Journal of Convex Analysis
Volume 1 (1994), No. 2, 151–172

On Lower Semicontinuity in BV Setting

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Received 25 January 1994
Revised manuscript received 23 September 1994

Dedicated to the memory of Professor Lamberto Cesari
with deepest affection and gratitude

1. Introduction

We discuss here the lower semicontinuity for integral functionals of the type

$$I[u,v] = \int_G F(x, u(x), v(x))\, dx \tag{1.1}$$

with respect to a sequence $(u_k, v_k)_{k\geq 0}$ subjected to the constraints

$$(x, u_k(x)) \in A, \quad v_k(x) \in Q(x, u_k(x)) \qquad \text{a.e. in } G, \quad k \in \mathbb{N} \tag{1.2}$$

where $G \subset \mathbb{R}^\nu$ is a bounded open set and $Q : A \times \mathbb{R}^m \to 2^{\mathbb{R}^N}$, $A \subset \mathbb{R}^{\nu+n}$, is a given multifunction.

A great deal of research was devoted to this subject for the topology of L_p-convergence of $(u_k)_{k\geq 0}$ and weak L_q-convergence of $(v_k)_{k\geq 0}$. We only mention Cesari [9e] and for the free case (i.e. $Q(t,x) = \mathbb{R}^N$, $(t,x) \in A$) we refer to [2,5,7,8,9e,12,13,15,16,17,18,23] where also a list of references can be found.

We are interested here in a result which involves a weaker topology; more precisely, it should not require any additional convergence assumption on the differential elements of the highest order when applied to the functionals of the calculus of variations. The reason is that this kind of theorems fit in optimization problems where BV (not necessarily continuous) solutions are expected, since the compactness results on BV do not involve the weak convergence of the gradients, as it occurs in Sobolev's spaces.

To this purpose we introduce here the mean value (mv) condition (see Section 2). Roughly speaking a sequence $(v_k)_{k\geq 0}$ of summable functions satisfies (mv) provided for a.e. $t_0 \in G$

$$\lim_{h\to 0} \lim_{k\to +\infty} \fint_{B(t_0,h)} [v_k(t) - v_0(t)]\, dt = 0.$$

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It is easy to see that (mv) is weaker than L_1-convergence, but the main point is that (mv) is the proper assumption for our setting. In fact (see Proposition 3.7) if $(x_k)_{k\in\mathbb{N}}$ is a sequence in $W^{1,1}(G,\mathbb{R}^n)$ which L_1–converges to a BV function x_0, then we get that a subsequence of the gradients $(\mathcal{D}x_k)_{k\geq 0}$ satisfies (mv). Here $\mathcal{D}x$ denotes the "essential gradient" of the BV function x, i.e. the density of the absolutely continuous part of the distributional derivative with respect to the Lebesgue measure.

The lower semicontinuity results we present here are obtained as a consequence of a closure theorem for an orientor field of type (1.2) with respect to the convergence

$$u_k \ \mathrm{L}_1\text{–converges to } u_0 \qquad \text{and} \qquad (v_k)_{k\in\mathbb{N}} \text{ satisfies (mv).} \tag{1.3}$$

Our main lower semicontinuity theorem is the following (see Theorem 5.1)

Main result. *Assume that* A *is closed and* F *is non negative.*
Let $(u_k, v_k)_{k\geq 0}$ *be a sequence of summable functions such that*

i) $(t, u_k(t)) \in A, \quad v_k(t) \in Q(t, u_k(t)) \qquad$ *a.e. in* $G, \quad k \in \mathbb{N}$;

ii) u_k L_1–*converges to* u_0 *and* $(v_k)_{k\in\mathbb{N}}$ *satisfies* (mv).

iii) *Suppose that the multifunction*

$$\widetilde{Q}_0(t,x) = \{(y^0, y): \ y_0 \geq F(t,x,y), \ y \in Q(t,x)\}$$

satisfies property (Q) *and* ($\mathrm{w}\widetilde{\mathrm{F}}$) *at the point* $(t_0, u_0(t_0))$, *for a.e.* $t_0 \in G$.

Then $(t, u_0(t)) \in A, \quad v_0(t) \in Q(t, u_0(t)) \qquad$ *a.e. in* G *and*

$$\liminf_{k\to+\infty} I[u_k, v_k] \geq I[u_0, v_0].$$

Property (Q), introduced by Cesari in 1966 (see [9e]), is an intermediate condition between upper semicontinuity and Kuratowski property. As we recall in Section 6, it is a seminormality assumption on the integrand and hence implies that $F(t,x,\cdot)$ is convex.

Condition ($\mathrm{w}\widetilde{\mathrm{F}}$) acts on the second variable, it was introduced in [11c] as a variant of Cesari's Lipschitz condition (F). Actually, we prove here (see Section 6) that ($\mathrm{w}\widetilde{\mathrm{F}}$) is really a weakening of assumption (F).

Some noteworthy particular cases of our functionals are the following

$$I[x] = \int_G F(t, (\mathcal{U}x)(t), (\mathcal{L}x)(t))\, dt \tag{1.4}$$

where $\mathcal{U}$ and $\mathcal{L}$ are given operators not necessarily linear. For example

$$\mathcal{L}x = \mathcal{D}x, \quad \text{or} \quad \mathcal{L}x = \operatorname{div} x \quad \text{or} \quad \mathcal{L}x = \mathcal{D}[\Psi(\cdot, x(\cdot))]$$

with Ψ a Lipschitzian function.

For the details and further examples we refer to Section 7.

We wish to recall that the lower semicontinuity for functionals of type (1.4), with respect to the weak topology in Sobolev's spaces, was studied by E.Rothe [23], G.Fichera [18], Cesari [9c,d] and Cesari-D.E.Cowles [12].

A particular class of integrals to which our theory apply, is the following

$$I[u] = \int_G |< a(t), D[\Psi(t, u(t))] > + \Phi(t, u(t))| \, dt$$

where a is continuous, Ψ is Lipschitzian and Φ is a Carathéodory function such that $|\Phi(t, x)| \leq \phi(t)$, with ϕ summable.

They take their source from problems of conservation laws. In order to prove existence results for the relaxed functional on BV, Cesari [5d] adopted a suitable tranformation which allows to reduce the functional I to standard integrals of the calculus of variations.

Now, in force of the present formulation, where abstract operators are involved, we can deal with functional I directly with remarkable advantages both in the assumptions and in the proofs.

Note that also functionals of the type

$$I[u] = \int_G [< a(t), D[\Psi(t, u(t))] > + \Phi(t, u(t))]^+ \, dt$$

can be handled in the same way.

Finally, we wish to mention that our research is partially motivated by a study of a variational model for the plastic deformation of beams and plates, where functionals of type (1.4) and BV solutions are involved ([11e,14,3a,b,4].)

2. Preliminaries

Let ν, m and p be given integers. Let $G \subset \mathbb{R}^\nu$ be a bounded open set.

According to standard notations, we denote by $L_1(G, \mathbb{R}^m)$ the space of summable functions $x : G \to \mathbb{R}^m$, by $W^{p,1}(G, \mathbb{R}^m)$ the Sobolev space of the functions $x \in L_1(G, \mathbb{R}^m)$ whose distributional derivatives up to the order p are summable functions and by $BV(G, \mathbb{R}^m)$ the space of the functions $x \in L_1(G, \mathbb{R}^m)$ which are of bounded variation in the sense of Cesari [9a].

For $m = 1$ we will briefly write L_1, $W^{p,1}$ and BV, respectively.

A BV function x admits an "essential gradient" [26], i.e. has a.e. partial derivatives $\frac{\partial x^i}{\partial t_j}$ computed by usual incremental quotients disregarding the values taken by x on a suitable null set. Moreover the "essential gradient" coincides with the density of the absolutely continuous part of the distributional derivative with respect to the Lebesgue measure. We denote by $\mathcal{D}x = \left(\frac{\partial x^i}{\partial t_j},\ i = 1, ..., m,\ j = 1, ..., \nu\right)$ and call $\mathcal{D}x$ the gradient of x.

