontinuous.

b') the multifunction L' defined on T by

$$L'(t) := \text{hypo} f(t,.) := \{(x,\lambda) \mid f(t,x) \geq \lambda\}$$

is Effros-measurable ('hypo' is an abbreviation for 'hypograph').

Thus, it would be clearer in definition 2.1 to use the more precise terms 'epi-normal integrand' and 'epi-Effros-measurable integrand' which explicitely refer to the epigraph. Similarly for the symmetric, but not equivalent, definition involving a') and b'), the term 'hypo-normal integrand' could be employed. It is easy to check that an integrand f is epi-normal if and only if $-f$ is hypo-normal.

Remark 2.3. On the other hand, using propositon III. 13 and lemma VII. 1 of [4], it is readily seen that any epi-normal integrand (resp. hypo-normal integrand) f is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable. Further, the converse implication holds with respect to the σ-field $\hat{\mathcal{T}}$ of universally measurable subsets of T. Indeed, from lemma VII. 1 in [4] we know that an extended-real valued function f is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable if and only if the graph of its associated epigraphical (resp. hypographical) multifunction belongs to the σ-field $\mathcal{T} \otimes \mathcal{B}(X \times \mathbb{R})$. Therefore by Auman-Von Neuman-Sainte Beuve's projection theorem (see theorem III. 23 of [4] or [20]), it can be seen that such a $\mathcal{T} \otimes \mathcal{B}(X)$ - measurable integrand f is, at the same time, epi-normal and hypo-normal with respect to $\hat{\mathcal{T}}$ (see remark 2.2 above). Moreover, the joint measurability of a normal integrand f implies that for any measurable function u from T into X, the function $f(.,u(.))$ is $\mathcal{T}$-measurable, which is needed in many situations, for example when one wants to define the integral functional associated with f.

Remark 2.4. It is worthwhile to mention that a lower semicontinuous normal integrand is also called a 'random lower semicontinuous function', because it can be viewed as a random variable with values in the space of all lower semicontinuous functions from X into the extended reals, endowed with a suitable topology (see for instance [19], [1], ...).

Now we shall need the following known result of which a short and general proof is provided for convenience.

Proposition 2.5. *If f is a Caratheodory integrand, that is, a real valued integrand satisfying the following conditions :*

(1) *for every $x \in X, f(.,x)$ is measurable*

(2) *for every $t \in T, f(t,.)$ is continuous,*

then it is a normal integrand.

Proof. Let F be the epigraphical multifunction of f and let P be a countable dense subset of X. Then, using hypothesis (2), one can see that, for any open subset U of X and for any open interval $I = (a, b)$ of $\mathbb{R}$,

$$\begin{aligned} F^-(U \times I) &= \{t \in T \mid \mathrm{epi} f(t,.) \cap (U \times I) \neq \emptyset\} = \{t \in T \mid f(t,x) < b \text{ for some } x \in U\} \\ &= \bigcup_{y \in P \cap U} \{t \in T \mid f(t,y) < b\}. \end{aligned}$$

Further, observe that, due to hypothesis (1), each subset of the above union is a member of the σ-field $\mathcal{T}$. Finish the proof by recalling that each open subset of $X \times \mathbb{R}$ can be written as a countable union of open sets of the form $U \times I$, using the fact that $\mathbb{R}$ has a countable basis consisting of open intervals. □

Remark 2.6. It can be observed that our proof does not involve the norm of X and even works in any separable topological space. Further, it also shows that if, instead of condition (2), we only assume that, for every $t \in T$, $f(t,.)$ is upper semicontinuous then f is an Effros-measurable integrand.

3. Effros measurability of multifunctions with values in X^*

In order to study the Effros measurability of the conjugate and the subdifferential of an integrand, we begin by examining the properties of the trace of the Effros σ-field $\mathcal{E}_{s^*}$ on $\mathcal{C}^*$. We have already denoted this σ-field by $\mathcal{E}_{s^*} \cap \mathcal{C}^*$; it is generated by the subsets

$$V^- := \{C \in \mathcal{C}^* \mid C \cap V \neq \emptyset\}$$

where V ranges over the set of s^*-open subsets of X^*. The following result is expressed in terms of multifunctions defined on an arbitrary abstract measurable space.

