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The Hausdorff Metric Topology, the Attouch-Wets Topology, and the Measurability of Set-Valued Functions

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Let $\langle X, d\rangle$ be a separable metric space. A set-valued function $\Gamma: S \rightrightarrows X$ defined on a measurable space S whose values are nonempty closed subsets of X is declared measurable provided for each open subset V of X, $\{s \in S: \Gamma(s) \cap V \neq \emptyset\}$ is a measurable subset of S. In this paper, we look at the relationship between measurability so defined and the Borel measurability of Γ, viewed as a single-valued function into the nonempty closed subsets of X, equipped with either the Hausdorff metric topology or with the Attouch-Wets topology. Our analysis rests on cardinality arguments in conjunction with the representation of these hyperspaces as weak topologies. Applications are given to convex-valued multifunctions.

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1. Introduction

Let $\langle X, d\rangle$ be a separable metric space, and let $\langle S, \mathcal{A}\rangle$ be a measurable space, i.e., a set S equipped with a sigma algebra of subsets $\mathcal{A}$. By a *set-valued function* or *multifunction* Γ from S to X, we mean a function that assigns to elements of S non-empty closed subsets of X. We denote such a function by $\Gamma: S \rightrightarrows X$. Measurability of a single-valued function $f: S \to X$ is defined in the usual way: for each open subset V of X, $f^{-1}(V)$ is in $\mathcal{A}$. Measurability for a set-valued function is usually defined as follows: for each open subset V of X, $\{s \in S : \Gamma(s) \cap V \neq \emptyset\}$ belongs to $\mathcal{A}$. As is well known this is equivalent to the measurability for each $x \in X$, of the single-valued function $s \to d(x, \Gamma(s))$. When

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X is Polish, this is also equivalent to the existence of a *Castaing representation* for the set-valued function [14, 24], i.e., a sequence of measurable selectors $\langle f_n \rangle$ for Γ such that for each $s \in S$, $\Gamma(s) = \mathrm{cl}\ \{f_n(s) : s \in S\}$.

On the other hand, we may equip the nonempty closed subsets $CL(X)$ with a topology τ, i.e., a prescribed collection of open sets, and view Γ as a single-valued function from S to $CL(X)$. Then Γ is measurable from this perspective provided for each $\mathcal{U} \in \tau$, we have $\Gamma^{-1}(\mathcal{U}) \in \mathcal{A}$. These two approaches sometimes give the same notion of measurability. It has long been known that for compact-valued multifunctions, this is true provided τ is the familiar Hausdorff metric topology [14, p. 62]. For a general closed-valued multifunction, Hess [21–22] has shown that this is true for the weaker Wijsman topology, which is the weakest topology τ on $CL(X)$ such that for each $x \in X$, $A \to d(x, A)$ is τ-continuous.

Let us call a topology τ on $\mathcal{F} \subset CL(X)$ *measurably compatible* provided for each multifunction Γ with values in $\mathcal{F}$, measurability of Γ in the above two senses coincide. When τ is the Hausdorff metric topology, separability of $\mathcal{F}$ is the key issue. Not only does this guarantee measurable compatiblity, but in this case, one can find a measurably compatible topology τ' on $CL(X)$ such that $\langle \mathcal{F}, \tau \rangle$ is a subspace of $\langle CL(X), \tau' \rangle$. Using cardinality arguments and the continuum hypothesis, we show that measurable compatibility fails without separability (in the special case that $\mathcal{F} = CL(X)$, where separability amounts to total boundedness of X, the continuum hypothesis need not be assumed). Separability of $\mathcal{F}$ when the elements of $\mathcal{F}$ are convex sets is of particular interest, for in this case, measurability of multifunctions with values in $\mathcal{F}$ can be expressed in terms of support functionals. A parallel analysis is performed for the Attouch-Wets topology, a recent variant of the Hausdorff metric topology.

2. Preliminaries

We write $\mathbb{N}$ for the set of natural numbers. If A is a set, then 2^A will denote the power set of A, the set of all subsets of A. We denote by c the cardinality of the continuum. In the sequel $\langle X, d \rangle$ will always be a separable metric space. If $x \in X$ and $t > 0$, then the open ball with center x and radius t will be denoted by $S_t[x]$. More generally, if A is a nonempty subset of X, we write $S_t[A]$ for the *enlargement* $\bigcup_{a \in A} S_t[a]$ of the set A of radius t. If A and B are nonempty subsets of X, we define the *gap* $D_d(A, B)$ between A and B and the *excess* of A over B by the formulas

$$D_d(A, B) = \inf_{a \in A} d(a, B) \text{ and } e_d(A, B) = \sup_{a \in A} d(a, B).$$

Gap and excess are both extensions of the usual distance between a point and a set, as $d(x, B) = e_d(\{x\}, B) = D_d(\{x\}, B)$. The Hausdorff distance between A and B is given by

$$\begin{aligned} H_d(A, B) &= \max\{e_d(A, B), e_d(B, A)\} \\ &= \inf\{\varepsilon > 0 : A \subset S_\varepsilon[B] \text{ and } B \subset S_\varepsilon[A]\}. \end{aligned}$$

It can also be shown that $H_d(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)|$, so that Hausdorff distance is just uniform distance between distance functionals [17]. We denote the Hausdorff metric topology by τ_{H_d}, and the Wijsman topology determined by d by τ_{W_d}. It is well-known that different admissible metrics give the same Hausdorff metric topologies if and only if they are uniformly equivalent. The situation for the Wijsman topology, as recently resolved in [16], is more complex.

In the sequel, $K(X)$ will denote the family of nonempty compact subsets of the metric space X. We will have occasion to consider normed linear spaces, and in this setting $C(X)$ will denote the nonempty closed convex subsets of X. For $A \in C(X)$, its support functional $\delta^*(\cdot, A): X^* \to (-\infty, +\infty]$ is defined by the formula $\delta^*(x^*, A) = \sup\{x^*(a): a \in A\}$. If $A \subset X$, then span A will be the linear subspace generated by A, and clco A will be its closed convex hull.

