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Structure of Efficient Sets for Strictly Quasi Convex Objectives

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This paper studies the weak efficient set (*WEff P*) of a minimization problem P with k objectives defined on a convex set X of $\mathbb{R}^n$. These objectives are continuous and belong to the class of so-called strictly quasiconvex functions, which contains, in particular, convex as well as linear fractional functions. When k is greater than n, it is of interest to replace the original problem by several subproblems, having at most n objectives. We show that if *WEff P* is bounded, the knowledge of the efficient sets of such subproblems, completely determines *WEff P*.

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1. Introduction

We consider the multiobjective problem

$$(P) \qquad \min_{x\in X}(f_1,\cdots,f_k)$$

where X is a closed convex set in $\mathbb{R}^n$, $f_i : X \longrightarrow \mathbb{R}$, $i = 1,\cdots,k$ and we focus our attention on the structure of the weak efficient set *WEff P*. In [7], it is shown that if the objectives are continuous and convex, the set *WEff P* is determinable from the spatial structure of the efficient sets of subproblems having at most n objectives. The purpose of this paper is to extend these results to a class of non convex objectives. Namely, we consider the class of strictly quasiconvex functions, which has been previously introduced in [2]. This class contains in particular convex functions as well as linear fractional functions, which gives a wide range of applications. See for instance [5][6] for many examples and an extensive bibliography on fractional programming in the scalar case. The paper is divided into two sections. In section 2 we state the problem and give definitions used in the sequel. In section 3 we extend several results, known in the convex case, to the strictly quasiconvex case. Then we deduce Theorem 3.7, which is the main result of this paper.

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2. Basic concepts

Recall that X being a convex set in $\mathbb{R}^n$, a functional $f : X \longrightarrow \mathbb{R}$ is quasiconvex on X iff for all points x, y in X and $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \max(f(x), f(y)).$$

This class has been extensively studied in the literature, see for instance [1] for many examples, properties and a bibliography.

We say that $f : X \longrightarrow \mathbb{R}$ is strictly quasiconvex [2] iff for every x, y in X and $\lambda \in]0,1[$ one has :

$$f(\lambda x + (1-\lambda)y) < max(f(x), f(y)) \text{ if } f(x) \neq f(y)$$

and

$$f(\lambda x + (1-\lambda)y) \leq f(x) \text{ if } f(x) = f(y)\cdot$$

In particular, linear fractional functions of the form $f(x) = \dfrac{a.x+s}{b.x+t}$ where $a, b \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, are strictly quasiconvex on every convex set X contained in their domain. (Here . stands for the scalar product in $\mathbb{R}^n$)

Convexity implies strict quasiconvexity and strict quasiconvexity implies quasiconvexity.

As an immediate consequence of the definition, it is easy to prove that if $x = \sum_{i=1}^{l} \lambda_i x_i,\ \lambda_i > 0,\ i = 1, \cdots, l,\ \sum_{i=1}^{l} \lambda_i = 1$, and $f(x_i) \leq z$ for $i = 1, \cdots, l$ then $f(x) < z$ whenever $\{f(x_i) : i = 1, \cdots, l\}$ is not a singleton.

Consider the problem

$$(P) \qquad \min_{x \in X} F(x)$$

with $F = (f_1, \cdots, f_k)$, each $f_i,\ i = 1, \cdots, k$ being strictly quasiconvex. We refer to a subproblem P_I of P when only a nonempty subset $I \subset \{1, \cdots, k\}$ of all objectives, is considered.

The notation $|I|$ stands for the cardinality of I.

Recall that the sets of minimal points and weakly minimal points of P are defined by

$$MinF(X) = \{z \in F(X) : (z - F(X)) \cap (\mathbb{R}^k_+ \backslash \{0\}) = \emptyset\}$$

$$WMinF(X) = \{z \in F(X) : (z - F(X)) \cap \operatorname{int} \mathbb{R}^k_+ = \emptyset\}\cdot$$

The corresponding efficient sets in the argument space are:

$$\mathit{Eff}\, P = \{x \in X : F(x) \in MinF(X)\}$$

$$\mathit{WEff}\, P = \{x \in X : F(x) \in WMinF(X)\}.$$

3. Determination of *WEff P*

In 1984, Lowe et al.[3] showed that, when all objectives are convex, *WEff P* is the union of efficient sets of all subproblems P_I , $I \subset \{1, \cdots, k\}, I \neq \emptyset$.

At first we give a similar result in the strictly quasiconvex case.