Proposition 3.1. *Let $(T, \mathcal{T})$ be a measurable space, X a separable Banach space and G a $\mathcal{C}^*$ - valued multifunction defined on $(T, \mathcal{T})$. Consider the three following statements :*
i) *for any s^*-open subset V of X^*, $G^-V := \{t \in T \mid G(t) \cap V \neq \emptyset\} \in \mathcal{T}$.*
ii) *for any w^*-compact subset K of X^*, $G^-K \in \mathcal{T}$.*
iii) *for any w^*-closed subset C of X^*, $G^-C \in \mathcal{T}$.*
Then, we have :
A) *In general, the following implications hold :* i) $\Rightarrow$ ii) $\Leftrightarrow$ iii).
B) *Moreover if X^* is s^*-separable, the three statements are equivalent.*

Proof. A) i) $\Rightarrow$ ii). Let d be the distance on X^* associated with the dual norm $||.||_*$ and K a w^*-compact subset. For any positive integer k, define the s^*-open subset V_k by

$$V_k := \{y \in X^* \mid d(y, K) < 1/k\}.$$

Obviously, we have $K = \bigcap_{k \geq 1} V_k$, and the equality

$$G^-K = \bigcap_{k \geq 1} G^-V_k \tag{3.1}$$

will yield the desired conclusion. In (3.1), the inclusion of the left-hand side in the right-hand side is clear. Conversly, take t in the right-hand side. For any k, one can find $y_k \in G(t) \cap V_k$ and $z_k \in K$ verifying

$$||y_k - z_k||_* < 1/k. \tag{3.2}$$

Since K, equipped with the w^*-topology, is a compact metrizable space (the metrizability follows from the separability of X), there exists a subsequence $(z_{k(i)})_{i\geq 1}$ of (z_k) which w^*-converges to some $z \in K$. From (3.2), we deduce that $z = w^* - \lim y_{k(i)}$ which proves $z \in G(t)$ and ends the proof of (3.1).

ii) $\Rightarrow$ iii) is an obvious consequence of the w^*-compactness of the closed balls of X^* and of the easy equality

$$G^- F \ = \ \bigcup_{k\geq 1} G^-\{F \cap B(0,k)\}.$$

iii) $\Rightarrow$ ii) is trivial.

B) Now, assuming that X^* is separable, it only remains to show the implication ii) $\Rightarrow$ i). The s^*-separability of X^* implies that each s^*-open subset V is the countable union of closed balls. Further, invoking once more the w^*-compactness of these balls and using the following easy identity

$$G^- V = \bigcup_{n\geq 1} G^- B_n$$

where the B_n's are closed balls, we obtain the desired conclusion. □

Remark 3.2. As soon as ii) is satisfied, dom$G$:= $\{t \in T \mid G(t) \neq \emptyset\}$ is a member of $\mathcal{T}$, because we clearly have

$$\mathrm{dom} G \ = \ \bigcup_{k\geq 1} G^- B(0,k).$$

Remark 3.3. For any multifunction G with values in $\mathcal{C}^*$, i) trivially implies the following property :

iv) for any w^*-open subset W of X^*, $G^- W \in \mathcal{T}$.

In other words, since the measurable space is arbitrary, we have the inclusion

$$\mathcal{E}_{w^*} \cap \mathcal{C}^* \subset \mathcal{E}_{s^*} \cap \mathcal{C}^* \tag{3.3}$$

Concerning the converse inclusion it may be observed that it certainly does not hold when X^* is not s^*-separable. Indeed, observe that the restriction of $\mathcal{E}_{s^*}$ (resp. $\mathcal{E}_{w^*}$) to X^*, regarded as a subspace of $\mathcal{C}^*$, can be identified with the Borel σ-field $\mathcal{B}(X^*, s^*)$ (resp. $\mathcal{B}(X^*, w^*)$). Since, as it is known, these Borel σ-fields are not equal when X^* is not s^*-separable, the equality

$$\mathcal{E}_{w^*} \cap \mathcal{C}^* = \mathcal{E}_{s^*} \cap \mathcal{C}^*$$

is not true when X^* is not s^*-separable. However we do not know if it is true when X^* is s^*-separable.

