Journal of Convex Analysis
Volume 1 (1994), No. 2, 259–264

A Reverse Convolution-Inequality

Friedemann Brock
Institut für Mathematik, Universität Leipzig,
Augustusplatz 9-10, D-04109 Leipzig, Germany.
e-mail : brock@mathematik.uni-leipzig.d400.de

Received 22 February 1994

We give a simple proof of a reverse convolution-inequality of two characteristic functions on $\mathbb{R}$ and their rearrangements and derive related inequalities for functions of several variables.

Keywords : Convolution, rearrangement

1991 Mathematics Subject Classification: 26A51, 26D15, 49K05

Consider the following problem : Let $A, B \subseteq [0, c]$ be two disjoint sets with given L-measures $|A|, |B|$, such that $|A| + |B| \leq c$. Where should A and B be placed, so that the expression

$$\int_{A\times B} \frac{dxdy}{(x-y)^2} \tag{1}$$

becomes minimal?

This problem was raised by G. Bouchitté and brought to my attention by G. Buttazzo and B. Kawohl. Since the integrand in (1) becomes small if x and y are far apart from each other, it is natural to conjecture that A and B should lie at opposite ends of the admissible interval $[0, c]$. The conjecture can be rewritten as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(x)\check{g}(y)h(x-y)\ dxdy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(y)h(x-y)\ dxdy \tag{2}$$

with $f(x) = \chi(A)(x)$, $g(y) = \chi(B)(y)$, $\hat{f}(x) = \chi([0, |A|])(x)$, $\check{g}(y) = \chi([c-|B|, c])(y)$ and $h(z) = z^{-2}$.

It should be pointed out that (2) does not just follow from the well known convolution-inequality

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(x)v(y)w(x-y)\ dxdy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} u^*(x)v^*(y)w^*(x-y)\ dxdy, \tag{3}$$

where u, v, w are nonnegative and measurable on $\mathbb{R}$, and u^*, v^*, w^* denote their symmetrically nonincreasing rearrangements (see [1], p.25).

ISSN 0944-6532 / \$ 2.50

In fact, if A and B are reflected across zero :

$$A_- := \{x \in \mathbb{R} \mid (-x) \in A\}$$
$$B_- := \{y \in \mathbb{R} \mid (-y) \in B\}$$

and if we set $u(x) := \chi(A \cup A_-)(x)$, $v(y) := \chi(B \cup B_-)(y)$, $w(z) = h(z)$ in (3), we conclude that

$$\begin{aligned} \int_{A \cup A_-} \int_{\mathbb{R} \setminus (B \cup B_-)} h(x-y)\ dxdy \geq & \int_{-|A|}^{|A|} \int_{c-|B|}^{+\infty} h(x-y)\ dxdy \\ & + \int_{-|A|}^{|A|} \int_{-\infty}^{-c-|B|} h(x-y)\ dxdy\ . \end{aligned} \tag{4}$$

In comparison with (2), there occur some additional terms in (4), and it is not clear to me how to deal with them.

Now we outline the contents of this paper. In Theorem 1.1 we give an elementary proof of the above conjecture (2) for weight-functions $h(x-y)$ which are symmetrically nonincreasing. Then this result is applied to convolutions of functions of one and several variables in Theorem 1.4 and 1.5, respectively.

Theorem 1.1. *Let A, B be L-measurable sets lying in an interval $[0, c], (c > 0)$, and let h be a measurable nonnegative symmetrically nonincreasing function on $\mathbb{R}$.*
Then :

$$\int_A \int_B h(x-y)\ dxdy \geq \int_0^{|A|} \int_{c-|B|+|A \cap B|}^{c+|A \cap B|} h(x-y)\ dxdy\ . \tag{5}$$

Remark 1.2. We cannot omit the terms $|A \cap B|$ on the right-hand side of (5). This can be seen from the following example :

Let $c = 4$, $A = B = [0, 1] \cup [3, 4]$ and :

$$h(x) = \begin{cases} 1 & \text{if } |x| \leq 3 \\ 0 & \text{if } |x| > 3 \end{cases}\ .$$

Open problem : It could be conjectured that the sharper inequality

$$\int_A \int_B h(x-y)\ dxdy \geq \int_0^{|A|} \int_{c-|B|}^{c} h(x-y)\ dxdy \tag{6}$$

holds for functions $h(z)$ which are *convex for positive z*. (Note that the function $h(z) = z^{-2}$ in (1) has just this property.) I could neither verify (6) nor find a counter-example.