Let $\langle W, \tau\rangle$ be a topological space. The associated *Borel field* $\mathcal{B}(W, \tau)$ is the smallest sigma algebra of subsets of W containing τ. Clearly $f: \langle S, \mathcal{A}\rangle \to W$ is measurable if and only if $B \in \mathcal{B}(W, \tau) \Rightarrow f^{-1}(B) \in \mathcal{A}$. Now let $\langle X, d\rangle$ be a separable metric space. Given a subfamily $\mathcal{F}$ of $CL(X)$, the smallest sigma algebra containing all sets of the form

$$\{F \in \mathcal{F}: F \cap V \neq \emptyset\}$$

where V is open in X is called the *Effros sigma algebra* $\mathcal{E}(\mathcal{F})$ [15]. Clearly, a multifunction $\Gamma: \langle S, \mathcal{A}\rangle \rightrightarrows X$ with values in $\mathcal{F}$ is measurable in the sense that for each open subset V of X, $\{s \in S: \Gamma(s) \cap V \neq \emptyset\}$ belongs to $\mathcal{A}$, if and only if the associated single-valued function from S to $\mathcal{F}$ is $\mathcal{A} - \mathcal{E}(\mathcal{F})$ measurable. Thus, if $\mathcal{F}$ is equipped with a topology τ, then measurability for multifunctions agrees with $\mathcal{A} - \mathcal{B}(\mathcal{F}, \tau)$-measurability for the transformation viewed as a single-valued function provided $\mathcal{E}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, \tau)$. The Theorem of Hess [21–22] alluded to in the introduction says this: for any separable metric space $\langle X, d\rangle$ and for any $\mathcal{F} \subset CL(X)$, we have $\mathcal{E}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, \tau_{W_d})$. In particular, the Borel field for the Wijsman topology is independent of the metric chosen for the space. As we shall see in section 3, this is not the case when τ_{W_d} is replaced by τ_{H_d}.

3. On the Borel Field of the Hausdorff Metric Topology

Let $\langle X, d\rangle$ be a separable metric space. As is well known, $\mathcal{E}(K(X)) = \mathcal{B}(K(X), \tau_{H_d})$. We first give, using cardinality arguments within ZFC, necessary and sufficient conditions for $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau_{H_d})$.

Theorem 3.1. *Let $\langle X, d\rangle$ be a separable metric space. The following are equivalent:*

(a) *$\langle X, d\rangle$ is a totally bounded metric space;*

(b) $\mathrm{card}(\tau_{H_d}) \leq c$*;*

(c) $\mathcal{E}(CL(X) = \mathcal{B}(CL(X), \tau_{H_d})$.

Proof. (a)⇒(c). Total boundedness of $\langle X, d\rangle$ is necessary and sufficient for the equality of the Wijsman and Hausdorff metric topologies [9]. By the Theorem of Hess, we have $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau_{W_d}) = \mathcal{B}(CL(X), \tau_{H_d})$.

(c)⇒(b). Suppose (b) fails. Then clearly $\mathrm{card}(\mathcal{B}(CL(X), \tau_{H_d})) > c$. On the other hand, by the Theorem of Hess, $\mathcal{E}(CL(X))$ is the Borel field of a separable metrizable space, namely $\langle CL(X), \tau_{W_d}\rangle$. In view of the standard inductive construction of the Borel sets, the Borel field for any such space has cardinality c [25, p. 347] (this does not use the continuum hypothesis). As a result, $\mathcal{E}(CL(X)) \neq \mathcal{B}(CL(X), \tau_{H_d})$.

(b)$\Rightarrow$(a). Suppose d is not totally bounded. Choose a sequence $\{x_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$ such that $d(x_i, x_k) > \varepsilon$ if $i \neq k$. Now write $B = \{x_n : n \in \mathbb{N}\}$. Then each element of 2^B is a closed subset of X, and when F_1 and F_2 are distinct elements of 2^B, we have $H_d(F_1, F_2) \geq \varepsilon$. Thus, the elements of 2^B form a uniformly discrete set in $\langle CL(X), \tau_{H_d}\rangle$, and so each element of 2^{2^B} is a closed subset in the hyperspace. Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are distinct elements of 2^{2^B} with say $\mathcal{F}_1 \not\subset \mathcal{F}_2$, then we have

$$e_{H_d}(\mathcal{F}_1, \mathcal{F}_2) \geq \varepsilon.$$

This shows that the family of ε-enlargements in $\langle CL(X), \tau_{H_d}\rangle$ whose centers run over 2^{2^B} are distinct sets. Since each enlargement is open in the Hausdorff metric topology, and the cardinality of 2^B is already c, we get $\mathrm{card}(\tau_{H_d}) \geq \mathrm{card} 2^{2^B} = 2^c$. We conclude that condition (b) fails. $\square$

We note that the implication (c)$\Rightarrow$(b) in the proof of Theorem 3.1 can be argued in another way, using the fact that the Effros Borel field is isomorphic with the Borel field of some subset of [0,1] (with the relative topology) (see, e.g., [15, p. 7].

Corollary 3.2. *Let $\langle X, d\rangle$ be a separable metric space. The following are equivalent:*

(a) *$\langle X, d\rangle$ is a totally bounded metric space;*

(b) *for each measurable space $\langle S, \mathcal{A}\rangle$, the measurability of a multifunction $\Gamma: S \rightrightarrows X$ implies its $\mathcal{A} - \mathcal{B}(CL(X), \tau_{H_d})$-measurability.*

Proof. (a)$\Rightarrow$(b). This is immediate from the coincidence of the Wijsman and Hausdorff metric topologies, and thus of their Borel fields.