The following proposition states a stability property of the measurability of $\mathcal{C}^*$-valued multifunctions under finite or countable intersections. It will be needed for proving our main results. Here, the s^*-separability of X^* is not needed.

Proposition 3.4. *If $(G_n)_{n\geq 1}$ is a sequence of $\mathcal{C}^*$-valued multifunctions satisfying property* ii) *above, then so does the multifunction $G := \bigcap_{n\geq 1} G_n$.*

Proof. Given a w^*-compact subset K of X^* consider, for any $n \geq 1$, the multifunction H_n defined by

$$H_n(t) := G_n(t) \cap K \quad t \in T$$

Each H_n is closed valued in the compact space (K, w^*) which, due to the separability of X, is also metrizable. Moreover each H_n satisfies ii). Consequently, corollary 4.3 of [11] or proposition III.4 of [4] implies that the multifunction H defined by

$$H(t) := \bigcap_{n\geq 1} H_n(t) \quad \in T$$

also satisfies ii). Thus remark 3.2 implies

$$G^- K = \mathrm{dom} H \in \mathcal{T}.$$

$\square$

A result similar to proposition 3.4 was given in [7] (proposition 4.3) when X^* is replaced by a locally convex topological vector space. For other related results on the Effros measurability of multifunctions, the reader may also consult [4], [11], [12], [17–18], [7–10]. On the other hand, observe that propositions 3.1 and 3.4 together show that, when X^* is s^*-separable, the $\mathcal{E}_{s^*}$-measurability of multifunctions is preserved under finite or countable intersections. More precisely we have the

Corollary 3.5. *If X^* is s^*-separable and $(G_n)_{n\geq 1}$ is a sequence of $\mathcal{C}^*$-valued multifunctions satisfying property* i) *above, then so does the multifunction $G := \bigcap_{n\geq 1} G_n$.*

4. Measurability of the conjugate and the subdifferential of a normal integrand

Now, consider an integrand f defined on $T \times X$, where X is a separable Banach space with *s^*-separable dual* X^*, and the multifunctions L and G respectively defined, by

$$L(t) := \mathrm{epi} f(t,.) \text{ and } G(t) := \mathrm{epi} f^*(t,.) \quad t \in T.$$

Recall that f is said to be an Effros-measurable integrand if multifunction L is Effros-measurable ; it is said to be normal if, in addition, $f(t,.)$ is s-lower semicontinuous, for every t in T. On the other hand, in order to simplify the notations in the rest of this section, since all multifunctions whose values lie is X^* will be C^*-valued, a multifunction being Effros-measurable with respect to $\mathcal{E}_{s^*} \cap \mathcal{C}^*$ will be declared *s^*-Effros-measurable*. Further an integrand g defined on $T \times X^*$ will be called a *w^*-lsc normal integrand* if it satisfies the following conditions :

$\alpha)$ for every $t \in T$, $g(t,.)$ is w^*-lower semicontinuous

$\beta)$ the multifunction $t \to \text{epi}g(t,.)$, from T into $2^{X^* \times \mathbb{R}}$, is s^*-Effros-measurable.

We begin by a result which provides an answer to the first problem presented in the introduction.

Theorem 4.1. *If f is a proper Effros measurable integrand defined on $T \times X$, then f^* is a w^*-lsc normal integrand on $T \times X^*$.*