Theorem 3.1. *Suppose that all objectives f_i, $i = 1, \cdots, k$ are strictly quasiconvex and upper semicontinuous along line segments in X then*

$$WEff\ P = \cup\{Eff\ P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset\}$$

Proof. Consider $x \in X$ such that $x \notin$ *WEff P*. There exists $y \in X$ with $f_i(y) < f_i(x)$ for all i and then, for each $I \subset \{1, \cdots, k\}, x \notin Eff\ P_I$.

Now, suppose that $x \notin \cup\{Eff\ P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset\}$. Taking $I = \{1, \cdots, k\}$, there must exist $i_1 \in I$ and $x_1 \in X$ such that

$$f_{i_1}(x_1) < f_{i_1}(x) \text{ and } f_i(x_1) \leq f_i(x), \quad i \in I. \tag{1}$$

Now if $I = I_1 = \{1, \cdots, k\} \backslash \{i_1\}$, there are $i_2 \in I_1$ and $x_2 \in X$ such that

$$f_{i_2}(x_2) < f_{i_2}(x) \quad \text{ and } \quad f_i(x_2) \leq f_i(x), \quad i \in I_1. \tag{2}$$

Setting $y_2 = \lambda x_1 + (1 - \lambda) x_2, \lambda \in]0, 1[$, and using the upper semicontinuity of f_{i_1} on the segment $[x_1, x_2]$, we have by (1), for $(1 - \lambda)$ small enough,

$$f_{i_1}(y_2) < f_{i_1}(x).$$

On the other hand, the strict quasiconvexity of f_{i_2} implies, for each $\lambda \in]0, 1[$

$$f_{i_2}(y_2) < f_{i_2}(x).$$

Further from the quasiconvexity of f_i 's , we have for each $\lambda \in]0, 1[$

$$f_i(y_2) \leq f_i(x), \quad i \in I_2 = \{1, \cdots, k\} \backslash \{i_1, i_2\}.$$

Suppose now, that we have obtained y_ℓ and $I_\ell = I \backslash \{i_1, \cdots, i_\ell\}$ such that $f_i(y_\ell) < f_i(x), \quad i \in \{i_1, \cdots, i_\ell\}$ and $f_i(y_\ell) \leq f_i(x), \quad i \in I_\ell$. Using $x \notin Eff\ P_{I_\ell}$ we get $i_{\ell+1} \in I_\ell$ and $x_{\ell+1} \in X$ satisfying

$$f_{i_{\ell+1}}(x_{\ell+1}) < f_{i_{\ell+1}}(x) \quad \text{ and } \quad f_i(x_{\ell+1}) \leq f_i(x), \quad i \in I_\ell.$$

Then from the upper semicontinuity of $f_{i_1}, \cdots, f_{i_\ell}$ on the segment $[y_l, x_{\ell+1}]$, the strict quasiconvexity of $f_{i_{\ell+1}}$ and the quasiconvexity of other objectives, we obtain $y_{\ell+1}$, a convex combination of y_ℓ and $x_{\ell+1}$, such that

$$f_i(y_{\ell+1}) < f_i(x), \quad i \in \{i_1, \cdots, i_{\ell+1}\}$$

and

$$f_i(y_{\ell+1}) \leq f_i(x), \quad i \in I_{\ell+1} = I \backslash \{i_1, \cdots, i_{\ell+1}\}.$$

As the number of objectives is k, we obtain finally $y_k \in X$ such that
$f_i(y_k) < f_i(x), \quad i \in \{1, \cdots, k\}$, which means that $x \notin WEff\, P$. □

Note that convex functions are automatically upper semicontinuous along line segments and also that the previous theorem remains valid if $X \subset E$, where E is a linear space without topology.

Now, following an idea developped by Ward [7] in the convex case, which uses Helly's Theorem, we give an extension of Theorem 3.1 when $n < k$.

Helly's Theorem : Let C_i, $i = 1, \cdots, m$ be a collection of convex sets in $\mathbb{R}^n$. If every subcollection of $n+1$ or fewer of these C_i has a nonempty intersection, then the entire collection of the m sets has a nonempty intersection.

Theorem 3.2. *Suppose that f_i, $i = 1, \cdots, k$ are strictly quasiconvex and upper semi-continuous along line segments, then*

$$WEff\, P = \cup\{Eff\, P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset, |I| \leq n+1\}$$

Proof. From Theorem 3.1 it is sufficient to consider the case $k > n+1$ and to prove the inclusion $\subset$.