(b)$\Rightarrow$(a). With no assumptions, $\Gamma: \langle CL(X), \mathcal{E}(CL(X))\rangle \rightrightarrows X$ defined by $\Gamma(A) = A$ is measurable. If X is not totally bounded, then by the Theorem of Hess, the inclusion $\tau_{W_d} \subset \tau_{H_d}$, and Theorem 3.1, we see that the inclusion $\mathcal{E}(CL(X)) \subset \mathcal{B}(CL(X), \tau_{H_d})$ must be proper. Thus the single-valued function $A \to A$ from $\langle CL(X), \mathcal{E}(CL(X))\rangle$ to $\langle CL(X), \mathcal{B}(CL(X), \tau_{H_d})\rangle$ is not measurable. $\square$

Corollary 3.3. *Let X be a separable metrizable space. Then X has two admissible metrics d and ρ such that $\mathcal{B}(CL(X), \tau_{H_d}) \neq \mathcal{B}(CL(X), \tau_{H_\rho})$ if and only if X is not compact.*

Proof. If X is compact, then X has a unique compatible uniformity, and so every admissible metric gives the same Hausdorff metric topology. Conversely, if X is noncompact, then X admits an unbounded metric d [20]. But X can also be embedded into a countable product of intervals and thus also admits a totally bounded metric ρ. By Theorem 3.1, the Borel fields for the induced Hausdorff metrics diverge. $\square$

We next show that the Borel field of the Hausdorff metric topology agrees with the Effros sigma algebra for certain subfamilies of $CL(X)$. A point of departure for our next result is the following classical fact: if $\langle X, d\rangle$ is separable, then $\langle K(X), H_d\rangle$ is separable (the finite subsets of a dense subset of X are dense in $K(X)$).

Theorem 3.4. *Let $\langle X, d\rangle$ be a separable metric space, and let $\mathcal{F}$ be a subfamily of $CL(X)$ that is separable with respect to the induced Hausdorff metric. Then $\mathcal{E}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, \tau_{H_d})$.*

Proof. We always have $\mathcal{E}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, \tau_{W_d}) \subset \mathcal{B}(\mathcal{F}, \tau_{H_d})$. The equality is the Theorem of Hess, whereas the second inclusion follows from $\tau_{W_d} \subset \tau_{H_d}$.

For the inclusion $\mathcal{B}(\mathcal{F}, \tau_{W_d}) \supset \mathcal{B}(\mathcal{F}, \tau_{H_d})$, consider a closed ball in $\langle \mathcal{F}, H_d\rangle$ with center $F_0 \in \mathcal{F}$, say $\{F \in \mathcal{F} : H_d(F, F_0) \leq \varepsilon\}$. As Hausdorff distance is a uniform distance between distance functionals, this closed ball may be written as

$$\bigcap_{x\in X} \{F \in CL(X) : |d(x, F) - d(x, F_0)| \leq \varepsilon\},$$

a closed set in the Wijsman topology. By second countability of $\langle \mathcal{F}, \tau_{H_d}\rangle$, each open subset of $\langle \mathcal{F}, \tau_{H_d}\rangle$ is a countable union of closed balls and is hence an F_σ-subset of $\langle \mathcal{F}, \tau_{W_d}\rangle$. In particular, each τ_{H_d}-open set belongs to $\mathcal{B}(\mathcal{F}, \tau_{W_d}) = \mathcal{E}(\mathcal{F})$. □

If one accepts the continuum hypothesis, it can be shown that equality of $\mathcal{E}(\mathcal{F})$ and $\mathcal{B}(\mathcal{F}, \tau_{H_d})$ implies that $\langle \mathcal{F}, \tau_{H_d}\rangle$ is separable. For if the hyperspace is nonseparable, then we can find a subset $\mathcal{W}$ of cardinality c of $\mathcal{F}$ and $\varepsilon > 0$ such that if $F_1 \in \mathcal{W}$ and $F_2 \in \mathcal{W}$ and $F_1 \neq F_2$, then $H_d(F_1, F_2) > \varepsilon$. For each subset $\mathcal{W}'$ of $\mathcal{W}$, form the ε-enlargement of $\mathcal{W}'$ in the Hausdorff metric in the space $\mathcal{F}$. Then the family of such ε-enlargements gives a subfamily of $\mathcal{B}(\mathcal{F}, \tau_{H_d})$ of cardinality 2^c, which cannot be contained in $\mathcal{E}(\mathcal{F})$.

Convex-valued multifunctions Γ whose values lie in a τ_{H_d}-separable subset are particularly tractable, for their measurability can be characterized in terms of the measurability, for each $x^* \in X^*$, of the associated single-valued functions $s \to \delta^*(x^*, \Gamma(s))$. Evidently, measurability of these functions is necessary for measurability of the initial multifunction, because

$$\{s \in S : \delta^*(x^*, \Gamma(s)) > t\} = \{s \in S : \Gamma(s) \cap \{x \in X : x^*(x) > t\} \neq \emptyset\}.$$

Before proceeding, we note that there are some other known cases where sufficiency also holds, e.g., when the values of Γ are bounded and X^* is strongly separable. For a counterexample in the general case, the reader may consult [5].

We require a well-known dual formula for the distance from a point $x \in X$ to a closed convex set A [23, p. 62]: $d(x, A) = \sup\{x^*(x) - \delta^*(x^*, A) : \|x^*\| \leq 1\}$. We also need the following elementary known fact whose proof we include for completeness (see, e.g., Chapter IX of Bourbaki [13]).

Lemma 3.5. *Let X be a separable metric space, and let $\{f_\alpha : \alpha \in \mathcal{I}\}$ be a family of continuous real-valued functions on X. Then there exists a countable subset $\mathcal{I}_0$ of $\mathcal{I}$ such that for each $x \in X$, $\sup\{f_\alpha(x) : \alpha \in \mathcal{I}_0\} = \sup\{f_\alpha(x) : \alpha \in \mathcal{I}\}$.*

Proof. For each $\alpha \in \mathcal{I}$, let $V_\alpha = \{(x, t) \in X \times R : t < f(x)\}$, and let $V = \bigcup\{V_\alpha : \alpha \in \mathcal{I}\}$.