Proof. The w^*-lower semicontinuity of $f^*(t,.)$, for each t in T, follows from the definition of conjugacy. It remains to show that the $\mathcal{C}^*(X^* \times \mathbb{R})$-valued multifunction G defined as above is s^*-Effros-measurable, i. e. satisfies condition i) of proposition 3.1. For this purpose consider $(u_k)_{k \geq 1}$, a Castaing representation of the $\mathcal{C}(X \times \mathbb{R})$-valued multifunction s-cl L (which is non empty valued, due to the properness of f). For any $k \geq 1$, we have $u_k = (v_k, a_k)$ where v_k (resp. a_k) is a measurable function defined on T, with values in X (resp. the reals). Moreover the following equality holds

$$\text{s-cl}L(t) = \text{s-cl}\{(v_k(t), a_k(t)) \mid k \geq 1\}.$$

Thus for every $(t, y) \in T \times X^*$ we can write

$$\begin{aligned} f^*(t,y) &= \sup\ [\langle y, x\rangle - f(t,x) \mid x \in X] \\ &= \sup\ [\langle y, x\rangle - r \mid (x,r) \in \text{epi}f(t,.)] \\ &= \sup\ [\langle y, x\rangle - r \mid (x,r) \in \text{s-cl epi}f(t,.)] \end{aligned}$$

because of the continuity of

$$(x, r) \to \langle y, x\rangle - r.$$

Then, the definition of a Castaing representation yields

$$f^*(t,y) = \sup_{k \geq 1} g_k(t,y) \tag{4.1}$$

where the integrands g_k are defined by

$$g_k(t,y) := \langle y, v_k(t)\rangle - a_k(t) \qquad (t,y) \in T \times X^* \quad k \geq 1.$$

For any $k \geq 1$, the multifunction G_k defined by

$$G_k(t) := \text{epi } g_k(t,.) \qquad t \in T$$

is $\mathcal{C}^*(X^* \times \mathbb{R})$-valued, because g_k is w^*-continuous.

Then proposition 2.5 applied to g_k , treated as a function from $T \times (X^*, s^*)$ into $\mathbb{R}$, shows that each G_k satisfies i). Finally, we obtain the s^*-Effros measurability of G by using (4.1) which yields the equality

$$G(t) = \bigcap_{k \geq 1} G_k(t), \tag{4.2}$$

and by invoking corollary 3.5. $\square$

Remark 4.2. In the above proof, it is worthwhile to observe that equalities (4.1) and (4.2) combined with corollary 3.5 imply the following general result : if $(g_k)_{k\geq 1}$ is a sequence of w^*-lower semicontinuous normal integrands defined on $T \times X^*$, then the integrand

$$g := \sup_{k\geq 1} g_k$$

has the same properties.

Proposition 4.3. *Let g be a w^*-lsc normal integrand defined on $T \times X^*$ and b a real-valued measurable function defined on T. Then the $\mathcal{C}^*$-valued multifunction G defined by*

$$G(t) := \{y \in X^* \mid g(t,y) \leq b(t)\}$$

is s^*-Effros-measurable.

Proof. Given a fixed w^*-closed set C, define the multifunction H, with closed values in $X^* \times \mathbb{R}$, by

$$H(t) := C \times \{r \in R \mid r \leq b(t)\} \quad t \in T.$$

In order to prove the s^*-Effros-measurability of H, it suffices to observe that, for any s^*-open set V of X^* and any open interval of $\mathbb{R}$, one has

$$H^-(V \times I) = \begin{cases} b^{-1}\{(\inf I, +\infty)\} & \text{if } C \cap V \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Further, noting the equality

$$G^-C = \{t \in T \mid H(t) \cap \text{epi } g(t,.) \neq \emptyset\}$$

and observing that, by corollary 3.5, the multifunction

$$t \to H(t) \cap \text{epi } g(t,.)$$

is s^*-Effros-measurable, we obtain the desired conclusion. □

Remark 4.4. In particular propositions 4.1 and 4.3 together imply that if f is normal integrand defined on $T \times X$ and b is a real-valued measurable function defined on T, then the $\mathcal{C}_{c^*}$-valued multifunction G defined as above, but with g replaced by f^*, is s^*-Effros-measurable.

The next proposition concerns the normality of the sum of two integrands on $T \times X^*$. It will be needed in the proof of our main result, but also has its own interest.