Given a point $t_0 \in G$ and a constant $h > 0$, we put

$$q_h(t_0) = [t_0 - h, t_0 + h] = \{t \in \mathbb{R}^\nu : \ t_{0j} - h \leq t_{0j} \leq t_{0j} + h,\ j = 1, ..., \nu\}$$

in the case the point t_0 is clearly determined, we briefly write $q_h(t_0) = q_h$.
Moreover, we denote by $|q_h|$ the area of the interval.
Given a function $x \in L_1(G, \mathbb{R}^m)$, for every $t_0 \in G$ and $h > 0$ sufficiently small, we put

$$⨍_{q_h} x(t)\, dt = |q_h|^{-1} \int_{q_h} x(t)\, dt.$$

Given a function $x^0 : R_0 \to \mathbb{R}$, where R_0 is a closed interval, for every subinterval $R = [a, b] = \{t \in \mathbb{R}^\nu : \ a_j \leq t_j \leq b_j, \ j = 1, ..., \nu\}$, we consider the difference of order ν relative to the 2^ν vertices of R, say

$$\Delta_R x^0 = x^0(b) - x^0(a) \qquad \text{if} \quad \nu = 1$$

$$\Delta_R x^0 = x^0(b_1, b_2) - x^0(b_1, a_2) - x^0(a_1, b_2) + x^0(a_1, a_2) \qquad \text{if} \quad \nu = 2$$

and so on.
The function x^0 is said to be of bounded variation in the sense of Vitali (VBV) [25] provided the interval function $\Delta_R x^0$ has bounded variation. A VBV function has a.e. superficial derivatives, say $\mathcal{D}^* x^0(t_0) = \lim_{h\to 0} (2h)^{-\nu} \Delta_{q_h} x^0(t_0)$ and $\mathcal{D}^* x^0$ is a summable function.

The function x^0 is said to be absolutely continuous in the sense of Vitali (VAC) [25] if the interval function $\Delta_R x^0$ is absolutely continuous; in this case we have $\Delta_R x^0 = \int_R \mathcal{D}^* x^0(t)\ dt$.

3. The mean-value condition

We introduce the following definition for a sequence $(v_k)_{k\geq 0}$ in $L_1(G, \mathbb{R}^m)$.

Definition 3.1. We say that $(v_k)_{k\geq 0}$ satisfies the *mean value* (mv) *condition* at a point $t_0 \in G$ provided

(mv) there exists a null set $H = H(t_0) \subset \mathbb{R}^+$ such that, for every number $\varepsilon > 0$ a constant $0 < h_0 = h_0(t_0, \varepsilon)$ can be determined in such a way that, for every $h \in]0, h_0[-H$, an integer $k_0 = k_0(t_0, \varepsilon, h)$ exists such that for every $k \geq k_0$

$$\left| ⨍_{q_h} v_k(t)\ dt - v_0(t_0) \right| < \varepsilon.$$

We say that $(v_k)_{k\geq 0}$ satisfies (mv) *condition* on G if (mv) holds at a.e. point $t_0 \in G$.

Let us observe that (mv) can be written:

(mv) there exists a null set $H = H(t_0) \subset \mathbb{R}^+$ such that for $i = 1, ..., m$

$$\lim_{h\to 0, h\notin H} \liminf_{k\to\infty} ⨍_{q_h} v_k^i(t) dt = \lim_{h\to 0, h\notin H} \limsup_{k\to\infty} ⨍_{q_h} v_k^i(t) dt = v_0^i(t_0).$$

The following criterions for (mv) condition can be easily proved.

Proposition 3.2. *Let* $(v_k)_{k\geq 0}$ *and* $(w_k)_{k\geq 0}$ *be two sequences which satisfy* (mv) *at* $t_0 \in G$, *then for every* $\alpha, \beta \in \mathbb{R}$ *the sequence* $(\alpha v_k + \beta w_k)_{k\geq 0}$ *satisfies* (mv) *at the same point.*

Proposition 3.3. *Let* $(v_k)_{k\geq 0}$ *be a sequence in* $L_1(G, \mathbb{R}^m)$ *and assume that*

i) *for a.e.* $t_0 \in G$ *there exists a constant* $h_0 = h_0(t_0) > 0$ *such that for a.e.* $h \in]0, h_0[$

$$\lim_{k\to+\infty} \int_{q_h} v_k(t)\ dt = \int_{q_h} v_0(t)\ dt.$$

Then $(v_k)_{k\geq 0}$ *satisfies* (mv) *on* G.

As an immediate consequence of Proposition 3.3 we also have

Corollary 3.4. *If* $v_k \rightharpoonup v_0$ *weakly in* $L_1(G, \mathbb{R}^m)$, *then the sequence* $(v_k)_{k\geq 0}$ *satisfies* (mv) *on* G.

Remark 3.5. On the converse, note that (mv) condition does not imply condition i) in Proposition 3.3 and, a fortiori, weak convergence in $L_1(G, \mathbb{R}^m)$. To this purpose, let us consider a sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,1}(G, \mathbb{R}^m)$ which L_1–converges to a function $u_0 \in BV(G, \mathbb{R}^m)$ and let $v_k = \mathcal{D}u_k$, $k \geq 0$, be the sequence of the gradients. Of course $(\mathcal{D}u_k)_{k\geq 0}$ does not satisfy condition i) since, otherwise the limit function would still belong to $W^{1,1}(G, \mathbb{R}^m)$. But, by virtue of Proposition 3.7 below, there exists a subsequence of the gradients which satisfies (mv) on G.

Still as a consequence of Proposition 3.3 we have

Corollary 3.6. *Let* $(u_k)_{k\geq 0}$ *be a sequence which converges in* $L_1(G, \mathbb{R}^m)$ *and let* $\Phi : A \to \mathbb{R}$ *be a Carathéodory function such that*

$$|\Phi(t, u_k(t))| \leq \phi(t), \qquad \text{a.e. in } G, \quad k \in \mathbb{N}$$

with $\phi \in L_1$.

Then the sequence $(\ \Phi(\cdot, u_k(\cdot))\ )_{k\geq 0}$ *satisfies* (mv) G.

Moreover, note that Lemma 2 and 6 in [11d] can be written in terms of (mv) condition as it follows.

Proposition 3.7. *Let* $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,1}(G, \mathbb{R}^n)$ *which* L_1*–converges to a function* $u_0 \in BV(G, \mathbb{R}^n)$.

Then there exists a subsequence of the gradients $(\mathcal{D}u_{s_k})_{k\geq 0}$ *which satisfies* (mv) *on* G.

Proposition 3.8. *Let* $(u_k^0)_{k\in\mathbb{N}}$ *be a sequence of VAC functions which converges pointwise a.e. in* R_0 *to a VBV function* u_0.

Then the superficial derivatives $(\mathcal{D}^* u_k)_{k\geq 0}$ *satisfies* (mv) *on* R_0.

Remark 3.9. Note that for $\nu = 1$ and $R_0 = [a, b]$ the concepts of VAC and VBV reduce to $W^{1,1}$ and BV respectively. Thus, by virtue of Proposition 3.8 the result of Proposition 3.7 improves as follows:

if $u_k : [a, b] \to \mathbb{R}^m$, $k \in \mathbb{N}$, *is a sequence of curves in* $W^{1,1}$ *which converges pointwise a.e. to a curve* $u_0 \in BV$, *then the derivatives* $(u_k')_{k \geq 0}$ *satisfies* (mv) *on* $[a, b]$.

As an application of Proposition 3.7 we can also prove the following criterion.

Proposition 3.10. *Let* $(u_k)_{k \in \mathbb{N}}$ *be a sequence in* $W^{1,1}(G, \mathbb{R}^n)$ *which* L_1*–converges to a function* $u_0 \in BV(G, \mathbb{R}^n)$.