Since $X \times R$ is Lindelöf, we can find a countable subfamily $\mathcal{I}_0$ of $\mathcal{I}$ such that $V = \bigcup\{V_\alpha : \alpha \in \mathcal{I}_0\}$. This subset $\mathcal{I}_0$ does the job. $\square$

Theorem 3.6. *Let X be a separable normed linear space, and let $\langle S, \mathcal{A}\rangle$ be a measurable space. Suppose $\mathcal{F}$ is a separable subspace of $\langle C(X), \tau_{H_d}\rangle$, and $\Gamma : S \rightrightarrows X$ has values in $\mathcal{F}$. Then Γ is measurable if and only if for each $x^* \in X^*$, $s \to \delta^*(x^*, \Gamma(s))$ is measurable.*

Proof. Only sufficiency is in question. Let $\{C_k : k \in \mathbb{N}\}$ be H_d-dense in $\mathcal{F}$. Writing E_k for $\{x^* \in X^* : \|x^*\| \leq 1 \text{ and } \delta^*(x^*, C_k) < +\infty\}$, we have for each $x \in X$,

$$d(x, C_k) = \sup\{x^*(x) - \delta^*(x^*, C_k) : x^* \in E_k\}.$$

Now for each $x^* \in E_k$, $x \to x^*(x) - \delta^*(x^*, C_k)$ is a continuous real-valued function. By Lemma 3.5, there exists a countable subset D_k of E_k such that for each $x \in X$,

$$d(x, C_k) = \sup\{x^*(x) - \delta^*(x^*, C_k) : x^* \in D_k\}.$$

Set $D = \bigcup_{k=1}^{\infty} D_k$. We claim that for each $x \in X$ and for each $A \in \mathcal{F}$, we have $d(x, A) = \sup\{x^*(x) - \delta^*(x^*, A) : x^* \in D\}$. We need only show that for each $t < d(x, A)$, there exists $x^* \in D$ with $x^*(x) - \delta^*(x^*, A) > t$. Choose $\varepsilon > 0$ such that $d(x, A) > t + \varepsilon$, and then choose C_k with $H_d(A, C_k) < \varepsilon/3$. Now choose $x^* \in D_k$ such that

$$d(x, C_k) - \frac{\varepsilon}{3} < x^*(x) - \delta^*(x^*, C_k).$$

Since $|\delta^*(x^*, C_k) - \delta^*(x^*, A)| \leq \|x^*\| \cdot H_d(C_k, A) < \varepsilon/3$, we get

$$x^*(x) - \delta^*(x^*, A) > x^*(x) - \delta^*(x^*, C_k) - \frac{\varepsilon}{3} > d(x, C_k) - \frac{2\varepsilon}{3} > d(x, A) - \varepsilon > t.$$

This establishes the claim.

Finally, fix $x \in X$. We may write for each $s \in S$

$$d(x, \Gamma(s)) = \sup\{x^*(x) - \delta^*(x^*, \Gamma(s)) : x^* \in D\}.$$

This means that $s \to d(x, \Gamma(s))$ is a countable supremum of measurable single-valued functions and is thus measurable. Measurability of the multifunction Γ now follows. $\square$

To close this section, we show that if $\mathcal{F}$ is a separable subspace of $\langle CL(X), \tau_{H_d}\rangle$, then the Hausdorfff metric topology restricted to $\mathcal{F}$ can be extended to a topology τ on $CL(X)$ such that $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau)$. This of course yields Theorem 3.4 above as a consequence. Our construction is based on ideas developed in [7], and takes into account the fact that many hyperspace topologies arise as weak topologies determined by families of gap and excess functionals (perhaps varying the metric in the process) [9, 11]. In particular, the Hausdorff metric topology is the weak topology on $CL(X)$ determined by the family $\{D_d(B, \cdot) : B \in CL(X)\} \cup \{e_d(B, \cdot) : B \in CL(X)\}$. In this representation, the family $\{D_d(B, \cdot) : B \in CL(X)\}$ may be replaced by $\{e_d(\cdot, B) : B \in CL(X)\}$ [11].

Theorem 3.7. *Let $\langle X, d\rangle$ be a separable metric space, and let $\mathcal{F}$ be a separable subspace of $\langle CL(X), \tau_{H_d}\rangle$. Then there exists a second countable metrizable topology τ on $CL(X)$ whose subspace topology on $\mathcal{F}$ agrees with the Hausdorff metric topology, and such that $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau)$.*

Proof. Since the Hausdorff metric topology for $\mathcal{F}$ is second countable, each base for the relative topology contains within it a countable base. As a result, we can find a countable subfamily $\mathcal{D}$ of $CL(X)$ such that $\{D_d(B, \cdot) : B \in \mathcal{D}\} \cup \{e_d(B, \cdot) : B \in \mathcal{D}\}$ as a family of functions on $\mathcal{F}$ generates the relative topology. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X, and let τ be the topology on $CL(X)$ generated by

$$\{d(x_n, \cdot) : n \in \mathbb{N}\} \cup \{D_d(B, \cdot) : B \in \mathcal{D}\} \cup \{e_d(B, \cdot) : B \in \mathcal{D}\},$$

viewed as a family of functions on $CL(X)$. As a weak topology, τ is automatically completely regular [20]. Since the family $\{d(x_n, \cdot) : n \in \mathbb{N}\}$ separates points in the hyperspace, τ is Hausdorff. Since the family of generating functionals is countable, the topology is second countable. Thus, by the Urysohn metrization theorem, the topology is second countable and metrizable.

Since $d(x_n, \cdot)$ is τ-continuous for each $n \in \mathbb{N}$, it follows that $d(x, \cdot)$ is τ-continuous for each $x \in X$, and so τ contains the Wijsman topology determined by d. This alone implies that $\mathcal{E}(CL(X)) \subset \mathcal{B}(CL(X), \tau)$.

