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Examples of Convex Functions and Classifications of Normed Spaces

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We study various properties of convex functions and their connections to the structure of the spaces on which they are defined. In particular, it is shown boundedness properties of convex functions on various bornologies are related to sequential convergence in dual topologies. Convex functions whose subdifferentials have range with nonconvex interior are constructed on nonreflexive spaces, and we exhibit examples of convex functions on infinite dimensional spaces whose subdifferentials have sparse domains.

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1. Introduction

It is often important to know when convex functions are bounded on a certain class of sets, for example, to show the Moreau-Yosida approximations of a convex function converge uniformly on bounded sets it is necessary and sufficient to have the function bounded on bounded sets. In a different direction, in the study of monotone operators it is of interest to know when the interior of the range of the subdifferential of a convex function is convex. The goal of this note is to provide limiting examples for these and other properties of convex functions and their subdifferentials. As a byproduct, several

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Banach space properties will be characterized by the existence (or nonexistence) of lower semicontinuous convex functions whose subdifferentials possess certain properties.

While it may not be surprising that a Banach space is finite dimensional if and only if every continuous convex function is bounded on bounded sets, the reason this is true is not because bounded closed sets are compact, but rather because weak star and norm convergence agree sequentially only in duals of finite dimensional spaces. Indeed, if one can construct a continuous convex function on each infinite dimensional space that is unbounded on some bounded set, then one obtains the Josefson-Nissenzweig theorem as a corollary (see Lemma 2.3). Moreover, spaces on which continuous convex functions are bounded on weakly compact sets are characterized by the coincidence of Mackey and weak star convergence for sequences in the dual space, rather than by the Schur property. A complete discussion of these results and corresponding results for different bornologies is presented in section two.

In the third section, a characterization of reflexive spaces is given via convexity properties of the interior of the range of subdifferential mappings. Using recent results of Moors ([11]) and Fonf ([7]) respectively, we also observe that spaces with the Radon-Nikodým property and respectively incomplete normed spaces can be characterized in terms of the size of the range of subdifferentials of certain convex functions.

The fourth section builds on an example of Phelps ([13]) to construct examples of convex functions on infinite dimensional Banach spaces whose domains are large but have subdifferentials with relatively small domains.

We will work with real Banach spaces, and occasionally with incomplete real normed linear spaces. For a normed linear space X, its continuous dual will be denoted by X^*, we also let $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$. For a convex function f, the *domain of* f is the set $dom(f) := \{x : f(x) < \infty\}$; by convention we will not allow functions to take the value $-\infty$. The *subdifferential* of f at x is defined by

$\partial f(x) := \{\Lambda \in X^* : \langle \Lambda, y - x\rangle \leq f(y) - f(x) \text{ for all } y \in X\}$. The domain of ∂f is defined by $dom(\partial f) := \{x : \partial f(x) \neq \emptyset\}$ and the range of ∂f is defined by $R(\partial f) := \{\Lambda \in X^* : \Lambda \in \partial f(x) \text{ for some } x \in X\}$.

2. Boundedness properties of convex functions

We will call a collection β of bounded subsets of X a *bornology* if it contains the compact sets and if it is closed under addition and scalar multiplication. The topology τ_β on X^* will denote the topology of uniform convergence on β-sets. The classes of interest to us will be the bounded, weakly compact and compact sets, which in a Banach space respectively generate the norm, Mackey and bounded weak star topology on X^*, where the latter agrees with the weak star topology on bounded sets.

Before proceeding to our main results, we record the following well-known facts for completeness.

Fact 2.A. *Let X be a locally convex topological vector space. Then the following are equivalent:*

(a) *X is barrelled;*

(b) *every lsc convex function on X is continuous throughout the interior of its domain;*

(c) *every lsc seminorm on X is continuous.*

In particular, lsc convex everywhere finite functions on Banach spaces are continuous; this can be shown in a straightforward fashion using the Baire category theorem. Throughout our discussion, continuous functions will be everywhere finite.

Fact 2.B. *Let f be a continuous convex function on a normed space. Then f is bounded on β-sets if and only if ∂f is bounded on β-sets, where β is a bornology.*

Proof. First, $dom(\partial f) = X$ since f is continuous and convex, this means f is bounded below on bounded sets.

$\Rightarrow$: If ∂f is unbounded on some $W \in \beta$, we choose $\{w_n\}_{n=1}^{\infty} \subset W$ such that $\Lambda_n \in \partial f(w_n)$ and $\|\Lambda_n\| > n^2$. Choose $h_n \in X$ such that $\|h_n\| \leq \frac{1}{n}$ and $\langle \Lambda_n, h_n\rangle \geq n$. Now $K = \{h_n\}_{n=1}^{\infty} \cup \{0\}$ is compact and so $W + K \in \beta$. However, $f(w_n + h_n) \geq f(w_n) + n \geq \inf_W f + n$ and so f is unbounded on $W + K$.