Assume that $A \subset \mathbb{R}^{\nu+n}$ *is a set whose projection in the t-space* $\mathbb{R}^\nu$ *contains* G *and* $\Psi : A \to \mathbb{R}^m$ *is a Lipschitzian function.*

Then there exists a subsequence of the gradients

$$(\ D[\Psi(\cdot, u_{s_k}(\cdot))] \ )_{k \geq 0}$$

which satisfies (mv) *on* G.

Proof. Note that the sequence $\Psi_k : G \to \mathbb{R}^m$ defined by $\Psi_k(t) = \Psi(t, u_k(t))$, $k \geq 0$, satisfies the assumptions of Proposition 3.7. $\square$

4. A closure result

Let $A \subset \mathbb{R}^{\nu+n}$, with $n \in \mathbb{N}$, be a set whose projection in the t-space $\mathbb{R}^\nu$ contains G. Let $Q : A \to 2^{\mathbb{R}^m}$ be a multifunction with nonempty values and let us consider the multivalued equation

$$(t, u(t)) \in A, \qquad v(t) \in Q(t, u(t)), \qquad \text{a.e. in } G. \tag{4.1}$$

We denote by S_Q the set of the measurable solutions of (4.1) and also use the notation $S_Q^1 = S_Q \cap L_1(G, \mathbb{R}^{n+m})$.

We recall that multifunction Q is said to satisfy Cesari's *property* (Q) at a point $(t_0, x_0) \in A$, provided [9e]

$$Q(t_0, x_0) = \bigcap_{\sigma > 0} \text{cl co } \{\cup \ Q(t, x), \ |t - t_0| \leq \sigma, \ |x - x_0| \leq \sigma\}. \tag{Q}$$

Note that if (Q) holds, then the set $Q(t_0, x_0)$ is necessarely closed and convex.

Moreover, in [11c] (see also [11d]) the following condition on multifunction Q was introduced.

Let $\mathcal{S} = (u_k, v_k)_{k \in \mathbb{N}}$ be a given sequence in S_Q^1, we say that Q satisfies *condition* (wF) *at a point* $(t_0, x_0) \in A$, *with respect to the sequence* $\mathcal{S}$, provided (wF) given any number $\varepsilon > 0$, there exist two numbers $0 < \sigma = \sigma(t_0, x_0, \mathcal{S}, \varepsilon) \leq \varepsilon$ and $0 < h_0 = h_0(t_0, x_0, \mathcal{S}, \varepsilon) \leq \sigma$ such that for a.e. $0 < h < h_0$ there exist a subsequence $(s_k)_{k\in\mathbb{N}}$ and a sequence $(\overline{u}_{s_k}, w_{s_k})_{k\in\mathbb{N}}$ in S_Q with the property that for every $k \in \mathbb{N}$

$$|\overline{u}_{s_k}(t) - x_0| \leq \sigma \qquad \text{a.e. in } q_h \qquad \text{and} \qquad \left| \fint_{q_h} [w_{s_k}(t) - v_{s_k}(t)]\, dt \right| \leq \varepsilon.$$

Besides equation (4.1), also the following type of multivalued equations are involved in problems of the calculus of variations (see [9e])

$$(t, u(t)) \in A, \quad (v^0(t), v(t)) \in \widetilde{Q}(t, u(t)), \qquad \text{a.e. in } G. \tag{4.$\widetilde{1}$}$$

where $\widetilde{Q} : A \to 2^{\mathbb{R}^{m+1}}$ has the following property:

$$\text{if} \quad (y^0, y) \in \widetilde{Q}(t, x) \quad \text{and} \quad y' > y^0, \quad \text{then} \quad (y', y) \in \widetilde{Q}(t, x).$$

Let $S_{\widetilde{Q}}$ denote the set of the solutions of (4.$\widetilde{1}$) and again we put $S^1_{\widetilde{Q}} = S_{\widetilde{Q}} \cap L_1(G, \mathbb{R}^{n+1+m})$.

Note that condition (wF) modifies as follows (see [11c,d]).

Given a sequence $\mathcal{S} = (u_k, v^0_k, v_k)_{k\in\mathbb{N}}$ in $S^1_{\widetilde{Q}}$, we shall say that $\widetilde{Q}$ satisfies *condition* (w$\widetilde{\text{F}}$) *at a point* $(t_0, x_0) \in A$, *with respect to the sequence* $\mathcal{S}$, provided

(w$\widetilde{\text{F}}$) given any number $\varepsilon > 0$, there exist two numbers $0 < \sigma = \sigma(t_0, x_0, \mathcal{S}, \varepsilon) \leq \varepsilon$ and $0 < h_0 = h_0(t_0, x_0, \mathcal{S}, \varepsilon) \leq \sigma$ such that for a.e. $0 < h < h_0$ there exist a subsequence $(s_k)_{k\in\mathbb{N}}$ and a sequence $(\overline{u}_{s_k}, w^0_{s_k}, w_{s_k})_{k\in\mathbb{N}}$ in $S_{\widetilde{Q}}$ with the property that for every $k \in \mathbb{N}$

$$|\overline{u}_{s_k}(t) - x_0| \leq \sigma \qquad \text{a.e. in } q_h \qquad \text{and}$$

$$\left| \fint_{q_h} [w_{s_k}(t) - v_{s_k}(t)]\, dt \right| \leq \varepsilon \qquad\qquad \fint_{q_h} [w^0_{s_k}(t) - v^0_{s_k}(t)]\, dt \leq \varepsilon.$$

We are ready to state the following closure result which can be proved by the same technique adopted for Theorem 3 in [11d].

Theorem 4.1. *(A closure result). Assume that* A *is closed.*

Let $\widetilde{Q} : A \to 2^{\mathbb{R}^{m+1}}$ *be a multifunction with nonempty values and let* $(u_k, v^0_k, v_k)_{k\geq 0}$ *be a given sequence.*

Suppose that

i) $(u_k, v^0_k, v_k) \in \mathcal{S}^1_{\widetilde{Q}}, \quad k \in \mathbb{N};$

ii) u_k L_1 *–converges to* u_0*;*

iii) $(v_k^0, v_k)_{k\geq 0}$ *satisfies* (mv) *on* G;

iv) *for a.e.* $t_0 \in G$, *multifunction* $\widetilde{Q}$ *has properties* (Q) *and* (w$\widetilde{\mathrm{F}}$), *with respect to the sequence* $(u_k, v_k^0, v_k)_{k\in\mathbb{N}}$, *at the point* $(t_0, u_0(t_0))$

v) *if* $(y^0, y) \in \widetilde{Q}(t,x)$ *and* $y' > y^0$, *then* $(y', y) \in \widetilde{Q}(t,x)$.

Then the limit function (u_0, v_0^0, v_0) *belongs to* $\mathcal{S}^1_{\widetilde{Q}}$.

Remark 4.2. Theorem 4.1 also holds in the case $A = G \times A_0$ with $A_0 \subset \mathbb{R}^n$ closed set.

Note that, in the particular case that $A = G \times A_0$ and multifunction $\widetilde{Q} : G \to 2^{\mathbb{R}^{m+1}}$ depends only on variable t, assumption (w$\widetilde{\mathrm{F}}$) is trivially satisfied.

Remark 4.3. As a particular case of Theorem 4.1 a closure result can be proved for equation (4.1), where assumption v) is obviously omitted and condition (w$\widetilde{\mathrm{F}}$) is replaced by (wF).

Again, in the particular case that $A = G\times A_0$ and multifunction $Q : G \to \mathbb{R}^m$ depends only on variable t, assumption (wF) is trivially satisfied.

Moreover, (wF) holds in the case $Q(t,x) = \mathbb{R}^n$, $\quad (t,x) \in A = G \times A_0$.

For these and other conditions assuring assumptions (wF) and (w$\widetilde{\mathrm{F}}$) we refer to [11c,d] and Section 6.