By second countability of τ, it is clear that its Borel field is generated by any system of generators that yield a subbase for the topology through countable unions, countable intersections, and complementation. Obviously, one subbase for the topology consists of all sets of the following form, where $t > 0$:

$$\begin{array}{l}
\{A \in CL(X) : d(x_n, A) < t\},\ \{A \in CL(X) : d(x_n, A) > t\}\ (n \in \mathbb{N}),\\
\{A \in CL(X) : D_d(B, A) < t\},\ \{A \in CL(X) : D_d(B, A) > t\}\ (B \in \mathcal{D}),\\
\{A \in CL(X) : e_d(B, A) < t\},\ \{A \in CL(X) : e_d(B, A) > t\}\ (B \in \mathcal{D}).
\end{array}$$

Since distance functionals are in particular gap/excess functionals with fixed left argument, the proof will be completely provided that we can show for each $B \in CL(X)$, the sets $\{A \in CL(X) : D_d(B, A) < t\}$ and $\{A \in CL(X) : e_d(B, A) > t\}$ belong to the Effros sigma algebra. To this end, let $\{b_n : n \in \mathbb{N}\}$ be a countable dense subset of B. We compute

$$\{A \in CL(X) : D_d(B, A) < t\} = \{A \in CL(X) \colon A \cap S_t[B] \neq \emptyset\} \in \mathcal{E}(CL(X)),$$

and

$$\begin{aligned}
\{A \in CL(X) : e_d(B, A) > t\} &= \bigcup_{n=1}^{\infty} \{A \in CL(X) : d(b_n, A) > t\}\\
&= \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \{A \in CL(X) : A \subset \{y : d(y, b_n) \geq t + 1/i\}\} \in \mathcal{E}(CL(X)).
\end{aligned}$$

This gives $\mathcal{E}(CL(X)) \supset \mathcal{B}(CL(X), \tau)$, as required. $\square$

4. On the Attouch-Wets Topology

The Hausdorff metric topology is a flexible tool for analysis only when restricted to bounded sets. For example, in the plane, the sequence of lines $\{L_n\}_{n=1}^{\infty}$ where L_n has equation $y = x/n$ ought to converge to the horizontal axis, but this fails with respect to the Hausdorff metric topology. Measurability considerations aside, the Hausdorff metric topology for unbounded closed sets is simply too strong.

The strongest topology that seems to be appropriate for unbounded sets is the so-called *Attouch-Wets topology* τ_{AW_d}, called elsewhere the *bounded-Hausdorff topology*, that has been under intensive investigation over the last five years [1, 2, 3, 4, 6, 8, 10, 11]. From a function space perspective, this is the topology of uniform convergence of distance functionals on bounded subsets of X. It can also be expressed in terms of enlargements [4]; fixing $x_0 \in X$, a compatible uniformity for the topology has as a base all sets of the form

$$\{(A, B) \in CL(X) \times CL(X) : A \cap S_n[x_0] \subset S_{1/n}[B] \text{ and } B \cap S_n[x_0] \subset S_{1/n}[A]\}.$$

Since the topology is Hausdorff and the above uniformity has a countable base, the Attouch-Wets topology is metrizable.

In this section we look at some of the questions posed in the last section with the Hausdorff metric topology replaced by the Attouch-Wets topology. The key tool in such an analysis is the following representation of the Attouch-Wets topology, paralleling the one announced for the Hausdorff metric topology in section 3: τ_{AW_d} is the weak topology on $CL(X)$ induced by $\{D_d(B, \cdot) : B \in CL_b(X)\} \cup \{e_d(B, \cdot) : B \in CL_b(X)\}$, where $CL_b(X)$ denotes the closed and bounded nonempty subsets of X [11].

In terms of what is true, there are no surprises, and only one result requires a new proof: there are complications in producing an analogue of Theorem 3.1. We begin with a technical fact.

Lemma 4.1. *Let* $\mathbb{N}$ *denote the set of positive integers. There exists a map* $\alpha \to I_\alpha$ *from* $2^{\mathbb{N}}$ *in itself such that if* $\alpha \neq \beta$*, then* $I_\alpha \not\subset I_\beta$ *and* $I_\beta \not\subset I_\alpha$*.*

Proof. For each $\alpha \in 2^{\mathbb{N}}$, let

$$I_\alpha = \{2n : n \in \alpha\} \cup \{2n + 1 : n \notin \alpha\}.$$

Let α and β be distinct elements of $2^{\mathbb{N}}$. Without loss of generality, we may assume $\alpha \backslash \beta \neq \emptyset$. Taking $n \in \alpha \backslash \beta$, we have $2n \in I_\alpha \backslash I_\beta$ and $2n + 1 \in I_\beta \backslash I_\alpha$, and thus none of the inclusions can occur. □

Theorem 4.2. *Let* $\langle X, d\rangle$ *be a separable metric space. The following are equivalent:*

(a) *each bounded subset of* X *is totally bounded;*

(b) $\mathrm{card}(\tau_{AW_d}) \leq c$*;*

(c) $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau_{AW_d})$.

Proof. (a)$\Rightarrow$(c). Total boundedness of bounded subsets of X is necessary and sufficient for the equality of the Wijsman and Attouch-Wets topologies [12]. By the Theorem of Hess, we have $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau_{W_d}) = \mathcal{B}(CL(X), \tau_{AW_d})$.

(c)$\Rightarrow$(b). This is argued just as in the proof of Theorem 3.1.