$\Leftarrow$: If f is unbounded on $W \in \beta$, it must be unbounded above on W. Thus we fix $w_0 \in W$ and choose $w_n \in W$ such that $f(w_n) \to \infty$. For $\Lambda_n \in \partial f(w_n)$ we have $\langle \Lambda_n, w_0 - w_n\rangle \to -\infty$ and so $\|\Lambda_n\| \to \infty$. $\square$

Lemma 2.1. *Let β be a bornology on some Banach space X. Suppose $\|x_n^*\| = 1$, $x_n^* \xrightarrow{\tau_\beta} 0$ and let f be defined by $f(x) := \sum_{n=1}^{\infty} f_n(\langle x_n^*, x\rangle)$, where $f_n : \mathbb{R} \to [0,\infty)$ is defined by $f_n(t) := 0$ if $|t| \leq \frac{1}{2}$ and $f_n(t) := n(|t| - \frac{1}{2})$ otherwise. Then:*

(a) *f is a continuous convex function, in fact for each $W \in \beta$, the restriction of f to $W + \frac{1}{4}B_X$ is Lipschitz.*

(b) *If x_n^* does not converge to 0 in the β_1-topology on X^* for some other bornology β_1 on X, then f is unbounded on some β_1-set.*

Proof. (a) Let $W \in \beta$, choose n_0 such that $\sup_W |x_n^*| < \dfrac{1}{4}$ for all $n > n_0$. Then $f(v) = \sum_{n=1}^{n_0} f_n(\langle x_n^*, v\rangle)$ for all $v \in W + \frac{1}{4}B_X$. Because each of the functions $f_n \circ x_n^*$ is Lipschitz, this shows f is a finite sum of Lipschitz functions on $W + \frac{1}{4}B_X$. Therefore the restriction of f to $W + \frac{1}{4}B_X$ is Lipschitz.

(b) Let $W \in \beta_1$ be such that $\limsup_n\{\sup_W x_n^*\} > \epsilon$. Now choose a subsequence $\{x_{n_k}\}_k$ such that $\sup_W x_{n_k}^* > \epsilon$ for all k. Letting $W_1 = \frac{2}{\epsilon}W$ we can choose $w_k \in W_1$ such that $\langle x_{n_k}^*, w_k\rangle \geq 2$. Then $f(w_k) \geq n_k$, and so f is unbounded on W_1. $\square$

Observe that a category argument will show that one cannot have homogeneous convex functions as given by the above lemma.

Theorem 2.2. *Let X be a Banach space. The following are equivalent:*

(a) *X is finite dimensional;*

(b) *weak star and norm convergence agree sequentially in X^*;*

(c) *each continuous convex function on X is bounded on bounded sets;*

(d) *for each continuous convex function f on X, ∂f is bounded on bounded sets.*

Proof. It is clear that (a) implies (c) and the equivalence of (c) and (d) follows from Fact 2.B. If (b) does not hold, then there is a sequence in S_{X^*} that converges weak* to 0. Hence Lemma 2.1, with β the compact sets and β_1 the bounded sets, shows that (c) is not satisfied. The implication (b) $\Rightarrow$ (a) is precisely the Josefson-Nissenzweig theorem (see [6, Chapter XII]). □

To develop results for boundedness properties of convex functions on other bornologies, we will need the following lemma.

Lemma 2.3. *Suppose $f : X \to \mathbb{R}$ is continuous convex, $\{x_n\}_{n=1}^{\infty}$ is bounded and $f(x_n) \to \infty$. For n such that $f(x_n) > \inf_X f$, let $\phi_n = x_n^*/\|x_n^*\|$ where $x_n^* \in \partial f(x_n)$. Then $\phi_n \xrightarrow{\tau_\beta} 0$, if f is bounded on β-sets.*

Proof. Because $\|x_n\| \le M$ for all n and some $M > 0$, we have $\langle \phi_n, -x_n \rangle \ge -M$ for all n. Suppose ϕ_n does not converge to 0 in the τ_β-topology. Then by passing to a subsequence if necessary, we can find $W \in \beta$ such that $\sup_W \phi_n > \epsilon$ for all n. Let $W_1 = \frac{M}{\epsilon} W$ and fix $w_n \in W_1$ such that $\langle \phi_n, w_n \rangle \ge M$. Consequently $\langle \phi_n, w_n - x_n \rangle \ge 0$ for all n and so $\langle x_n^*, w_n - x_n \rangle \ge 0$ for all n. Therefore

$$f(w_n) \ge f(x_n) + \langle x_n^*, w_n - x_n \rangle \ge f(x_n) \quad \text{for all} \quad n.$$

Since $f(x_n) \to \infty$, this contradicts the boundedness of f on W_1. □

Remark. Observe that Lemma 2.3 justifies the use of the highly nontrivial Josefson-Nissenzweig theorem in the proof of Theorem 2.2. Indeed, suppose f is a continuous convex function on a Banach space that is unbounded on a bounded set. Because f is continuous, it is bounded on compact sets. Thus we may apply Lemma 2.3, with β the compact sets, to produce a sequence in S_{X^*} that converges weak star to 0.