5. Application to the lower semicontinuity of integral functionals

Let $M = \{(t,x,y) : \ y \in Q(t,x), \ (t,x) \in A\}$ denote the graph of multifunction Q.

We consider a function $F : M \to \mathbb{R}$ such that, for every $(u,v) \in \mathcal{S}^1_Q$, $\quad F(\cdot, u(\cdot), v(\cdot))$ is measurable and $F^-(\cdot, u(\cdot), v(\cdot)) \in L_1$.

Let $I : \mathcal{S}^1_Q \longrightarrow \mathbb{R}\cup(+\infty)$ be the functional defined by

$$\begin{aligned} &I(u,v) = \int_G F(t,u(t),v(t))\,dt \qquad \text{if} \quad F^+(\cdot,u(\cdot),v(\cdot)) \in L_1 \\ &I(u,v) = +\infty \qquad \text{elsewhere.} \end{aligned}$$

As an application of closure Theorem 4.1 we shall prove here some lower semicontinuity theorems for the functional I.

To this end, let us consider the multifunction $\widetilde{Q}_0 : A \to 2^{\mathbb{R}^{m+1}}$ defined by

$$\widetilde{Q}_0(t,x) = \{(y^0, y) : \ y_0 \geq F(t,x,y), \ y \in Q(t,x)\}. \tag{5.1}$$

Theorem 5.1. *(A lower semicontinuity result). Assume that A is closed.*

Let $(u_k, v_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{S}^1_Q$ and let (u_0, v_0) be a function in $L_1(G, \mathbb{R}^{n+m})$ such that

i) *u_k L_1 –converges to u_0;*

ii) *$(v_k)_{k\geq 0}$ satisfies* (mv) *on G.*

Suppose that

iii) *a function $\lambda \in L_1$ exists such that $F_k(t) = F(t, u_k(t), v_k(t)) \geq \lambda(t)$, a.e. in G, $k \geq 0$;*

iv) *for a.e. $t_0 \in G$ multifunction $\widetilde{Q}_0$ has properties* (Q) *and* (w$\widetilde{\mathrm{F}}$), *with respect to the sequence $(u_k, F_k, v_k)_{k\in\mathbb{N}}$, at the point $(t_0, u(t_0))$.*

Then the couple (u_0, v_0) lies in $\mathcal{S}^1_Q$ and we have

$$\liminf_{k\to+\infty} \int_G F(t, u_k(t), v_k(t))\ dt \geq \int_G F(t, u_0(t), v_0(t))\ dt.$$

Proof. Note that it is not restrictive to assume that $\liminf\limits_{k\to+\infty} I(u_k, v_k) = \lim\limits_{k\to+\infty} I(u_k, v_k) < +\infty$ and $\sup\limits_{k\in\mathbb{N}} I(u_k, v_k) = W < +\infty$.

Thus, the functions $F_k(\cdot) = F(\cdot, u_k(\cdot), v_k(\cdot))$, $k \in \mathbb{N}$ are summable.

Fixed an interval $R_0 = [a_0, b_0] \supset G$, let us extend F_k to R_0 by putting $F_k(t) = 0$, $t \in R_0 - G$, $k \in \mathbb{N}$.

Then let $\phi_k : R_0 \to \mathbb{R}$, $k \in \mathbb{N}$, be the sequence defined by

$$\phi_k(t) = \int_{[a_0,t]} F_k(\tau)\ d\tau.$$

Observe that $(\phi_k)_{k\in\mathbb{N}}$ is a sequence of VAC functions which have equi-bounded variation in the sense of Vitali: in fact, for every finite partition $D = [R]$ of the interval R_0, we have

$$\sum_{R\in D} |\Delta_R\ \phi_k| = \sum_{R\in D} \left| \int_R \phi_k(t)\ dt \right| \leq \int_{R_0} \phi_k(t)\ dt + 2\int_{R_0} |\lambda(t)|\ dt \leq W + 2\int_G |\lambda(t)|\ dt.$$

Thus, by virtue of Helly's theorem ([20], see also [11c], proof of Theorem 1"), a function $\phi_0 \in$ VBV exists such that (for a suitable subsequence)

$$\phi_k \longrightarrow \phi_0 \quad \text{pointwise in } R_0.$$

Taking into account of Proposition 3.8, we can deduce that the sequence of superficial derivatives

$$(\mathcal{D}^*\phi_k)_{k\in\mathbb{N}} \quad \text{satisfies (mv) on G.} \tag{5.2}$$

It is easy to see that

$$\mathcal{D}^*\phi_k(t) = F_k(t) \quad \text{a.e. in G,} \quad k \in \mathbb{N} \tag{5.3}$$

and moreover

$$(u_k, F_k, v_k) \in \mathcal{S}^1_{\widetilde{Q}_0}, \quad k \in \mathbb{N}. \tag{5.4}$$

From $(5.2) - (5.4)$ and the assumptions i), ii) and iv), by virtue of Theorem 4.1, we get that $(u_0, \mathcal{D}^* \phi_0, v_0) \in \mathcal{S}^1_{\widetilde{Q}_0}$ or equivalently

$$(u_0, v_0) \in \mathcal{S}^1_Q \qquad \text{and} \qquad \mathcal{D}^* \phi_0(t) \geq F(t, u_0(t), v_0(t)) \quad \text{a.e. in G.} \tag{5.5}$$

From assumption iii), we deduce that $\phi_0(t) \geq \hat{\lambda}(t)$, a.e. in R_0, where $\hat{\lambda}(t) = \lambda(t)$ on G, and $\hat{\lambda}(t) = 0$ on $R_0 - G$.

Let us consider the function $\Phi_0(t) = \phi_0(t) - \int_{[a_0,t]} \hat{\lambda}(\tau)\, d\tau$, $t \in R_0$; since the interval function $\Delta_R \Phi_0$ is additive and non-negative, we have ([22], Theorem III.1.28)

$$\Delta_{R_0} \Phi_0 = \phi_0(b) - \int_{R_0} \hat{\lambda}(t)\, dt \geq \int_{R_0} \mathcal{D}^* \Phi_0(t)\, dt \geq \int_G \mathcal{D}^* \phi_0(t)\, dt - \int_G \lambda(t)\, dt.$$

Finally, from (5.5), we deduce that

$$I(u_0, v_0) \leq \int_G \mathcal{D}^* \phi_0(t)\, dt \leq \phi_0(b) = \lim_{k \to +\infty} \phi_k(b) = \lim_{k \to +\infty} I(u_k, v_k)$$

which concludes the proof. □

We wish to remark that, in the case both multifunction Q and integrand F (and hence multifunction $\widetilde{Q}_0$) do not depend on variable x, then Theorem 5.1 reduces to the following result.

Theorem 5.2. *(A lower semicontinuity result). Assume that* $A = G \times A_0$ *with* A_0 *closed.*

Let $(v_k)_{k \in \mathbb{N}}$ *be a sequence in* $\mathcal{S}^1_Q$ *and let* v_0 *be a function in* $L_1(G, \mathbb{R}^m)$.

Suppose that

i) $(v_k)_{k \geq 0}$ *satisfies* (mv) *on* G.

ii) *a function* $\lambda \in L_1$ *exists such that* $F(t, v_k(t)) \geq \lambda(t)$, *a.e. in* G, $k \geq 0$;

iii) *multifunction* $\widetilde{Q}_0$ *has property* (Q) *a.e. in* G.

Then $v_0 \in \mathcal{S}^1_Q$ *and*

$$\liminf_{k \to +\infty} \int_G F(t, v_k(t))\, dt \geq \int_G F(t, v_0(t))\, dt.$$

Note that assumptions (Q) in Theorems 5.1 and 5.2 implies that multifunction Q is convex valued and the integrand F is convex in the last variable. For further remarks on the assumptions on the integrand, see Section 6.