From Theorem 1.1 we immediately conclude a

Corollary 1.3. *Let A, B, h, c be as in* Theorem 1.1. *Then :*

$$\int_A \int_B h(x-y)\ dxdy \geq \int_0^{|A|} \int_{2c-|B|}^{2c} h(x-y)\ dxdy\ . \tag{7}$$

Proof of Theorem 1.1

First we observe that it suffices to prove (5) for the case that A, B consist of a finite number of closed intervals. Then the assertion follows by approximation.

Next we assume that A, B are disjoint. Then (5) reduces to (6). We prove (6) by induction over the total number m of the disjoint intervals of $A \cup B$.

Here and in the following we use a simple argument which can easily be derived from the properties for the function h :

Let M, M', N, N' be intervals with $|M| = |M'|, |N| = |N'|$,

$$\text{dist}\ \{S(M); S(N)\} \ \leq\ \text{dist}\ \{S(M'); S(N')\} \tag{8a}$$

and with $S(M) := |M|^{-1} \int_M x dx$ denoting the centre of gravity. Then :

$$\int_M \int_N h(x-y)\ dxdy \geq \int_{M'} \int_{N'} h(x-y)\ dxdy\ . \tag{8b}$$

1.) Let $m = 2$, i.e. A and B are intervals. Then (6) follows from (8a,b).

2.) Assume that (6) is proved for any $c > 0$ and $m \leq k$, $(k \in \mathbb{N},\ k \geq 2)$.

3.) Now let $m = k+1$. We denote by $[a, b]$ the interval on the far right of $A \cup B$. We can assume that $[a, b] \subseteq B$ and set $B' := B \setminus [a, b]$. Then $A \cup B'$ has only k disjoint intervals all lying in the interval $[0, a]$. Applying 2.), we get

$$\int_A \int_{B'} h(x-y)\ dxdy \geq \int_0^{|A|} \int_{a-|B'|}^{a} h(x-y)\ dxdy\ . \tag{9}$$

Further we have because of (8a,b) :

$$\int_A \int_a^b h(x-y)\ dxdy \geq \int_0^{|A|} \int_a^b h(x-y)\ dxdy\ . \tag{10}$$

Adding (9) and (10) and again using (8a,b) we conclude

$$\begin{aligned}\int_A \int_B h(x-y)\ dxdy &\geq \int_0^{|A|} \int_{a-|B|}^{a} h(x-y)\ dxdy \\ &\geq \int_0^{|A|} \int_{c-|B|}^{c} h(x-y)\ dxdy\end{aligned}$$

which proves (6).

Next let A, B be sets as above which are no longer disjoint. If then $[a, b] \subseteq A \cap B$ with $0 \leq a < b \leq c$, we define new sets A', B' by "shifting" :

$$\begin{aligned}A' &:= (A \cap [0, b]) \cup \{x \in \mathbb{R} \mid (x - b + a) \in (A \cap [b, c])\}, \\ B' &:= (B \cap [0, a]) \cup \{x \in \mathbb{R} \mid (x - b + a) \in (B \cap [a, c])\}.\end{aligned}$$

Then because of (8a,b) we get

$$\int_A \int_B h(x-y)\ dxdy \geq \int_{A'} \int_{B'} h(x-y)\ dxdy\ .$$

We can repeat the above argument step by step to derive two disjoint sets A'', B'' with $|A''| = |A|, |B''| = |B|$ and $A'', B'' \subseteq [0, c + |A \cap B|]$. With the help of (5) we conclude then, that

$$\begin{aligned}\int_A \int_B h(x-y)\ dxdy &\geq \int_{A''} \int_{B''} h(x-y)\ dxdy \\ &\geq \int_0^{|A|} \int_{c+|A\cap B|-|B|}^{c+|A\cap B|} h(x-y)\ dxdy\ .\end{aligned}$$

This concludes the proof of Theorem 1.1.