The following theorem, in particular, shows each continuous convex function on an Asplund space (see [14]) is bounded on bounded sets provided it is bounded on weakly compact sets.

Theorem 2.4. *For a Banach space X, the following are equivalent.*

(a) *$X \not\supset \ell_1$.*

(b) *Mackey and norm convergence agree sequentially in X^*.*

(c) *Every continuous convex function on X that is bounded on weakly compact sets is bounded on bounded sets.*

(d) *∂f is bounded on bounded sets for each continuous convex function f with ∂f bounded on weakly compact sets.*

Proof. The equivalence of (a) and (b) was shown by Ørno ([12]); see also [1, Theorem 5]. It follows from Fact 2.B that (c) and (d) are equivalent.

(b) $\Rightarrow$ (c): We suppose (b) holds and that f is a continuous convex function satisfying $f(x_n) \to \infty$ where $\{x_n\}_{n=1}^{\infty}$ is bounded. Then $\{\phi_n\}_n$ given by Lemma 2.3 does not converge in norm to 0. Therefore it does not converge Mackey to 0 by the hypothesis. Consequently Lemma 2.3, with β the weakly compact sets, shows f is unbounded on some weakly compact set.

(c) $\Rightarrow$ (b): Suppose (b) fails. Then there is a sequence $\{x_n^*\}_{n=1}^{\infty} \subset S_{X^*}$ that converges Mackey to 0. Using this sequence with β the weakly compact sets and β_1 the bounded sets in Lemma 2.1, shows that there is a continuous convex function bounded on weakly compact sets but unbounded on some bounded set; that is (c) fails. □

A Banach space is said to have the *Schur property* if its weakly compact sets are norm compact. The principal example of a space with the Schur property is $\ell_1(\Gamma)$ for any set Γ.

Theorem 2.5. *A Banach space X has the Schur property if and only if every lsc convex function on X is bounded on weakly compact subsets of the interior of its domain.*

Proof. A lsc convex function is continuous on the interior of its domain (see Fact 2.A) and thus if X has the Schur property such a function is bounded on weakly compact sets of its domain. Conversely, suppose X does not have the Schur property, then there is a separable subspace $Z \subset X$ which fails to be Schur. Using the w^*-sequential compactness of B_{Z^*}, it is not hard to construct $\{z_n, z_n^*\}_{n=1}^{\infty} \subset Z \times Z^*$ such that $z_n \overset{w}{\to} 0$, $z_n^* \overset{w^*}{\to} 0$, $\|z_n^*\| = 1$ while $\langle z_n^*, z_n \rangle = 1$ for all n (see [2, Theorem 3.4]). Let $\tilde{z}_n^*$ be a Hahn-Banach extension of z_n^* to all of X. Define $f(x) := \sum_{n=1}^{\infty} f_n(\langle \tilde{z}_n^*, x \rangle)$ where the f_n's are as in Lemma 2.1. Now f is unbounded on the weakly compact set $\{z_n\}_{n=1}^{\infty} \cup \{0\}$. On the other hand arguing as in Lemma 2.1(a), one can check f is continuous on a set containing $Z + \frac{1}{4} B_X$ (or one may directly check f is finite on this set and apply Fact 2.A). □

It turns out that the Schur property does not characterize the spaces for which everywhere defined continuous convex functions are bounded on weakly compact sets. To see this, we will need the following lemma.

Lemma 2.6. *Suppose $f : X \to \mathbb{R}$ is continuous and convex, $\{x_n\}_{n=1}^{\infty}$ is bounded and $f(x_n) \to \infty$. For n such that $f(x_n) > \min_X f$, we let $\phi_n = x_n^* / \|x_n^*\|$ where $x_n^* \in \partial f(x_n)$. Then $\limsup_{n\to\infty} \langle \phi_n, x_n \rangle > 0$.*

Proof. By Lemma 2.3 we know $\phi_n \overset{w^*}{\to} 0$, and thus $\{\phi_n\}_{n=1}^{\infty}$ is not relatively norm compact. Hence by passing to a subsequence, we may assume there exists $\epsilon > 0$ such that

$d(\phi_n, E_{n-1}) > \epsilon$ where $E_{n-1} = span\{\phi_k : 1 \leq k \leq n-1\}$. To see this, observe if $\limsup_{n\to\infty}\{\sup_k d(\phi_k, E_n)\} = 0$, then $\{\phi_n\}_{n=1}^{\infty}$ would have a finite ϵ-net for each $\epsilon > 0$ because it is a bounded set and because finite dimensional spaces have norm compact balls. This would contradict the fact $\{\phi_n\}_{n=1}^{\infty}$ is not relatively norm compact.