Theorems 4.1 and 5.1 represent an extension of the closure and lower semicontinuity results in [11a,b,c,d] which are given in the particular case $v_k = \mathcal{D}u_k$, $k \geq 0$. To point out the interest of the present research we refer to Sections 7 and 8 where various new applications are presented.

Moreover, we wish to mention that the present paper is inspired by the results given in [9c,d,12] for control problems. Our theorems can be considered as a translation of those results in BV setting. Note that here a weaker topology is taken into consideration; more precisely, in [9c,d,12] weak-L_1 convergence of the differential elements is adopted, whereas the present theorems do not require any convergence assumptions on the differential elements of the highest order (see Section 7 for details).

6. Some remarks on assumptions (Q) and (w$\widetilde{\mathrm{F}}$)

In order to illustrate the operativity of our semicontinuity result, we shall present some conditions on the integrand F which assure that the assumptions on multifunction $\widetilde{Q}_0$ are satisfied. Let us start by recalling some well-known results (see [9e] for details).

Property (Q) is intermediate between Kuratowski condition (K) and upper semicontinuity. As we already observed in [11b,c], assumption (Q) can not be replaced by weaker condition (K) in the present setting.

We recall that if $\widetilde{Q}_0$ satisfies property (Q) at the point (t_0, x_0), then $\widetilde{Q}_0(t_0, x_0)$ is a closed and convex set.

Let us assume now that $Q(t,x) = \mathbb{R}^n$, $(t,x) \in A$, thus $\widetilde{Q}_0(t,x) = \text{epi } F(t,x,\cdot)$, $(t,x) \in A$. In this case $\widetilde{Q}_0(t_0,x_0)$ is closed and convex iff $F(t_0,x_0,\cdot)$ is lower semicontinuous and convex.

Moreover, the following result holds [9e].

Proposition 6.1. *Multifunction* $\widetilde{Q}_0(t,x) = \text{epi } F(t,x,\cdot)$, $(t,x) \in A$, *satisfies property* (Q) *at the point* $(t_0,x_0) \in A$ *iff* F *is seminormal at the same point, i.e.*

(s/n) *for every* $\varepsilon > 0$ *and* $y_0 \in \mathbb{R}^m$, *a constant* $\sigma = \sigma(t_0,x_0,y_0,\varepsilon) > 0$ *and an affine function* $z : \mathbb{R}^m \to \mathbb{R}$ *exist such that*

$F(t_0,x_0,y_0) < z(y_0) + \varepsilon$

$F(t,x,y) \geq z(y)$, *for every* $y \in \mathbb{R}^m$ *and* $(t,x) \in A$ *with* $|t-t_0| < \sigma$, $|x-x_0| < \sigma$.

For further conditions assuring property (Q) we refer to [9e], we only recall here the following criteria for seminormality (see [9b, 28, 27]).

For more recent results about seminormality, see [2].

Proposition 6.2. *Assume that* $F : A \times \mathbb{R}^m \to \mathbb{R}_0^+$ *is continuous and* $F(t,x,\cdot)$ *is convex,* $(t,x) \in A$. *Given a point* $(t_0,x_0) \in A$, *if there exists an affine function* $w : \mathbb{R}^m \to \mathbb{R}$ *such that*

$$F(t_0,x_0,y) \geq w(y), \quad y \in \mathbb{R}^m \quad \text{and} \quad \lim_{|y| \to +\infty} [F(t_0,x_0,y) - w(y)] = +\infty$$

then F *is seminormal at the point* (t_0,x_0).

Proposition 6.3. *Assume that* $F : A \times \mathbb{R}^m \to \mathbb{R}_0^+$ *is continuous and* $F(t,x,\cdot)$ *is convex,* $(t,x) \in A$. *Then the function* $F_\varepsilon : A \times \mathbb{R}^m \to \mathbb{R}_0^+$, $\varepsilon > 0$ *defined by*

$$F_\varepsilon(t,x,y) = F(t,x,y) + \varepsilon|y|$$

is seminormal at every point $(t_0, x_0) \in A$.

Proof. Note that F_ε satisfies the assumptions of Proposition 6.2 with $z(y) = 0, y \in \mathbb{R}^m$.

Condition $(\mathrm{w}\widetilde{\mathrm{F}})$ acts on the variable x; indeed, as we already observed, it is trivial provided $\widetilde{Q}_0$ does not depend on x.
It was introduced in [11d] as a weakening of conditions (F_i'), $i = 1, \ldots, 3$ in [11c]. They can be considered as variants of Cesari's Lipschitz-type property (F) ([9e]) which was proposed in Cesari-Suryanarayana [13] as an important tool in optimal control theory. The Authors also showed that these assumptions are rather natural, easy to verify and actually satisfied in many applications. See also Cesari-Angell [10].

In what follows we assume that $A = G \times A_0$, with A_0 closed, and $Q : G \to 2^{\mathbb{R}^n}$. Moreover, let $\mathcal{S} = (u_k, v_k)_{k\in\mathbb{N}}$ be a sequence in S_Q^1 and let $u_0 \in L_1(G, \mathbb{R}^n)$ be such that u_k L_1–converges to u_0.

The main consequence of condition (F) is the following result (see [13a])

Proposition 6.4. *Suppose that* F *satisfies conditions*

(C) *for every* $\varepsilon > 0$ *a compact set* $K_\varepsilon \subset G$ *exists such that* $\operatorname{meas}(G - K_\varepsilon) < \varepsilon$, *and* $F_{/K_\varepsilon \times A_0 \times \mathbb{R}^m}$ *is continuous;*

(F) *for a.e.* $t \in G$, *every* $u_1, u_2 \in A_0$ *and every* $k \in \mathbb{N}$,

$$|F(t, u_1, v_k(t)) - F(t, u_2, v_k(t))| \leq C\ \phi(|u_1 - u_2|)$$

where $C > 0$ *is a constant and* $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a non decreasing function such that* $\phi(0+) = 0$, $\phi(\zeta) \leq c|\zeta|$, $c \geq 0$, *for all* $\zeta \geq \zeta_0 \geq 0$.

Then

$$\lim_{k\to+\infty} \int_G |F(t, u_k(t), v_k(t)) - F(t, u_0(t), v_k(t))|\ dt = 0. \tag{$\mathcal{D}$}$$

We called our condition $(\mathrm{w}\widetilde{\mathrm{F}}) =$ weak (F) since we thought that it could be implied by assumption (F) even if we were not able to prove this result. Now we can finally justify this terminology (see Propositions 6.5 and 6.7 below).

Proposition 6.5. *If* F *satisfies conditions* (C) *and* (F), *then*

$(\mathcal{R})$ *for every* $\varepsilon > 0$, *every compact set* $K \subset G$ *with* $\operatorname{meas}(G - K) < \varepsilon$ *and a.e.* $t_0 \in K$

$$\limsup_{h\to 0}\ \limsup_{k\to+\infty}\ |q_h|^{-1} \int_{q_h - K} [F(t, u_0(t_0), v_k(t)) - F(t, u_k(t), v_k(t))]\ dt \leq 0.$$

Proof. By virtue of Proposition 6.4

$$\delta_k(\cdot) = F(\cdot, u_0(\cdot), v_k(\cdot)) - F(\cdot, u_k(\cdot), v_k(\cdot)) \longrightarrow 0 \qquad \text{in } L_1$$

then we deduce that (see also Proposition 3.3), for a.e. $t_0 \in G$

$$\lim_{h\to 0} \lim_{k\to +\infty} ⨍_{q_h} |\delta_k(t)|\, dt = 0. \tag{6.1}$$

Let $\varepsilon > 0$ be fixed and let $K \subset G$ be a compact set with $\text{meas}\,(G - K) < \varepsilon$. By virtue of well-known density results [19], for a.e. $t_0 \in K$

$$\lim_{h\to 0} ⨍_{q_h} |u_0(t) - u_0(t_0)|\, dt = 0 \qquad \lim_{h\to 0} \frac{\text{meas}\,(q_h - K)}{|q_h|} = 0. \tag{6.2}$$