The above results can be generalized to convolution inequalities of functions.

Theorem 1.4. *Let h, c be as in* Theorem 1.1 *and let f, g be nonnegative measurable functions on the interval $[0, c]$ with*

$$\text{supp}\ \{f > 0\} \cap\ \text{supp}\ \{g > 0\} = \emptyset. \tag{11}$$

We denote by $\hat{f}(x)$ the monotone nonincreasing rearrangement of $f(x)$ with respect to $x = 0$ and by $\check{g}(x)$ the monotone nondecreasing rearrangement of g with respect to $x = c$. Then :

$$\int_0^c \int_0^c \hat{f}(x)\check{g}(y)h(x-y)\ dxdy \leq \int_0^c \int_0^c f(x)g(y)h(x-y)\ dxdy\ , \tag{12}$$

as long as one of the integrals in (12) *converges.*

Proof. We introduce the characteristic functions of the level sets of $f, g, \hat{f}, \check{g}$:

$$\begin{gathered}\chi\left(\{(x,\alpha) \mid f(x) > \alpha\}\right)(x,\alpha) =: \chi\left(\{f(x) > \alpha\}\right)(x,\alpha)\ , \\ \chi\left(\{(y,\beta) \mid g(y) > \beta\}\right)(y,\beta) =: \chi\left(\{g(y) > \beta\}\right)(y,\beta)\ , \\ \chi(\{(x,\alpha) \mid \hat{f}(x) > \alpha\})(x,\alpha) = \\ \chi\left(\{(x,\alpha) \mid x \in [0, |\{f(\cdot) > \alpha\}|]\}\right)(x,\alpha) =: \chi\left([0, |\{f(x) > \alpha\}|]\right)(x,\alpha), \\ \chi\left(\{(y,\beta) \mid \check{g}(y) > \beta\}\right)(y,\beta) = \\ \chi\left(\{(y,\beta) \mid y \in [c - |\{g(\cdot) > \beta\}|, c]\}\right)(y,\beta) =: \chi\left([c - |\{g(y) > \beta\}|, c]\right)(y,\beta)\ .\end{gathered}$$

We have :

$$\begin{aligned} f(x) &= \int_0^\infty \chi(\{f(x) > \alpha\})(x,\alpha)\ d\alpha\ , \\ \hat{f}(x) &= \int_0^\infty \chi([0, |\{f(x) > \alpha\}|])(x,\alpha)\ d\alpha\ , \\ g(y) &= \int_0^\infty \chi(\{g(y) > \beta\})(y,\beta)\ d\beta\ , \\ \check{g}\,(y) &= \int_0^\infty \chi([c - |\{g(y) > \beta\}|, c])(y,\beta)\ d\beta\ . \end{aligned}$$

Then after a change of the order of integration (12) takes the form

$$\int_0^\infty\int_0^\infty \left\{ \int_0^c\int_0^c \chi(\{f(x)>\alpha\})\chi(\{g(y)>\beta\})(y,\beta)h(x-y)\ dydx \right\} d\alpha d\beta \ \geq$$
$$\int_0^\infty\int_0^\infty \left\{ \int_0^c\int_0^c \chi([0,|\{f(x)>\alpha\}|])\chi([c-|\{g(y)>\beta\}|,c])(y,\beta)h(x-y)\ dydx \right\} d\alpha d\beta.$$

Setting $A := \{x \mid f(x)>\alpha\}$, $B := \{y \mid g(y)>\beta\}$, $(\alpha,\beta>0)$, we conclude that $A\cap B=\emptyset$ because of (11). Therefore we can apply Theorem 1.1 on the inner integrals $\{\ldots\}$ in (13). This proves the assertion.

Now we introduce the following partition in $\mathbb{R}^n$:

$$x=(x_1,\ldots,x_n),\ (x_1,\ldots,x_{n-1}) =: x',\ x_n =: y,$$
$$\xi=(\xi_1,\ldots,\xi_n),\ (\xi_1,\ldots\xi_{n-1}) =: \xi',\ \xi_n =: \eta.$$

We denote by $h:\mathbb{R}^n\to\mathbb{R}_0^+$ a L-measurable function which is symmetrically nonincreasing with respect to y.