Replacing ϕ_n with $\frac{\phi_n}{\epsilon}$ we have

$$d(\phi_n, E_{n-1}) > 1 \quad \text{for all} \quad n. \tag{2.1}$$

We suppose $\limsup_{n\to\infty}\langle \phi_n, x_n\rangle \leq 0$, since otherwise there is nothing more to show. By passing to a further subsequence we may assume

$$\langle \phi_n, x_n\rangle < \frac{1}{2^n} \quad \text{for all} \quad n;$$

clearly (2.1) still holds for this subsequence. By induction we can choose $\{h_k\}_{k=1}^{\infty} \subset 2B_X$ and a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ such that:

$$\langle \phi_{n_k}, h_j\rangle = 0 \quad \text{for} \quad j \geq k+1; \tag{2.2}$$

$$\langle \phi_{n_k}, h_k\rangle = 2; \tag{2.3}$$

$$|\langle \phi_{n_k}, h_j\rangle| \leq \frac{1}{2^k} \quad \text{for} \quad j \leq k-1. \tag{2.4}$$

Indeed, supposing ϕ_{n_j}, h_j have been chosen for $j \leq k-1$, by w^*-convergence we choose $n_k > n_{k-1}$ such that $|\langle \phi_{n_k}, h_j\rangle| \leq \frac{1}{2^k}$ for all $j \leq k-1$. By (2.1), $\phi_{n_k} \notin E_{n_k-1} + B_{X^*}$ which is weak* closed because it is the sum of a weak* compact ball with a weak* closed subspace. Thus we may choose $h_k \in S_X$ separating $E_{n_k-1} + B_{X^*}$ and ϕ_{n_k}. Multiplying h_k by an appropriate scalar with absolute value not exceeding 2, we have $\langle \phi_j, h_k\rangle = 0$ for $j \leq n_k - 1$ and $\langle \phi_{n_k}, h_k\rangle = 2$. Now let $h = \sum_{k=1}^{\infty} 2^{-k} h_k$. Then

$$\langle \phi_{n_k}, h\rangle = \langle \phi_{n_k}, 2^{-k}h_k\rangle + \sum_{j<k} 2^{-j}\langle \phi_{n_k}, h_j\rangle + \sum_{j>k} 2^{-j}\langle \phi_{n_k}, h_j\rangle. \tag{2.5}$$

Now, (2.2) implies $\sum_{j>k} 2^{-j}\langle \phi_{n_k}, h_j\rangle = 0$, while (2.4) implies

$$\sum_{j<k} 2^{-j}\langle \phi_{n_k}, h_j\rangle \geq \sum_{j<k} 2^{-j}(-2^{-k}) > -2^{-k}.$$

Combining this with (2.3) and (2.5) yields

$$\langle \phi_{n_k}, h\rangle > 2 \cdot 2^{-k} - 2^{-k} = 2^{-k}.$$

Thus $\langle \phi_{n_k}, h - x_{n_k}\rangle > 0$ and hence $\langle x_{n_k}^*, h - x_{n_k}\rangle > 0$ as well. Consequently

$$f(h) \geq f(x_{n_k}) + \langle x_{n_k}^*, h - x_{n_k}\rangle \geq f(x_{n_k}).$$

Because $f(x_k) \to \infty$, this contradicts the fact f is finite valued. $\square$

Since ℓ_∞ is a Grothendieck space with the Dunford-Pettis property (see [6, p. 103, 113]), it follows that weak star and Mackey convergence agree sequentially in ℓ_∞^*. Therefore, the next theorem shows ℓ_∞ is a space that does not have the Schur property and yet each continuous convex function is bounded on weakly compact sets.

Theorem 2.7. *Let X be a Banach space. The following are equivalent:*

(a) *weak star and Mackey convergence agree sequentially in X^*;*

(b) *each continuous convex function on X is bounded on weakly compact subsets of X;*

(c) *∂f is bounded on weakly compact sets for each continuous convex f.*

Proof. Fact 2.B shows (b) and (c) are equivalent. To prove (a) implies (b) we suppose there is a continuous convex function that is not bounded on some weakly compact set W. We choose $\{x_n\}_{n=1}^{\infty} \subset W$ such that $f(x_n) \to \infty$. For ϕ_n as in Lemma 2.3, we have that $\phi_n \xrightarrow{w^*} 0$ by Lemma 2.3. However, by Lemma 2.6, $\langle \phi_n, x_n \rangle \not\to 0$ and thus ϕ_n does not converge Mackey to 0. The implication (b) $\Rightarrow$ (a) follows from Lemma 2.1. $\square$

More generally, using Lemmas 2.1, 2.3 and 2.6 as in the above theorem, one can prove the following result (compare also with Theorems 2.2 and 2.4).