Let $t_0 \in K$ be fixed in such a way that assumption (F) and (6.2) hold. Denoted by $E = \{t \in G : \ |u_0(t_0) - u_0(t)| \geq \zeta_0\}$, from (F) we have

$$\begin{aligned}
&|q_h|^{-1} \int_{q_h - K} [F(t, u_0(t_0), v_k(t)) - F(t, u_0(t), v_k(t))]\, dt \leq \\
\leq &\frac{C}{|q_h|} \int_{q_h - K} \phi(|u_0(t_0) - u_0(t)|)\, dt \leq \\
\leq &\frac{Cc}{|q_h|} \int_{(q_h - K)\cap E} |u_0(t_0) - u_0(t)|\, dt + \frac{C}{|q_h|} \int_{(q_h - K) - E} \phi(\zeta_0) \leq \\
\leq &Cc ⨍_{q_h} |u_0(t_0) - u_0(t)|\, dt + C\phi(\zeta_0) \frac{\text{meas}\,(q_h - K)}{|q_h|}
\end{aligned}$$

and hence, from (6.2) we deduce

$$\limsup_{h\to 0} \limsup_{k\to +\infty} |q_h|^{-1} \int_{q_h - K} [F(t, u_0(t_0), v_k(t)) - F(t, u_k(t), v_k(t))]\, dt \leq 0. \tag{6.3}$$

The assertion is an immediate consequence of (6.1) and (6.3). □

Remark 6.6. Let us consider the following weakening of condition $(\mathcal{R})$

(w$\mathcal{R}$) *for every* $\varepsilon, \varrho > 0$, *every compact set* $K \subset G$ *with* $\text{meas}\,(G - K) < \varrho$ *and a.e.* $t_0 \in K$, *there is a constant* $0 < h' = h'(\varepsilon, \varrho, K, t_0, u_0(t_0), \mathcal{S})$ *such that for a.e.* $0 < h < h'$ *a subsequence* $(s_k)_{k\in\mathbb{N}}$ *exists with the property that*

$$|q_h|^{-1} \int_{q_h - K} [F(t, u_0(t_0), v_{sk}(t)) - F(t, u_{sk}(t), v_{sk}(t))]\, dt \leq \varepsilon.$$

Let us present now a criterion for $(\text{w}\widetilde{\text{F}})$.

Proposition 6.7. *If F satisfies condition* (w$\mathcal{R}$)*, then $\widetilde{Q}_0$ satisfies condition* (w$\widetilde{\mathrm{F}}$)*, with respect to $\mathcal{S}$, at $(t_0, u_0(t_0))$ for a.e. $t_0 \in G$.*

Proof. From the assumption u_k L_1–converges to u_0, we deduce that there is a subsequence, say still (k), such that $u_k \to u_0$ a.e. in G.

Then, by virtue of Egoroff's and Lusin's theorems, for every $\varepsilon > 0$, a compact set K_ε exists such that meas $(G - K_\varepsilon) < \varepsilon$ and the following conditions hold

$$u_{0/K_\varepsilon} \quad \text{is continuous} \tag{6.4}$$

$$u_k \to u_0 \quad \text{uniformly on} \quad K_\varepsilon. \tag{6.5}$$

Let $\varepsilon > 0$ be fixed.

We consider the sequence $\varepsilon_m = \frac{\varepsilon}{2m}$, $m \in \mathbb{N}$. Let K_m denote the compact set corresponding to ε_m and let $N_m \subset K_m$ be the null subset of the points t_0 to which (w$\mathcal{R}$) does not apply; note that $N = \bigcup_m N_m \cup (\bigcap_m (G - K_m))$ is a null set.

Let $t_0 \in G - N$ be fixed.

Then there exists $\overline{m} = \overline{m}(t_0) \in \mathbb{N}$ such that $t_0 \in K_{\overline{m}} - N_{\overline{m}}$. From (6.4) and (6.5) we deduce that a constant $\overline{h} = \overline{h}(\varepsilon, t_0, u_0) > 0$ and an integer $\overline{k} = \overline{k}(\varepsilon, (u_k)_{k \geq 0})$ exist such that for every $0 < h < h_0$ and every $k > \overline{k}$, if $t \in K_{\overline{m}} \cap q_h$

$$|u_k(t) - u_0(t_0)| < |u_k(t) - u_0(t)| + |u_0(t) - u_0(t_0)| < \varepsilon_{\overline{m}} + \varepsilon_{\overline{m}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{6.6}$$

Moreover, let $0 < h' = h'(\varepsilon, t_0, u_0(t_0), \mathcal{S}) \leq \varepsilon$ be a constant such that (see Remark 6.6) for a.e. $0 < h < \min(\overline{h}, h')$ a subsequence $(s_k)_{k \in \mathbb{N}}$ exists with the property that $s_k \geq \overline{k}$, $k \in \mathbb{N}$, and

$$|q_h|^{-1} \int_{q_h - K_{\overline{m}}} [F(t, u_0(t_0), v_{sk}(t)) - F(t, u_{sk}(t), v_{sk}(t))] \, dt \leq \varepsilon_{\overline{m}} < \varepsilon \qquad (6.7).$$

Now, let us consider the sequence $(\overline{u}_{s_k}, w^0_{s_k}, w_{s_k})_{k \in \mathbb{N}}$ defined by

$$\overline{u}_{s_k} = \begin{cases} u_0(t_0) & t \in q_h - K_{\overline{m}} \\ u_{s_k}(t) & t \in q_h \cap K_{\overline{m}} \end{cases}$$

$$w_{s_k}(t) = v_{s_k}(t) \qquad w^0_{s_k}(t) = F(t, \overline{u}_{s_k}(t), v_{s_k}(t)) \qquad t \in G.$$

Of course the sequence $(\overline{u}_{s_k}, w^0_{s_k}, w_{s_k})_{k \in \mathbb{N}}$ lies in $S_{\widetilde{Q}_0}$ and moreover, from (6.6) and (6.7), we get

$$|\overline{u}_{s_k}(t) - u_0(t_0)| \leq \varepsilon \qquad t \in q_h$$

$$\fint_{q_h} [w^0_{s_k}(t) - f(t, u_{s_k}(t), v_{s_k}(t))] \, dt =$$

$$= |q_h|^{-1} \int_{q_h - K_{\overline{m}}} [F(t, u_0(t_0), v_{sk}(t)) - F(t, u_{sk}(t), v_{sk}(t))] \, dt \leq \varepsilon$$

which proves the assertion. $\square$

Let us prove another sufficient condition for $(\mathcal{R})$.

Proposition 6.8. *If F satisfies the condition*

(F$''$) *two functions $\Phi : A_0 \to \mathbb{R}$ and $\psi \in L_1$ exist such that $\Phi(u_k)$ converges to $\Phi(u_0)$ in L_1 and for a.e. $t_0 \in G$ a constant $\rho = \rho(t_0, u_0(t_0), \mathcal{S}) > 0$ exists such that for a.e. $t \in G$ with $|t - t_0| < \rho$ and $k \in \mathbb{N}$ we have*

$$|F(t, u_0(t_0), v_k(t)) - F(t, u_k(t), v_k(t))| \leq |\Phi(u_0(t_0)) - \Phi(u_k(t))| + |\psi(t_0) - \psi(t)|,$$

then F satisfies condition $(\mathcal{R})$.