(b)$\Rightarrow$(a). Let B be a bounded subset of X that is not totally bounded. Choose a sequence $\{x_n\}_{n=1}^{\infty}$ in B and $\varepsilon > 0$ such that $d(x_i, x_k) > \varepsilon$ if $i \neq k$. Consider the family $\mathcal{I} = \{I_\alpha : \alpha \in 2^{\mathbf{N}}\}$ of Lemma 4.1, and for each α, write

$$F_\alpha = \{x_n : n \in I_\alpha\}.$$

By construction, each F_α is bounded and has no accumulation points and is thus closed. By the properties of $\mathcal{I}$, $\mathcal{F} = \{F_\alpha : \alpha \in 2^{\mathbf{N}}\}$ has cardinality c, and $\alpha \neq \beta$ implies both $F_\alpha \not\subset F_\beta$ and $F_\beta \not\subset F_\alpha$. Notice also that when $\alpha \neq \beta$, we have $e_d(F_\alpha, F_\beta) \geq \varepsilon$, for taking $x_0 \in F_\alpha \backslash F_\beta$ (a nonempty set), we have $d(x_0, F_\beta) \geq \varepsilon$.

Consider for each $\Lambda \in 2^{2^{\mathbf{N}}}$ the set

$$\Sigma_\Lambda = \bigcup_{\alpha \in \Lambda} \{F \in CL(X) : e_d(F_\alpha, F) < \varepsilon\}.$$

Since excess functionals with fixed left argument are τ_{AW_d}-continuous [11, p. 516], each Σ_Λ is τ_{AW_d}-open. We claim that all the sets Σ_Λ are distinct. Let Λ and Λ' be distinct elements of $2^{2^{\mathbf{N}}}$, and take without loss of generality $\beta \in \Lambda \backslash \Lambda'$. Then $e_d(F_\alpha, F_\beta) \geq \varepsilon$ for each $\alpha \in \Lambda'$, and so $F_\beta \notin \Sigma_{\Lambda'}$. Thus, $F_\beta \in \Sigma_\Lambda \backslash \Sigma_{\Lambda'}$, establishing the claim. We have shown that

$$\operatorname{card}(\tau_{AW_d}) \geq \operatorname{card}\{\Sigma_\Lambda : \Lambda \in 2^{2^{\mathbf{N}}}\} = 2^c,$$

and so (b) fails provided (a) does. □

The following results are proved using obvious modifications of arguments used in the proofs in section 3.

Corollary 4.3. *Let $\langle X, d \rangle$ be a separable metric space. The following are equivalent:*

(a) *bounded subsets of X are totally bounded;*

(b) *for each measurable space $\langle S, \mathcal{A} \rangle$, the measurability of a multifunction $\Gamma : S \rightrightarrows X$ implies its $\mathcal{A} - \mathcal{B}(CL(X), \tau_{AW_d})$-measurability.*

Theorem 4.4. *Let $\langle X, d \rangle$ be a separable metric space, and let $\mathcal{F}$ be a separable subspace of $\langle CL(X), \tau_{AW_d} \rangle$. Then there exists a second countable metrizable topology τ on $CL(X)$ whose subspace topology on $\mathcal{F}$ agrees with the Attouch-Wets topology, and such that $\mathcal{E}(CL(X)) = \mathcal{B}(CL(X), \tau)$.*

From Theorem 4.4, it follows that for each separable subfamily $\mathcal{F}$ of $CL(X)$ with respect to the induced Attouch-Wets topology, we have $\mathcal{E}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, \tau_{AW_d})$, and the converse holds, assuming the continuum hypothesis.

We close this section with an investigation of the size of the Attouch-Wets topology, restricted to the family of closed convex sets. First, a structural lemma.

Lemma 4.5. *Let X be an infinite dimensional normed linear space. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in X such that whenever $\{m, n_1, n_2, n_3, \ldots, n_k\}$ are distinct positive integers, then $d(a_m, \mathrm{span}\{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}) \geq 1$.*

Proof. It suffices to construct for each $n \geq 2$ a subset $A_n = \{a_1, a_2, a_3, \ldots, a_n\}$ of X such that

(i) $d(a_j, \mathrm{span}(A_n - \{a_j\}) \geq 1 \qquad (j \leq n)$,

(ii) $A_n \subset A_{n+1}$.

This will be done inductively. Suppose $A_1, A_2, A_3, \ldots, A_n$ have been so constructed. Fix $j \leq n$; since $d(a_j, \mathrm{span}(A_n \backslash \{a_j\}) \geq 1$, we can separate $\mathrm{span}(A_n \backslash \{a_j\})$ from $S_1[a_j]$ by a norm one element y_j of X^*. Clearly, both $A_n \backslash \{a_j\} \subset y_j^{-1}(0)$ and $d(a_j, y_j^{-1}(0)) \geq 1$. Also, let y_{n+1} be a norm one functional with $A_n \subset y_{n+1}^{-1}(0)$. It is easily verified that $y_1, y_2, \ldots, y_{n+1}$ are linearly independent, and so there exists $a_{n+1} \in \bigcap_{j=1}^{n} y_j^{-1}(0)$ with $y_{n+1}(a_j) = 1$ [18, p. 421]. Since $\|y_{n+1}\| = 1$, we have $d(a_{n+1}, \mathrm{span}\, A_n) \geq 1$. By construction, for each $j \leq n$, we have $d(a_j, y_j^{-1}(0)) \geq 1$, and since $\{a_{n+1}\} \cup (A_n \backslash \{a_j\}) \subset y_j^{-1}(0)$, we get $d(a_j, \mathrm{span}(A_{n+1} \backslash \{a_j\}) \geq 1$ for $j = 1, 2, \ldots, n$. □

Theorem 4.6. *Let X be a separably infinite dimensional normed linear space. Then both the Attouch-Wets topology and the Hausdorff metric topology restricted to $C(X)$ have 2^c elements.*

Proof. The finer Hausdorff metric topology has at most 2^c elements, for if $\{x_n : n \in \mathbb{N}\}$ is a countable dense subset of X, then all sets of the form $\mathrm{clco}\{x_n : n \in M\}$ where $M \subset \mathbb{N}$ are τ_{H_d}-dense in $C(X)$, and each open set in the hyperspace can be written as a union of balls with rational radii whose centers are of this form. It remains to show that the coarser Attouch-Wets topology has at least this many elements.