Theorem 2.8. *For a Banach space X with bornologies $\beta \subset \beta_1$, the following are equivalent.*

(a) *Each continuous convex function bounded on β-sets is bounded on β_1-sets.*

(b) *τ_β and τ_{β_1} convergence agree sequentially in X^*.*

We refer the reader to [1] for additional characterizations (in terms of differentiability) of the dual sequential convergence notions studied in this section.

Remark. One can also form dual versions of the results just discussed. For example, a dual version of Theorem 2.2 is: *A Banach space X has the Schur property if and only if every continuous weak*-lsc convex function on X^* is bounded on bounded sets.* A dual version of Theorem 2.7 is: *A Banach space X has the Dunford-Pettis property if and only if every continuous weak*-lsc convex function on X^* is bounded on weakly compact sets.* A straightforward variation of Lemma 2.3 in which the x_n^*'s there are ϵ-subgradients (any fixed $\epsilon > 0$ will do) can be used to prove these results. Indeed, w^*-lsc, continuous convex functions on X^* have ϵ-subgradients from X at each point in X^*, therefore the dual proofs can now be performed as the originals without difficulty.

3. Properties of the range of the subdifferential mapping

We begin with a result that will allow us to characterize reflexive spaces via various properties of the range of the subdifferential mapping, and more generally properties of maximal monotone operators. We shall say a function f is *coercive* if $\lim_{\|x\| \to \infty} \dfrac{f(x)}{\|x\|} = \infty$.

Theorem 3.1. *A Banach space X is reflexive if the interior of $R(\partial f)$ is convex for each coercive continuous convex function f on X.*

Proof. Suppose X is nonreflexive and $p \in X$ with $\|p\| = 5$ and $p^* \in Jp$ where J is the duality map, the subdifferential of $\frac{1}{2}\|\cdot\|^2$. Define

$$f(x) := \max\left\{\frac{1}{2}\|x\|^2, \quad \|x-p\| - 12 + \langle p^*, x\rangle, \quad \|x+p\| - 12 - \langle p^*, x\rangle\right\}$$

for $x \in X$.

For a function g defined by $g := \max\{\phi_j : 1 \leq j \leq n\}$, where ϕ_j are continuous convex functions, one has $\partial g(x) = conv \bigcup\{\partial\phi_j(x) : \phi_j(x) = g(x)\}$. Thus we have

$$\partial f(p) = B_{X^*} + p^*, \quad \partial f(-p) = B_{X^*} - p^*, \quad \partial f(x) = Jx \text{ for } x \in B_X \tag{3.1}$$

using inequalities like $\|p-p\| - 12 + \langle p^*, p\rangle = 13 > \frac{25}{2} = \frac{1}{2}\|p\|^2$. Moreover, $f(0) = 0$ and $f(x) > \frac{1}{2}\|x\|$ for $\|x\| > 1$, thus $\|\Lambda\| > \frac{1}{2}$ if $\Lambda \in \partial f(x)$ and $\|x\| > 1$. Combining this with (3.1) shows $R(\partial f) \cap \frac{1}{2}B_{X^*} = R(J) \cap \frac{1}{2}B_{X^*}$.

Let U_{X^*} denote the open ball in X^*. Now James' theorem gives us points $x^* \in \frac{1}{2}U_{X^*} \setminus R(J)$, thus $U_{X^*} \setminus R(\partial f) \neq \emptyset$. However from (3.1)

$$U_{X^*} \subset conv((p^* + U_{X^*}) \cup (-p^* + U_{X^*})) \subset conv\, \mathrm{int} R(\partial f)$$

so $R(\partial f)$ has nonconvex interior. □

Corollary 3.2. *A normed linear space X is reflexive if and only if every continuous convex function f on X has $intR(\partial f)$ convex.*

Proof. If X is reflexive, then $R(\partial f) = dom(\partial f^*)$ and $int\, dom(\partial f^*) = int\, dom(f^*)$ which is convex. For the converse, observe that the proof of Theorem 3.1 applies to any normed linear space for which $\frac{1}{2}U_{X^*} \setminus J(X) \neq \emptyset$. Let $\widetilde{X}$ denote the completion of X. We have $J(X) \subset J(\widetilde{X})$. Hence, if $\widetilde{X}$ is not reflexive, then by James' theorem $\frac{1}{2}U_{X^*} \setminus J(\widetilde{X}) \neq \emptyset$ and so $\frac{1}{2}U_{X^*} \setminus J(X) \neq \emptyset$. If $\widetilde{X}$ is reflexive, then we renorm $\widetilde{X}$ with a strictly convex norm (see e.g. [5, Section VII.2]) so that $J(x) \cap J(y) = \emptyset$ for $x \neq y$ where J is the duality mapping with respect to the strictly convex norm. Thus, $\frac{1}{2}U_{X^*} \setminus J(X) \neq \emptyset$ in this case as well. □