Proof. Observe that, by virtue of well-known density results [19], for a.e. $t_0 \in G$ we have

$$\lim_{h\to 0} \fint_{q_h} |\psi(t_0) - \psi(t)|\, dt = 0, \qquad \lim_{h\to 0} \fint_{q_h} |\Phi(u_0(t_0)) - \Phi(u_0(t))|\, dt = 0 \tag{6.8}$$

$$\text{and} \qquad \lim_{h\to 0} \lim_{k\to\infty} \fint_{q_h} |\Phi(u_0(t)) - \Phi(u_k(t))|\, dt = 0.$$

Let $t_0 \in G$ be fixed in such a way that (6.8) and (F$''$) are satisfied, then

$$\begin{aligned}
&\fint_{q_h} [F(t, u_0(t_0), v_k(t)) - F(t, u_k(t), v_k(t))]\ dt \leq \\
&\leq \fint_{q_h} |\Phi(u_0(t_0)) - \Phi(u_0(t))|\, dt + \fint_{q_h} |\Phi(u_0(t)) - \Phi(u_k(t))|\, dt + \fint_{q_h} |\psi(t_0) - \psi(t)|\, dt
\end{aligned}$$

and the assertion follows by virtue of (6.8). □

Remark 6.9. Note that, by virtue of Propositions 6.5 and 6.8, Proposition 6.7 applies to integrands of the type

1) $F(t, \cdot, y)$ Lipschitzian, $t \in G$, $y \in \mathbb{R}^m$;
2) $F(t, x, y) = f(t, x, y) + F_0(t, y)$ with f, F_0 Carathéodory and f satisfying assumption (F);
3) $F(t, x, y) = f(t, x, y) + F_0(t, y)$ with F_0 Carathéodory and f satisfying assumption (F$''$);
4) $F(t, x, y) = |f(t, x, y) + F_0(t, y)|$ with F_0 and f as above.

Moreover it is trivially satisfied in the case

5) F is bounded.

In other words, the multifunction $\widetilde{Q}_0$ associated to any integrands of these classes satisfies condition (w$\widetilde{\text{F}}$).

Finally, from Theorem 5.1 and Propositions 6.1, 6.3 and 6.7, the following lower semicontinuity result can be deduced.

Theorem 6.10. *(A lower semicontinuity result) Assume that* $A_0 \subset \mathbb{R}^n$ *is closed. Let* $\mathcal{S} = (u_k, v_k)_{k \geq 0}$ *be a sequence in* $L_1(\mathbb{R}^{n+m})$ *such that*

i) $u_k \in A_0$, *a.e. in* G, $k \in \mathbb{N}$;

ii) u_k L_1*–converges to* u_0;

iii) $(v_k)_{k \geq 0}$ *is* L_1*–equibounded and satisfies* (mv).

Suppose that $F : G \times A_0 \times \mathbb{R}^m \to \mathbb{R}_0^+$ *is continuous, satisfies condition* (w$\mathcal{R}$) *and* $F(x, y, \cdot)$ *is convex,* $x \in A_0$ *a.e.* $t \in G$.

Then $u_0(t) \in A_0$ *a.e. in* G *and*

$$\liminf_{k \to +\infty} \int_G F(t, u_k(t), v_k(t))\, dt \geq \int_G F(t, u_0(t), v_0(t))\, dt.$$

Proof. For every $\varepsilon > 0$, let $F_\varepsilon : G \times A_0 \times \mathbb{R}^m \to \mathbb{R}_0^+$ be the integrand defined by

$$F_\varepsilon(t, x, y) = F(t, x, y) + \varepsilon |y|.$$

By virtue of Propositions 6.3, 6.1 and 6.7, F_ε satisfies all the hypotheses of Theorem 5.1, thus

$$\liminf_{k \to +\infty} \int_G F_\varepsilon(t, u_k(t), v_k(t))\, dt \geq \int_G F_\varepsilon(t, u_0(t), v_0(t))\, dt \geq \int_G F(t, u_0(t), v_0(t))\, dt. \quad (6.9)$$

Now

$$\int_G F_\varepsilon(t, u_k(t), v_k(t))\, dt = \int_G F(t, u_k(t), v_k(t))\, dt + \varepsilon \int_G |v_k(t)|\, dt, \ \ k \in \mathbb{N}$$

and, since $(v_k)_{k \in \mathbb{N}}$ is equibounded in $L_1(G, \mathbb{R}^m)$, the assertion is an immediate consequence of (6.9). □

Remark 6.11. We wish to point out that if $F : G \times A_0 \times \mathbb{R}^m \to \mathbb{R}_0^+$ satisfies the assumptions of Theorem 8.1, then also the integrand $F_a : G \times A_0 \times \mathbb{R}^m \to \mathbb{R}_0^+$ defined by

$$F_a(t, x, y) = F(t, x, \langle a(t), y \rangle)$$

with $a : G \to \mathbb{R}^m$ continuous, satisfies the same assumptions.

7. Some remarkable particular cases

We wish to point out that the following formulations can be framed as particular cases of the present setting. Let us consider a multiple integral of the type

$$I[x] = \int_G F(t, (\mathcal{U}x)(t), (\mathcal{L}x)(t))\, dt \quad (7.1)$$

subjected to the constraints

$$(t, (\mathcal{U}x)(t)) \in A, \quad (\mathcal{L}x)(t) \in Q(t, (\mathcal{U}x)(t)) \qquad \text{a.e. in } G \tag{7.2}$$

where $\mathcal{U}$ and $\mathcal{L}$ are given operators and Q is a given multifunction.

Note that $(7.1) - (7.2)$ represents a rather abstract version of integral functional of the calculus of variations with constraints on the differential elements.

We recall that [9e] every optimal control problem can be reduced to a suitable problem of the calculus of variations with constraints on the gradient. By virtue of this remark, $(1.1) - (1.2)$ contains also the following formulation:

a cost functional

$$I[x, y] = \int_G f_0(t, (\mathcal{U}x)(t), y(t))\, dt \tag{7.3}$$

with state equation

$$(\mathcal{L}x)(t) = f(t, (\mathcal{U}x)(t), y(t)) \qquad \text{a.e. in } G \tag{7.4}$$

and possible constraints of the form

$$(t, (\mathcal{U}x)(t)) \in A\,, \quad (\mathcal{L}x)(t) \in Q(t, (\mathcal{U}x)(t)) \qquad \text{a.e. in } G. \tag{7.5}$$

Note that operators $\mathcal{U}$ appearing in the definition of I and in the constraints need not be the same, but may be thought of as different components of the same operator. Analogous for operator $\mathcal{L}$.

Optimal control problems of type $(7.3) - (7.5)$ were studied by Cesari [9c,d] and Cesari-Cowles [12], who also gave closure and lower closure results with respect to the weak topology in Sobolev's spaces.

Forthermore, lower semicontinuity theorems for integral functionals of type (7.1) had already been proved by Rothe [23] and Fichera [18], again in the same setting.

The aim of the present paper is to study the lower semicontinuity in BV setting and, as we shall show in what follows, we are able to prove the lower semicontinuity of $(7.1) - (7.2)$ (and hence $(7.3) - (7.5)$) with respect to a weaker topology which do not require any convergence assumptions on the differential elements of the highest order.

Before to enter in details, we wish to mention that the interest for formulation $(7.1) - (7.2)$ has recently increased since optimization problems of this type, with BV optimal solutions, were adopted as a mathematical model to describe the plastic deformation of beams and plates under different loads [11e,14,3a,b,4].

Let (S, w) be a topological space and let $\mathcal{U} : S \to L_1(G, \mathbb{R}^n)$ and $\mathcal{L} : S \to L_1(G, \mathbb{R}^m)$ be given operators.

Our lower semicontinuity results apply under the assumption that

if u_k *w-converges to* u_0, *then (for a suitable subsequence) we have that*
$\mathcal{U}u_k$ L_1*–converges to* $\mathcal{U}u_0$ *and*
$(\mathcal{L}u_k)_{k\geq 0}$ *is* L_1*–equibounded and satisfies* (mv).