Let $\{a_n\}_{n=1}^{\infty}$ be the sequence described in Lemma 4.5, and let $\mathcal{I} = \{I_\alpha : \alpha \in 2^{\mathbb{N}}\}$ be the family of indices as described in Lemma 4.1. For each α, form $C_\alpha = \mathrm{clco}\{a_n : n \in I_\alpha\}$. Then if $\alpha \neq \beta$, we have $e_d(C_\alpha, C_\beta) \geq 1$, as elements in C_β can be approximated by finite convex combinations of elements of $\{a_n : n \in I_\beta\}$. The trace of the sets Σ_Λ in the proof of Theorem 4.2 on $C(X)$ gives 2^c τ_{AW_d}-open sets in the relative topology. □

What happens if $C(X)$ is replaced by $CB(X)$, the family closed and bounded convex sets ? We conjecture that the statement of Theorem 4.6 remains valid, although we are not able to prove this in complete generality. The claim is true provided X is a Banach space with a Schauder basis [18, p. 71] (such spaces are automatically separable, but not conversely [19]).

Theorem 4.7. *Let X be a Banach space with a Schauder basis. Then both the Attouch-Wets topology and the Hausdorff metric topology restricted to $CB(X)$ have 2^c elements.*

Proof. Let $\{a_n : n \in \mathbb{N}\}$ be a Schauder basis of norm one elements; each x in X may be represented uniquely as a series: $x = \sum_{i=1}^{\infty} \alpha_i a_i$. For each $n \in \mathbb{N}$, define the coordinate functional $y_n \in X^*$ by $y_n(\sum_{i=1}^{\infty} \alpha_i a_i) = \alpha_n$, where $x = \sum_{i=1}^{\infty} \alpha_i a_i$. The convergence of the series to x and the fact that $\|a_n\| = 1$ for each n imply that $\{y_n(x) : n \in \mathbb{N}\}$ is bounded for each $x \in X$, and so by the uniform boundedness principle, $\{y_n : n \in \mathbb{N}\}$ is uniformly bounded by some $\mu > 0$. In this case, we get for all $n \in \mathbb{N}$, $d(a_n, \mathrm{span}\{a_i : i \in \mathbb{N}, i \neq n\}) \geq \mu^{-1}$. The arguments of the previous theorem now go through to the bounded case, since whenever $M \subset \mathbb{N}$, we have $\mathrm{clco}\{a_n : n \in M\} \in CB(X)$. □

5. On the separability of hyperspaces

As we remarked in section 3, when $\langle X, d\rangle$ is a separable metric space, then $\langle K(X), \tau_{H_d}\rangle$ is separable. More generally, $K(X)$ may be replaced by the closed and totally bounded subsets of X. One can also show that the family of closed sets whose intersection with each closed ball is totally bounded is separable in the Attouch-Wets topology τ_{AW_d} (this is a special case of Proposition 5.1 below). There is another positive result regarding τ_{AW_d}-separability that we mention in passing. If X is a normed linear space with strongly separable dual, then the family of closed flats in X of finite codimension is separable. This follows for example from the Walkup-Wets Isometry Theorem [27], stating the Hausdorff distance between the truncation of two cones in X by the unit ball U is the Hausdorff distance between the truncation of the polar cones in X^* by the dual unit ball U^*. Separability of subspaces with respect to the Attouch-Wets topology plays a role in the theory of random sets, as Attouch and Wets [3, Theorem 5.2] have given a multivalued law of large numbers for random convex sets (actually, for random convex lower semicontinuous functions) with values in a separable subspace.

One is lead to ask: is it possible to link separability of a family $\mathcal{F}$ in the Attouch-Wets topology with the separability of truncations of its members by fixed balls in the Hausdorff metric topology? In the setting of a normed linear space, we give sufficient condition which is also necessary in the convex case. The sufficient condition can easily be reformulated to apply to a general separable metric space, and we leave this task to the reader.

Let $\mathcal{F}$ be a family of nonempty closed subsets of a separable normed linear space X. For each positive integer k, let $\mathcal{F}^k = \{A \in \mathcal{F} : A \cap \mathrm{int}\, kU \neq \emptyset\}$.

Proposition 5.1. *Let X be a separable normed linear space, and let $\mathcal{F}$ be a family of nonempty closed subsets of X.*

(a) *If for each $k \in \mathbb{N}$ for which $\mathcal{F}^k$ is nonempty, $\{A \cap kU : A \in \mathcal{F}^k\}$ is τ_{H_d}-separable, then $\mathcal{F}$ is τ_{AW_d}-separable;*

(b) *If $\mathcal{F}$ is a τ_{AW_d}-separable family of closed convex sets, then for each $k \in \mathbb{N}$ for which $\mathcal{F}^k$ is nonempty, $\{A \cap kU : A \in \mathcal{F}^k\}$ is τ_{H_d}-separable.*

Proof. We prove (a) first. There exists $k_0 \in \mathbb{N}$ such that $\mathcal{F}^{k_0}$ is nonempty. For each $k \geq k_0$ let $\{A_{kj} : j \in \mathbb{N}\} \subset \mathcal{F}^k$ be chosen so that $\{A_{kj} \cap kU : j \in \mathbb{N}\}$ is τ_{H_d}-dense in $\{A \cap kU : A \in \mathcal{F}^k\}$. Now fix $A \in \mathcal{F}$ and $n \in \mathbb{N}$, and choose $k \geq n$ such that $A \cap \mathrm{int}\, kU \neq \emptyset$.