Remark. Suppose X is reflexive and f is a continuous, convex and coercive function. Then for any $\phi \in X^*$, $f - \phi$ has weakly compact level sets and thus attains its minimum. Consequently, $\phi \in R(\partial f)$, which shows $R(\partial f) = X^*$. On the other hand, Calvert and Fitzpatrick showed $R(\partial f) \neq X^*$ if a Banach space X is nonreflexive and f is continuous, convex and coercive ([4]).

Let us recall a maximal monotone operator T mapping X to subsets of X^* is said to be *coercive* provided there is a function $c : [0, \infty) \to (-\infty, \infty)$ such that $c(r) \to \infty$ when

$r \to \infty$ and $\langle x^*, x\rangle \geq c(\|x\|)\|x\|$ for each $(x, x^*) \in G(T)$. Combining Corollary 3.2 and the above remark with some properties of maximal monotone operators (see [16]), we obtain the following theorem.

Theorem 3.3. *Let X be a Banach space. The following are equivalent:*

(a) *X is reflexive;*

(b) *$R(\partial f) = X^*$ for some coercive continuous convex function f on X;*

(c) *$R(\partial f) = X^*$ for each coercive continuous convex function f on X;*

(d) *$R(T) = X^*$ for each coercive maximal monotone operator T on X;*

(e) *the interior of $R(\partial f)$ is convex for each coercive continuous convex function f on X;*

(f) *the interior of $R(T)$ is convex for each coercive maximal monotone operator T on X;*

(g) *the interior of $R(\partial f)$ is convex for each continuous convex function f on X;*

(h) *the interior of $R(T)$ is convex for each maximal monotone operator T on X.*

We do not know if X is reflexive provided $R(\partial f)$ has nonempty interior for each coercive continuous convex function f on X.

Recall that a Banach space is said to have the *Radon-Nikodým property* (RNP) if each bounded subset is dentable; see [3]. We will say a set is *generic* if it contains a dense G_δ-set.

Theorem 3.4. *A Banach space has the RNP if and only if $R(\partial f)$ is generic for each coercive convex lsc function f.*

Proof. Suppose X has the RNP and f is a coercive, convex, lsc function. Then $dom(f^*) = X^*$. Because X has the RNP, Collier's result ([3, Theorem 5.7.4]) shows f^* is Fréchet differentiable on a dense G_δ set—which is contained in $R(\partial f)$. To prove the converse, we suppose X does not have the RNP. By a recent result of Moors ([11, Theorem 4.4]), there is a norm $\|\cdot\|$ on X such that for its dual norm $\|\cdot\|^*$ on X^*, the set $S = \{\phi \in X^* : \partial\|\cdot\|^*(\phi) \cap X \neq \emptyset\}$ is not generic. Now define f by $f(x) = \frac{1}{2}\|x\|^2$. If $\phi \in R(\partial f)$, then there is an $x \in S_X$ such that $\langle \phi, x\rangle = \|\phi\|$. Thus $R(\partial f) \subset S$, and so $R(\partial f)$ cannot be generic. □

For general normed linear spaces we have the following result.

Theorem 3.5. *Let X be a normed linear space. The following are equivalent:*

(a) *X is a Banach space;*

(b) *∂f is maximal monotone for each lsc proper convex f;*

(c) *∂f is maximal monotone whenever $R(\partial f) \neq \emptyset$ and f is a proper lsc convex function;*

(d) *$dom(\partial f)$ is dense in $dom(f)$ for each proper lsc convex f;*

(e) *$R(\partial f) \neq \emptyset$ for each proper lsc convex f.*

Proof. Rockafellar's theorem ([14, Theorem 3.25]) shows (a) implies (b) while the Bronsted-Rockafellar theorem ([14, Theorem 3.18]) shows (a) implies (d). Both (b) implies (c), and (d) implies (e) are trivial.

(c) $\Rightarrow$ (a): Let X be an incomplete normed space. Then X has a separable incomplete subspace Z that is closed as a subset of X. By a recent result of Fonf ([7]) there is a closed bounded convex nonempty subset C of Z with no support points. Define g to be the indicator function of C, so g is 0 on C and ∞ elsewhere. Then g is lsc, because C is a closed subset of X. Now ∂g is not maximal monotone. Indeed, $\partial g(x) = Z^\perp$ for $x \in C$ and $\partial g(x) = \emptyset$ elsewhere. There is a proper maximal monotone extension of ∂g by taking $Tz = Z^\perp$ for all $z \in Z$.