For example, the following particular cases can be framed in our theory (see Propositions 3.2, 3.3, 3.7, 3.10 and Theorem 4.1):

(a) $(x_k)_{k\in\mathbb{N}}$ is a sequence in $W^{1,1}(G,\mathbb{R}^N)L_1$–converging to a function $u_0 \in BV(G,\mathbb{R}^N)$ and

$$\mathcal{U}x = x, \quad \mathcal{L}x = \mathcal{D}x \qquad \text{or} \qquad \mathcal{L}x = \text{div } x;$$

(b) $(x_k)_{k\geq 0}$ is the same sequence as in (a) and

$$(\mathcal{U}x)(t) = \Phi(t, x(t)), \qquad (\mathcal{L}x)(t) = \mathcal{D}[\Psi(t, x(t))]$$

where Φ and Ψ satisfies the assumptions of Corollary 3.6 and Proposition 3.10 respectively;

(c) $(x_k)_{k\in\mathbb{N}}$ is a sequence in $W^{p+1,1}(G,\mathbb{R}^N)$ which $W^{p,1}$–converges to a function $u_0 \in W^{p,1}(G,\mathbb{R}^N)$, with $\mathcal{D}^p u_0 \in BV(G,\mathbb{R}^M)$, and

$$\mathcal{U}x = (x, \mathcal{D}x, \mathcal{D}^2x, ..., \mathcal{D}^p x), \qquad \mathcal{L}x = \mathcal{D}^{p+1}x$$

where $\mathcal{D}^2x$ denotes the essential gradients of $\mathcal{D}x$ and so on;

(d) $(x_k)_{k\geq 0}$ is the same sequence as in (c) and

$\mathcal{U}$ is any operator on the variables $(x, \mathcal{D}x, \mathcal{D}^2x, ..., \mathcal{D}^p x)$ which is continuous with respect to L_1–topology,
$\mathcal{L}$ is a linear combination of the elements of $\mathcal{D}^{p+1}x$, for example $\mathcal{L}x = \text{div } (\mathcal{D}^p x)$;

(e) $(x_k)_{k\geq 0}$ is the same sequence as in (c) and

$$(\mathcal{U}x)(t) = \Phi(t, X(t)), \qquad (\mathcal{L}x)(t) = \mathcal{D}[\Psi(t, X(t))]$$

where $X = (x, \mathcal{D}x, \mathcal{D}^2x, ..., \mathcal{D}^p x)$ and Φ and Ψ are functions as in (b).

Finally we wish to mention that also the following operators can be framed in our setting

(f) $(x_k)_{k\geq 0}$ is a sequence in $L_1(G,\mathbb{R}^n)$ and

$$(\mathcal{U}x)(t) = \int_G K(t,\tau)\, x(\tau)\, d\tau, \qquad \mathcal{L}x = x$$

where K is a suitable kernel (see [18] and [23,24]).

Remark 7.1. Existence results for problems of the calculus of variations can be derived from the closure and lower semicontinuity statements of Sections 4 an 5. More precisely, let us assume that functional I is defined on $W^{1,1}(G,\mathbb{R}^N)$ and let us consider the Serrin-type extension to $BV(G,\mathbb{R}^N)$

$$J[x] = \inf_{\Gamma(x)} \liminf_{k\to+\infty} I[x_k]$$

where $\Gamma(x) = \left\{(x_k)_{k\in\mathbb{N}} : x_k \in W^{1,1}(G,\mathbb{R}^N),\text{ satisfies (7.2) and } x_k\ L_1\text{–converges to } x\right\}$,

if $\Gamma(x) = \phi$ we put $J[x] = +\infty$, as usual.

An analogous functional can be defined in the case I ranges over $W^{p,1}(G,\mathbb{R}^N)$.

By virtue of our lower semicontinuity results, we can deduce that the natural relation between I and J occurs

$$J[x] = I[x] \quad \text{on } W^{1,1}(G,\mathbb{R}^N) \qquad \text{and} \qquad J[x] \geq I[x] \quad \text{on} \quad BV(G,\mathbb{R}^N).$$

Moreover, by means of a standard process, we can prove the existence of minima for functional J over a class Ω satisfying suitable compactness conditions. We do not enter into detail here, but only refer to [9e] for the classical theory and to [11b,c,e] for more recent developments.

8. Applications to a class of integral functionals

Let us consider now functionals of the type

$$I[x] = \int_G |\langle a(t), \mathcal{D}[\Psi(t,x(t))]\rangle + \Phi(t,x(t))|\; dt$$

where $a : G \to \mathbb{R}^{\nu N}$, $\Phi : A \to \mathbb{R}$ and $\Psi : A \to \mathbb{R}^N$ are given functions.

This class of functionals takes its source from problems of conservation laws. Cesari [9f,g,h] discussed the related optimization problems under the assumptions that Φ is continuous and Ψ is of class C^1.

In order to get the existence of BV minima for the Serrin-type extension of I, Cesari adopted a suitable transformation which allows to reduce functional I to a standard integral of the type

$$I^*[x^*] = \int_G F(t, x^*(t), \mathcal{D}x^*(t))\; dt$$

to which the existence results of [11c,d] apply.

Now, by virtue of the present formulation where abstract operators are involved, we can deal with functional I directly, with remarkable advantages both in the assumptions and in the proof. We only mention here the lower semicontinuity result since, as we already mentioned (see Remark 6.6), the existence theorem is then a standard consequence.

Theorem 8.1. *Assume that* $A = G \times A_0$ *with* A_0 *closed.*

Let $(x_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,1}(G,\mathbb{R}^n)$ *and let* $x_0 \in BV(G,\mathbb{R}^n)$ *be such that*

i) $x_k(t) \in A_0$ *a.e. in* G, $k \in \mathbb{N}$;

ii) $(x_k)_{k\in\mathbb{N}}$ *has equibounded variation and* L_1*–converges to* x_0.

Suppose that

iii) a *is continuous,*

iv) Ψ *is Lipschitzian,*

v) Φ *is Carathéodory and there exists* $\phi \in L_1$ *such that* $|\Phi(t, x_k(t))| \leq \phi(t)$ *a.e. in* G, $k \in \mathbb{N}$.

Then the limit function satisfies $x_0(t) \in A_0$ *a.e. in* G, *and we have*

$$\liminf_{k \to +\infty} I[x_k] \geq I[x_0].$$

Proof. Let $\mathcal{U}$ and $\mathcal{L}$ be the operators defined by

$$(\mathcal{U}x)(t) = \Phi(t, x(t)), \qquad (\mathcal{L}x)(t) = \mathcal{D}[\Psi(t, x(t))].$$

By virtue of Proposition 3.10 and Corollary 3.6, we have that (for a suitable subsequence)

$$\mathcal{U}x_k \to \mathcal{U}x_0 \quad \text{in } L_1 \qquad \text{and} \qquad (\mathcal{L}x_k)_{k \in \mathbb{N}} \quad \text{satisfies (mv) on } G.$$

Moreover, it is easy to see that the sequence $(\mathcal{D}[\Psi(\cdot, x_k(\cdot))])_{k \in \mathbb{N}}$ has equibounded variation, thus $(\mathcal{L}x_k)_{k \in \mathbb{N}}$ is bounded in $L_1(G, \mathbb{R}^{\nu N})$.

Let us consider the integrand $F : G \times A_0 \times \mathbb{R}^{\nu N} \to \mathbb{R}_0^+$ defined by

$$F(t, x, y) = |\langle a(t), y \rangle + x|$$

of course it satisfies assumption of Theorem 8.1 (see also Remark 7.1) and the assertion follows as an immediate application of Proposition 6.5. $\square$

Remark 8.2. Note that a result analogous to that of Theorem 8.1 can be proved for the functional

$$I[x] = \int_G [\langle a(t), \mathcal{D}[\Psi(t, x(t))] \rangle + \Phi(t, x(t))]^+ \, dt.$$

We recall that the lower semicontinuity for integrals of the type

$$I[x] = \int_G [\langle a(t, x(t)), \mathcal{D}x(t) \rangle + b(t, x(t))]^+ \, dt$$

with respect to weak topology in Sobolew's spaces, was discussed in [17,1].

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