Consider $x \notin \cup\{Eff\ P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset, |I| \leq n+1\}$. Then for each $J \subset \{1, \cdots, k\}, J \neq \emptyset$ with $|J| \leq n+1$, we have $x \notin \cup\{Eff\ P_I : I \subset J, I \neq \emptyset\}$ and from Theorem 3.1, it follows that $x \notin WEff\, P_J$. Therefore there exists

$$x_J \in X \text{ such that } f_j(x_J) < f_j(x) \text{ for all } j \in J. \tag{3}$$

For each $i \in \{1, \cdots, k\}$ we define the closed convex set

$$C_i = \operatorname{conv}\{x_J : J \subset \{1, \cdots, k\}, J \neq \emptyset, |J| \leq n+1, \ \ i \in J\}.$$

It is clear from (3) that, for all $J \subset \{1, \cdots, k\}$ with $J \neq \emptyset, |J| \leq n+1$, $i \in J$, we have $f_i(x_J) < f_i(x)$ and the quasiconvexity of f_i entails that for every $y \in C_i$

$$f_i(y) < f_i(x). \tag{4}$$

Now, for a fixed J with $|J| \leq n+1$, the collection $\{C_i\ ,\ i \in J\}$ has x_J in common and from Helly's Theorem, there exists some y^* belonging to $\bigcap_{i=1}^{k} C_i$. Thus from (4), for each $i \in \{1, \cdots, k\}$

$$f_i(y^*) < f_i(x)$$

and $x \notin WEff\, P$. □

Recall that if $C \subset \mathbb{R}^n$, $\dim C$ denotes the dimension of the affine space generated by C. The following lemma will be useful in the sequel.

Lemma 3.3. *Let $C = \operatorname{conv}\{y_i \in \mathbb{R}^n : i = 1, \cdots, n+1\}$. Suppose that there exists $x \in C$ which cannot be written as a convex combination of fewer than $n+1$ points y_i, then $\dim C = n$ and $x \in \operatorname{int} C$.*

Proof. Suppose that $\dim C \leq n-1$. By Carathéodory's Theorem [4] , every point of C can be expressed as a convex combination of n elements y_i. This is a contradiction with the assumption about x. It remains to prove that $x \in \operatorname{int} C$. Suppose that $x = \sum_{i=1}^{n+1} \lambda_i y_i$, with all $\lambda_i > 0$. As $\dim C = n$, the vectors $y_{n+1} - y_i, i = 1, \cdots, n$ are independent. We consider the neighborhood of 0 in $\mathbb{R}^n$ defined by

$$N = \{z : z = \sum_{i=1}^{n} \gamma_i (y_{n+1} - y_i) \ , \ |\gamma_i| < \lambda_i \quad i = 1, \cdots, n, \quad |\sum_{i=1}^{n} \gamma_i| < \lambda_{n+1}\}.$$

We have for every $z \in N$,

$$x + z = \sum_{i=1}^{n} (\lambda_i - \gamma_i) y_i + (\lambda_{n+1} + \sum_{i=1}^{n} \gamma_i) y_{n+1} \in C.$$

Thus $x + N \subset C$ and $x \in \operatorname{int} C$. □

Lemma 3.4. *Suppose that $x \in$ WEff P_J and that for every $j \in J$ there exists y_j such that*

$$f_i(y_j) < f_i(x), \quad i \in J \backslash \{j\}, \tag{5}$$

then

$$x \in \operatorname{int} C \text{ where } C = \operatorname{conv}\{y_j : j \in J\}\cdot$$

Proof. For each $j \in J$, consider the closed, convex subset of X,

$$C_j = \operatorname{conv}\{y_k : k \in J \backslash \{j\}\}.$$

Obviously, by (5), $f_j(y_k) < f_j(x)$ for $k \in J \backslash \{j\}$ and the quasiconvexity of f_j implies

$$\forall y \in C_j \qquad f_j(y) < f_j(x). \tag{6}$$

From (6), for all $j \in J$, $x \notin C_j$ and since $x \in$ *WEff* P_J we have $\cap\{C_j : j \in J\} = \emptyset$.

Now let us define $C'_j = \operatorname{conv}(C_j, \{x\})$ for $j \in J$.

The intersection of $n+1$ of the sets $\{(C'_j)_{j \in J}, C\}$ is nonempty. Indeed, all C'_j contain x and if we take a collection of the form $\{(C'_j)_{j \in J \backslash \{j_0\}}, C\}$, all of these sets contain y_{j_0}.