Theorem 1.5. *Let $c>0$ and*

$$f,g:\mathbb{R}^n\to\mathbb{R}_0^+$$

be L-measurable functions with support in $\mathbb{R}^{n-1}\times[0,c]$. Let $\hat f(x)$ denote the monotone nonincreasing rearrangement of $f(x)$ in the direction y with respect to the hyperplane $\{y=0\}$, and let $\check g(\xi)$ denote the monotone nondecreasing rearrangement of $g(\xi)$ in the direction η with respect to the hyperplane $\{\eta=2c\}$. Then :

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(\xi)h(x-\xi)\ dxd\xi \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat f(x)\ \check g(\xi)h(x-\xi)\ dxd\xi. \tag{14}$$

Proof. Again we introduce characteristic functions :

$$\chi(\{(x,\alpha)\mid f(x)>\alpha\})(x,\alpha) =: \chi(f(x)>\alpha\})(x,\alpha)\ ,$$
$$\chi(\{(\xi,\beta)\mid g(\xi)>\beta\})(\xi,\beta) =: \chi(\{g(\xi)>\beta\})(\xi,\beta)\ ,$$
$$\chi(\{(x,\alpha)\mid \hat f(x)>\alpha\})(x,\alpha) =$$
$$\chi(\{(x,\alpha)\mid y\in[0,|\{f(x',\cdot)>\alpha\}|]\})(x,\alpha) =: \chi([0,|\{f(x)>\alpha\}|])(x,\alpha),$$
$$\chi(\{(\xi,\beta\mid \check g(\xi)>\beta\})(\xi,\beta) =$$
$$\chi(\{(\xi,\beta)\mid \eta\in[2c-|\{g(\xi',\cdot)>\beta\}|,2c]\})(\xi,\beta) =: \chi([2c-|\{g(\xi)>\beta\}|,2c])(\xi,\beta)\ .$$

Then using the representations

$$
\begin{aligned}
f(x) &= \int_0^\infty \chi(\{f(x) > \alpha\})(x, \alpha)\ d\alpha\ , \\
\hat{f}(x) &= \int_0^\infty \chi([0, |\{f(x) > \alpha\}|])(x, \alpha)\ d\alpha\ , \\
g(\xi) &= \int_0^\infty \chi(\{g(\xi) > \beta\})(\xi, \beta)\ d\beta\ , \\
\check{g}\,(\xi) &= \int_0^\infty \chi([2c - |\{g(\xi) > \beta\}|, 2c])(\xi, \beta)\ d\beta\ ,
\end{aligned}
$$

the inequality (14) takes the form

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \Big\{ \int_0^c \int_0^c \chi(\{f(x', y) > \alpha\})(x', y, \alpha) \cdot \\
&\qquad \cdot \chi(\{g(\xi', \eta) > \beta\})(\xi', \eta, \beta) \cdot h(x' - \xi', y - \eta)\ dy d\eta \Big\} \cdot dx' d\xi' d\alpha d\beta \geq \\
&\int_0^\infty \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \Big\{ \int_0^{2c} \int_0^{2c} \chi([0, |\{f(x', y) > \alpha\}|])(x', y, \alpha) \cdot \\
&\qquad \cdot \chi([2c - |\{g(\xi', \eta) > \beta\}|, 2c])(\xi', \eta, \beta) \cdot h(x' - \xi', y - \eta)\ dy d\eta \Big\} \cdot dx' d\xi' d\alpha d\beta\ .
\end{aligned}
\tag{15}
$$

If we set $A := \{y \mid f(x', y) > \alpha\}$, $B := \{y \mid g(\xi', \eta) > \beta\}$ with fixed $\alpha, \beta > 0$, $x', \xi' \in \mathbb{R}^{n-1}$, we can apply the Corollary on the inner integrals $\{\ldots\}$ in (15) for almost every $x', \xi' \in \mathbb{R}^{n-1}$, which proves (15).

Acknowledgment. This article was written during my stay at the University of Cologne in January 1994. I thank B. Kawohl for pointing out the problem and for helpful discussions.

References

[1] B. Kawohl : Rearrangements and Convexity of Level Sets in PDE, Springer Lecture Notes in Math. 1150 (1985).