Proposition 4.5. *If f_1 and f_2 are two proper Effros-measurable integrands defined on $T \times X$, then the convex w^*-lower semicontiuous integrand $f_1^* + f_2^*$ defined on $T \times X^*$ is s^*-Effros-measurable (thus, is a w^*-lsc normal integrand).*

Proof. Returning to equality (4.1) we see that there exists two sequences $(g_{1j})_{j\geq 1}$ and $(g_{2k})_{k\geq 1}$ of w^*-lsc normal integrands verifying, for any $(t,y) \in T \times X^*$,

$$f_1^*(t,y) = \sup_{j\geq 1} g_{1j}(t,y) \quad \text{and} \quad f_2^*(t,y) = \sup_{k\geq 1} g_{2k}(t,y).$$

Hence we have

$$f_1^*(t,y) + f_2^*(t,y) = \sup_{j\geq 1} \sup_{k\geq 1}\{g_{1j}(t,y) + g_{2k}(t,y)\}$$

which, by remark 4.2, yields the desired conclusion. □

Now, consider a normal proper integrand f defined on $T \times X$ and a function u from T into X satisfying, for every $t \in T, u(t) \in \mathrm{dom} f(t,.)$, and define the $\mathcal{C}_{c^*}$-valued multifunction D_u by

$$D_u(t) := \partial f(t,u(t)) = \{y \in X^* \mid f(t,x) \geq f(t,u(t)) + \langle y, x - u(t)\rangle,\ \forall x \in X\}.$$

For every $t \in T$, $D_u(t)$ is the subdifferential to $f(t,.)$ at $u(t)$. We say that D_u is the *subdifferential multifunction* associated to f and u. Here is the second main result of the present paper.

Theorem 4.6. *If f is a proper normal integrand on $T \times X$, then the $\mathcal{C}_{c^*}$-valued multifunction D_u is s^*-Effros-measurable.*

Proof. In order to show the s^*-Effros measurability of D_u , we begin by recalling the following standard equality, valid for every $t \in T$,

$$D_u(t) = \{y \in X^* \mid f^*(t,y) \leq \langle y, u(t)\rangle - f(t,u(t))\}. \tag{4.3}$$

By the end of remark 2.3 the real valued function $f(.,u(.))$ is measurable. Further, consider the integrand

$$(t,y) \to f^*(t,y) - \langle y, u(t)\rangle = f^*(t,y) + (\chi_{\{-u(t)\}})^* \tag{4.4}$$

where $\chi_{\{-u(t)\}}$ denotes the indicator function of the singleton $\{-u(t)\}$. Since the integrand

$$(t,x) \to \chi_{\{-u(t)\}}(x)$$

defined on $T \times X$ is normal, proposition 4.5 shows that the convex w^*-lower semicontinuous integrand defined on $T \times X^*$ by (4.4) is normal too. Consequently, the proof is finished by returning to (4.3) and by applying proposition 4.3. □

Remark 4.7. An alternate proof of proposition 4.6 can be done using neither proposition 4.5 nor the indicator function. Indeed consider the integrand h defined on $T \times X^*$ by

$$h(t,y) := f^*(t,y) - \langle y, u(t)\rangle$$

and return to (4.1) which gives

$$h(t,y) = \sup_{k\geq 1}[\langle y, v_k(t) - u(t)\rangle - a_k(t)] \qquad (t,y) \in T \times X^*.$$

By remark 4.2 this yields the desired conclusion.

Remark 4.8. One may also ask the following question : if f is an Effros-measurable integrand defined on $T \times X$, is the biconjugate integrand f^{**} normal ? An affirmative answer can be given if, for every $t \in T$, there exists at least one continuous affine function less than or equal to $f(t,.)$. Indeed in such a case theorem I.3 of [4] shows that

$$\mathrm{epi} f^{**}(t,.) = \mathrm{clco}\ \mathrm{epi} f(t,.)$$

where 'clco' denotes the closed convex hull operation in $X \times \mathbb{R}$. Thus, it only remains to invoke theorem 9.1 of [11], or theorem 1.5 in [13] (or corollary 3.2.3 in [8] where the completeness of X is not required), which shows that the epigraphical multifunction associated with f^{**} is Effros-measurable. Clearly, these arguments are valid even if X^* is not s^*-separable (in fact they remain valid when X is an arbitrary metrizable locally convex vector space).

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