Applying Helly's Theorem, there exists $z \in \left(\bigcap_{j \in J} C'_j \right) \bigcap C$. If $z \neq x$, then for each $j \in J$, $z = \lambda_j x + (1 - \lambda_j) y$, with $\lambda_j \in [0, 1[$, $y \in C_j$. From (6) and the strict quasiconvexity of f_j, for each $j \in J$, $f_j(z) < f_j(x)$ which contradicts $x \in$ *WEff* P_J. Thus $z = x$ and $x \in C$. Thus we can write $x = \sum_{j \in J} \lambda_j y_j$, $\lambda_j \geq 0$, $\quad \sum_{j \in J} \lambda_j = 1$.

Suppose that $\lambda_{j_0} = 0$ for some $j_0 \in J$. By definition of y_j, $f_{j_0}(y_j) < f_{j_0}(x)$, for all $j \in J \backslash \{j_0\}$ and f_{j_0} being quasiconvex, $f_{j_0}(x) \leq \max(f_{j_0}(y_j), j \in J\backslash\{j_0\}) < f_{j_0}(x)$, a contradiction. Thus every λ_j is strictly positive and using Lemma 3.3 we conclude $x \in \operatorname{int} C$. □

It is well known that the continuity of f_i's implies that *WEff* P is a closed set. Let us denote by bd(*WEff* P) the boundary of *WEff* P.

Theorem 3.5. *Suppose that f_i, $i = 1, \cdots, k$ are strictly quasiconvex and continuous. Then*

$$\operatorname{bd}(\mathit{WEff}\, P) \subset \cup\{\mathit{Eff}\, P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset, |I| \leq n\}.$$

Proof. By Theorem 3.1 it is sufficient to consider the case $k > n$.

Suppose that $x \in \operatorname{bd}(\mathit{WEff}\ P) \setminus \cup\{\mathit{Eff}\ P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset, |I| \leq n\}$. Using Theorem 3.2 we have $x \in \mathit{Eff}\, P_J$ for some $J \subset \{1, \cdots, k\}$, with $|J| = n+1$ and $x \notin \mathit{Eff}\, P_{J'}$ for all $J' \subset J, J' \neq \emptyset, J' \neq J$. In particular for every $j \in J$, $x \notin \cup\{\mathit{Eff}\ P_{J'} : J' \subset J\backslash\{j\}, J' \neq \emptyset\}$. But from Theorem 3.1 , this latter set is $\mathit{WEff}\, P_{J\backslash\{j\}}$. Thus

$$\forall j \in J \quad \exists y_j \in X \quad f_i(y_j) < f_i(x), i \in J\backslash\{j\}.$$

It follows from Lemma 3.4 that $x \in \operatorname{int}\ C$, where $C = \operatorname{conv}\{y_j : j \in J\}$. From Lemma 3.3, each $y \in \operatorname{bd}\ C$ can be written as a convex combination of only n points y_j, say $\{y_j : j \in J\backslash\{i\}\}$ and by the quasiconvexity of $f_i, f_i(y) < f_i(x)$. It follows that

$$\forall y \in \operatorname{bd}\ C \quad \max_{j\in J}(f_j(x) - f_j(y)) > 0\cdot \tag{7}$$

As $\max_{j\in J}(f_j(x) - f_j(y))$ is continuous with respect to y, it achieves its minimum on the compact set bd C.

Denote by $m = \min_{y\in \operatorname{bd}\ C} \quad \max_{j\in J}(f_j(x) - f_j(y))$, we have $m > 0$.

Now consider the neighborhood of x in $\mathbb{R}^n$ defined by

$$V = \{y \in C : \max_{j\in J}(f_j(x) - f_j(y)) < m\}\cdot$$

Note that by the definition of m, $V \cap \operatorname{bd}\ C = \emptyset$ and then $V \subset \operatorname{int} C$. We show that $V \subset \mathit{Eff}\, P_J$.

Suppose on the contrary that $y \in V\backslash \mathit{Eff}\, P_J$. Then

$$\exists u \in X \quad (f_i(y) - f_i(u))_{i\in J} \in \mathbb{R}^n_+\backslash\{0\}\cdot \tag{8}$$

As $y \in \operatorname{int} C$ and C being a compact set, we can find $t_0 = \max\{t > 0 : y + t(y - u) \in C\}$. We have $z = y + t_0(y - u) \in \operatorname{bd}\ C$ and by (7), there exists $i_0 \in J$ such that

$$f_{i_0}(x) - f_{i_0}(z) \geq m > 0\cdot \tag{9}$$

As $y \in V$ we have also $f_{i_0}(x) - f_{i_0}(y) < m$ and with (9)

$$f_{i_0}(y) - f_{i_0}(z) > 0\cdot \tag{10}$$

Now using the fact that y is a convex combination of u and z, and the strict quasiconvexity of f_{i_0}, we get either $f_{i_0}(u) = f_{i_0}(z) \geq f_{i_0}(y)$, which contradicts (10), or

$$f_{i_0}(u) \neq f_{i_0}(z) \text{ and } f_{i_0}(y) < \max(f_{i_0}(u), f_{i_0}(z)). \tag{11}$$

But $f_{i_0}(y) > f_{i_0}(z)$ by (10) and then (11) entails $f_{i_0}(y) < f_{i_0}(u)$, a contradiction with (8).