(e) $\Rightarrow$ (a): Let Z and C be as above and let $y \in Z \setminus \{0\}$. Define $h(x) := \inf\{t : x + ty \in C\}$ for $x \in X$. As usual we take $\inf \emptyset = \infty$. Then h is a lsc, proper and convex function, and the inf is attained when it is finite. Suppose $x^* \in \partial h(x)$. Then for $z \in C$ we have $h(z) \le 0$ and $h(z - h(x)y) \le h(x)$. Now

$$\langle x^*, z - h(x)y - x\rangle \le h(z - h(x)y) - h(x) \le 0$$

and $x + h(x)y \in C$ so x^* supports C at $x + h(x)y$. Thus $x^* \in Z^\perp$.

On the other hand $h(x+y) = h(x) - 1$ so $\langle x^*, y\rangle \le h(x+y) - h(x) = -1$ and since $y \in Z$ we obtain the contradiction $x^* \notin Z^\perp$. Therefore $\partial h(x) = \emptyset$ for all x. □

4. Functions whose subdifferentials have small domain

We just saw that Fonf's result implies there is a proper lsc convex function f on each incomplete space with $dom(\partial f) = \emptyset$. In general, $dom(\partial f)$ is not convex on spaces of dimension 2 or larger: indeed, one can consider $f(x, y) = \max\{|x|, 1 - \sqrt{y}\}$ on $\mathbb{R}^2$. Our next examples show that although $dom(\partial f)$ is dense in $dom(f)$ in Banach spaces, in other ways $dom(\partial f)$ can be quite small. These examples were motivated by an example of Phelps ([13, Example 3]) and they build on the techniques therein. In what follows, we will say X has a separable quotient, if it has a separable infinite dimensional quotient.

Example 4.1. Let X be a Banach space with a separable quotient. Then there exist proper lsc convex functions f and g such that:

(a) $dom(f) = dom(g)$ and $dom(f)$ is dense in X;

(b) ∂f and ∂g are at most single-valued;

(c) $dom(\partial f) \cap dom(\partial g) = \emptyset$.

Proof. Let Z be a subspace of X such that X/Z is separable and infinite dimensional. According to [10, Proposition 1.f.3], X/Z admits a Markushevich basis, thus we can find a biorthogonal collection $\{x_n, x_n^*\}_{n=1}^\infty$ such that $x_n^* \in Z^\perp$, $\|x_n\| = 1$ for all n and $span(\{x_n\}_{n=1}^\infty \cup Z)$ is norm dense in X. Now define f and g as follows:

$$f(x) := \sum_{n=1}^{\infty} (n\langle x_n^*, x\rangle)^2 \quad \text{and} \quad g(x) := f(x - y) \;\; \text{where } y = \sum_{n=1}^{\infty} n^{-\frac{7}{4}} x_n.$$

Let us begin by verifying (a). First, $span(\{x_n\}_{n=1}^\infty \cup Z) \subset dom(f)$ and so $dom(f)$ is dense in X. If $x \in dom(f)$, then

$$f(x \pm y) = \sum_{n=1}^{\infty} n^2(\langle x_n^*, x\rangle \pm n^{-\frac{7}{4}})^2 \leq \sum_{n=1}^{\infty} n^2(2\langle x_n^*, x\rangle^2 + 2n^{-\frac{7}{2}}) < \infty. \tag{4.1}$$

It follows from (4.1) that $x - y \in dom(f)$ if and only if $x \in dom(f)$; consequently $dom(f) = dom(g)$.

For (b) it suffices to check ∂f is at most single-valued. Suppose $\Lambda_1,\ \Lambda_2 \in \partial f(u)$ for some $u \in X$. Now for $z \in Z$ and x_n fixed,

$$f(u + t(x_n + z)) = f(u) + n^2(2t\langle x_n^*, u\rangle + t^2).$$

This is a differentiable function in t and hence $\Lambda_1,\ \Lambda_2$ agree on $t(x_n + z)$. Because $span(\{x_n\}_{n=1}^{\infty} \cup Z)$ is norm dense in X, this means $\Lambda_1 = \Lambda_2$.