Choosing A_{kj} with $H_d(A_{kj} \cap kU,\ A \cap kU) < 1/n$, we get

$$A \cap \operatorname{int} nU \subset A \cap kU \subset S_{1/n}[A_{kj} \cap kU] \subset S_{1/n}[A_{kj}],$$

and

$$A_{kj} \cap \operatorname{int} nU \subset A_{kj} \cap kU \subset S_{1/n}[A \cap kU] \subset S_{1/n}[A].$$

This proves that $\{A_{kj} : k \geq k_0 \text{ and } j \in \mathbb{N}\}$ is τ_{AW_d}-dense in $\mathcal{F}$.

For the partial converse (b), let $\mathcal{F}$ be a τ_{AW_d}-separable family of closed convex sets, with $\mathcal{F}_0$ a countable τ_{AW_d}-dense subset of $\mathcal{F}$. We rely on the following theorem of Beer and Lucchetti [10]: in the convex case, if $A = \tau_{AW_d}\text{-}\lim A_n$ and $B = \tau_{AW_d}\text{-}\lim B_n$ and $A \cap \operatorname{int} B \neq \emptyset$, then $A \cap B = \tau_{AW_d}\text{-}\lim(A_n \cap B_n)$. Let $A \in \mathcal{F}^k$; by density, there exists a sequence $\langle A_n \rangle$ in $\mathcal{F}_0$ with $A = \tau_{AW_d}\text{-}\lim A_n$. Clearly, $A \cap \operatorname{int} kU \neq \emptyset \Rightarrow A_n \cap \operatorname{int} kU \neq \emptyset$ eventually. With $B = B_1 = B_2 = \ldots = kU$, we get

$$A \cap kU = \tau_{AW_d}\text{-}\lim(A_n \cap kU) = \tau_{H_d}\text{-}\lim(A_n \cap kU),$$

as Attouch-Wets convergence of a sequence of bounded convex sets to a bounded limit implies its Hausdorff metric convergence. This shows that $\{A \cap kU : A \in \mathcal{F}_0^k\}$ is a countable τ_{H_d}-dense subset of $\{A \cap kU : A \in \mathcal{F}^k\}$, and the proof is complete. □

We have no idea whether convexity is really needed in the converse.

References

[1] H. Attouch, R. Lucchetti, R. Wets: The topology of the ρ-Hausdorff distance, Annali Mat. Pura. Appl. 160 (1991), 303–320.

[2] H. Attouch, R. Wets: Quantitative stability of variational systems, I. The epigraphical distance. Trans. Amer. Math. Soc. 328 (1991), 695-730.

[3] H. Attouch, R. Wets: Epigraphical processes: laws of large numbers for random lower semi-continuous functions, preprint, 1994.

[4] D. Azé, J.-P. Penot: Operations on convergent families of sets and functions, Optimization 21 (1990), 521–534.

[5] A. Barbati: A measurability criterion for closed, convex and non-empty valued multifunctions, preprint.

[6] G. Beer: Conjugate convex functions and the epi-distance topology, Proc. Amer. Math. Soc. 108 (1990), 117–126.

[7] G. Beer: Topologies on closed and closed convex sets and the Effros measurability of set valued functions, Sém. d'Anal. Convexe Montpellier (1991), exposé N° 2.

[8] G. Beer, A. Di Concilio: Uniform continuity on bounded sets and the Attouch-Wets topology, Proc. Amer. Math. Soc. 109 (1990), 427–436.

[9] G. Beer, A. Lechicki, S. Levi, S. Naimpally: Distance functionals and the suprema of hyperspace topologies, Annali Mat. Pura Appl. 162 (1992), 367–381.

[10] G. Beer, R. Lucchetti: The epi-distance topology: continuity and stability results with applications to convex optimization problems, Math. Operations Research 17 (1992), 715–726.

[11] G. Beer, R. Lucchetti: Weak topologies for the closed subsets of a metrizable space, Trans. Amer. Math. Soc. 335 (1993), 805–822.

[12] G. Beer, R. Lucchetti: Well-posed optimization problems and a new topology for the closed subsets of a metric space, Rocky Mountain J. Math. 23 (1993), 1197–1220.

[13] N. Bourbaki: Topologie générale, Chapter IX (2nd edition), Hermann, Paris, 1958.

[14] C. Castaing, M. Valadier: Convex analysis and measurable multifunctions, Lecture Notes in Math. 580, Springer-Verlag, Berlin, 1977.

[15] J.P.R. Christensen: Topology and Borel structure, North-Holland, Amsterdam, 1974.

[16] C. Constantini, S. Levi, J. Zieminska: Metrics that generate the same hyperspace convergence, Set-valued analysis 1 (1993), 141–158.

[17] B. Cornet: Topologies sur les fermés d'un espace métrique, Cahiers de mathématiques de la décision 7309, Université de Paris Dauphine, 1973.

[18] N. Dunford, J. Schwartz: Linear operators, Part I, Wiley, New York, 1958.

[19] P. Enflo: A counterexample to the approximation problem in Banach spaces, Acta Mat. 130 (1973), 309–317.

[20] R. Engelking: General topology, Heldermann Verlag, Berlin 1989.

[21] C. Hess: Quelques résultats sur la mesurabilité des multifunctions à valeurs dans un espace métrique séparable, Sém. d'Anal. Convexe Montpellier (1986), exposé N° 1.

[22] C. Hess: Loi de probabilité des ensembles aléatoires à valeurs fermées dans un espace métrique séparable, C. R. Acad. Sci. Paris 296, série I (1983), 883–886.

[23] R. Holmes: A course in optimization and best approximation, Lecture Notes in Math. 257, Springer-Verlag, New York, 1972.

[24] E. Klein, A. Thompson: Theory of correspondences, Wiley, New York, 1984.

[25] K. Kuratowski: Topology, vol. 1, Academic Press, New York, 1966.

[26] A. Lechicki, S. Levi: Wijsman convergence in the hyperspace of a metric space, Boll. Un. Mat. Ital. 5-B (1987), 435–452.

[27] D. Walkup, R. Wets: Continuity of some convex-cone valued mappings, Proc. Amer. Math. Soc. 18 (1967), 229–235.