Thus we have proved that $V \subset \mathit{Eff}\ P_J$, which means that $x \in \mathrm{int}(\mathit{Eff}\ P_J)$. As $\mathit{Eff}\ P_J \subset \mathit{WEff}\ P_J$, we have got a contradiction with the assumption $x \in \mathrm{bd}(\mathit{WEff}\ P_J)$ and the proof is complete. □

In the following we adopt the notations of Ward [7] :

$$r[x, v] = \{x + tv : t \geq 0\} \quad x \in X, v \in \mathbb{R}^n,$$
$$U(P, n) = \cup\{\mathit{Eff}\ P_I : I \subset \{1, \cdots, k\}, I \neq \emptyset, |I| \leq n\}$$
$$S(P, n) = \{x \in X \backslash U(P, n) : \forall v \neq 0,\ r[x, v] \cap U(P, n) \neq \emptyset\}.$$

The result of (lemma 4, [7]) remains true in the strictly quasiconvex case.

Lemma 3.6. *Suppose that f_i, $i = 1, \cdots, k$ are strictly quasiconvex, then $S(P, n) \subset \mathit{Eff}\ P$.*

Proof. Assume that $x \in S(P, n) \backslash \mathit{Eff}\ P$. Then, there exists $y \in X$ such that

$$(f_i(x) - f_i(y))_{i=1,\cdots,k} \in \mathbb{R}^k_+ \backslash \{0\}$$

and we can find $z \in r[x, x - y] \cap U(P, n)$. As $x \neq z$ and $x \neq y$, $x = \lambda z + (1 - \lambda)y$ with $\lambda \in]0, 1[$. If $f_i(y) = f_i(z)$ then the strict quasiconvexity of f_i implies $f_i(x) \leq f_i(z)$. If $f_i(y) \neq f_i(z)$ the strict quasiconvexity of f_i implies $f_i(x) < max(f_i(y), f_i(z))$ and with $f_i(x) \geq f_i(y)$ one has $f_i(x) < f_i(z)$. Therefore, in all cases, $f_i(x) \leq f_i(z), i = 1, \cdots, k$. As $z \in \mathit{Eff}\ P_I$ for some I such that $|I| \leq n$, we get also $x \in \mathit{Eff}\ P_I$, contradicting $x \notin U(P, n)$. □

Theorem 3.7. *Let f_i, $i = 1, \cdots, k$ be strictly quasiconvex, continuous functions and suppose that $\mathit{WEff}\ P$ is bounded, then $\mathit{WEff}\ P = U(P, n) \cup S(P, n)$.*

Proof. The inclusion $\supset$ is clear from Lemma 3.6 and Theorem 3.2.

For $\subset$ observe that $\mathit{WEff}\ P = \mathrm{bd}(\mathit{WEff}\ P) \cup \mathrm{int}(\mathit{WEff}\ P)$. From Theorem 3.5 follows $\mathrm{bd}(\mathit{WEff}\ P) \subset U(P, n)$.

Now we prove that $\mathrm{int}(\mathit{WEff}\ P) \backslash U(P, n) \subset S(P, n)$.

Consider $x \in \mathrm{int}(\mathit{WEff}\ P) \backslash U(P, n)$ and an halfline $r[x, v]$. Since $\mathit{WEff}\ P$ is bounded, there exists $z \in r[x, v] \cap \mathrm{bd}(\mathit{WEff}(P))$, $z \neq x$. By Theorem 3.5, $z \in r[x, v] \cap U(P, n)$ and then $x \in S(P, n)$. □

4. Conclusion

The subset $S(P,n)$ is a kind of convex hull of $U(P,n)$ and the knowledge of $U(P,n)$ completely determines $S(P,n)$. Thus it is sufficient to solve subproblems P_I with at most n criteria, to obtain *WEff* P. In $\mathbb{R}^2$ a graphical representation of *WEff* P can be obtained, even with a great number of criteria, as soon as bicriteria subproblems can be solved.

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