To prove (c) we first observe that $x \notin dom(\partial f)$ if there is a $C > 0$ such that

$$|\langle x_n^*, x\rangle| > Cn^{-\frac{7}{4}} \quad \text{for infinitely many } n. \tag{4.2}$$

Indeed, suppose $x \in dom(f)$ satisfies (4.2). Let m be such that $|\langle x_m^*, x\rangle| > Cm^{-\frac{7}{4}}$, and let $h_m = \langle x_m^*, x\rangle x_m$. Then $\|h_m\| = |\langle x_m^*, x\rangle|$ and $\langle x_m^*, x - h_m\rangle = 0$ and thus

$$\begin{aligned}\frac{f(x - h_m) - f(x)}{\|h_m\|} &= \frac{\sum_{n\neq m}(n\langle x_n^*, x\rangle)^2 - \sum_{n=1}^{\infty}(n\langle x_n^*, x\rangle)^2}{\|h_m\|} \\ &= \frac{-m^2\langle x_m^*, x\rangle^2}{\|h_m\|} \leq -Cm^{\frac{1}{4}}.\end{aligned}$$

Hence $x \notin dom(\partial f)$ provided (4.2) holds. Finally if $x \in dom(\partial f)$, then (4.2) fails to hold, and it is easy to see $x - y$ satisfies (4.2) and so $x \notin dom(\partial g)$. $\square$

We recall that $x \in K$ is said to be a *proper support* point of the closed convex set K if there exists $\Lambda \in X^*$ such that $\langle \Lambda, x\rangle = \inf_K \Lambda < \sup_K \Lambda$. The next example contrasts the distinction between nonsupport points and interior.

Example 4.2. If X is a Banach space having a separable quotient, then there is a lsc convex function f on X whose domain K is densely spanning and $dom(\partial f)$ is contained in the proper support points of K.

Proof. Choose $\{x_n, x_n^*\}_{n=1}^{\infty}$ and Z as in Example 4.1. Let $K = \cap_{n=1}^{\infty}\{x : 0 \leq \langle x_n^*, x\rangle \leq 4^{-n}\}$. Then K is densely spanning and we let δ_K be the indicator function of K. Now we define f by

$$f(x) := \delta_K(x) + \sum_{n=1}^{\infty} 2^n\langle x_n^*, x\rangle.$$

We claim $\langle x_n^*, \bar{x}\rangle = 0$ eventually provided $\bar{x} \in dom(\partial f)$. Indeed, if $\langle x_{n'}^*, \bar{x}\rangle = \epsilon_{n'}$ for some subsequence, then $f(\bar{x} - \epsilon_{n'}x_{n'}) - f(\bar{x}) \leq -2^{n'}\epsilon_{n'}$; hence in this case $\partial f(\bar{x}) = \emptyset$.

Now, if $\bar{x} \in K$ satisfies $\langle x^*_{n_0}, \bar{x}\rangle = 0$, then $\bar{x}$ is a proper support point of K because $\bar{k} = \bar{x} + 4^{-n_0} x_{n_0} \in K$ and $\langle x^*_{n_0}, \bar{k}\rangle > 0$. □

Remark. (a) We do not know if functions as in the above examples can be constructed in every infinite dimensional Banach space. However, if the assumption of densely spanning domains is dropped, then of course one can construct such examples on every infinite dimensional separable subspace of a Banach space. Moreover, it is still unknown whether every Banach space has a separable quotient. Classes of spaces that are known to have separable quotients are weakly compactly generated spaces, and more generally spaces with Markushevich bases. The space ℓ_∞ is an example of a space with a separable quotient but no Markushevich basis (see [8, 10]).

(b) If a space admits an uncountable biorthogonal system, then it has a closed convex set consisting only of proper support points ([9]) and it is still unknown whether such a set can be constructed in each nonseparable Banach space. However, no such set exists in a separable space and so Example 4.2 is not redundant. Moreover, it is not clear how to construct densely spanning support sets in nonseparable spaces when densely spanning biorthogonal systems do not exist.

The previous two examples provide us with the following characterization of finite dimensional Banach spaces. Let us recall that the *relative interior* of a convex set is its interior relative to its affine hull; see [15, Section 6].

Theorem 4.3. *For a Banach space X, the following are equivalent.*

(a) *X is finite dimensional.*

(b) *$dom(\partial f) \cap dom(\partial g) \neq \emptyset$ whenever f, g are proper lsc and $dom(f) = dom(g)$.*

(c) *For every lsc convex function f with compact domain K, $dom(\partial f)$ is not contained in the proper support points of K.*

Proof. (a) $\Rightarrow$ (b): This is well-known because convex sets have nonempty relative interior in $\mathbb{R}^n$ and the relative interior of the domain of a convex function is contained in the domain of its subdifferential; see [15, Theorems 6.2 and 23.4].

(a) $\Rightarrow$ (c): This is also well-known because points in the relative interior of $dom(f)$ are not proper support points of $dom(f)$.

(b) $\Rightarrow$ (a): If X is infinite dimensional, then it contains a separable subspace E so we may apply Example 4.1 on E.

(c) $\Rightarrow$ (a): If X is infinite dimensional, then we consider a separable subspace E with a Schauder basis $\{x_n, x^*_n\}_{n=1}^\infty$ (see [6, p. 39]). Using this basis in place of the biorthogonal system in the proof of Example 4.2, we see that the domain K of the function f constructed there is compact. □

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