#A12 INTEGERS 9 (2009), 139-140

A NOTE ON THE Q-BINOMIAL RATIONAL ROOT THEOREM

Ying-Jie Lin[1]
Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China
yingjie6183032@163.com

Abstract

We show that a theorem obtained by K. R. Slavin can be easily deduced from the q-Lucas theorem.

Received: 11/19/08, Revised: 2/23/09, Accepted: 3/3/09

1. The Main Result

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \displaystyle\prod_{j=1}^{k} \frac{1-q^{n-j+1}}{1-q^j}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Slavin [3] proved the following q-binomial rational root theorem and used it to derive other new product theorems.

Theorem 1 (q-binomial rational root). *For $k, m, n \in \mathbb{Z}$, $n > 0$ and $0 \le k \le n$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{e^{-2i\pi m/n}} = \begin{cases} \dbinom{(m,n)}{(m,n)k/n} & \text{if } n \mid km, \\ 0, & \text{otherwise,} \end{cases}$$

where $i^2 = -1$ and (m, n) is the greatest common divisor of m and n.

Slavin's proof of Theorem 1 is quite long. In this note, we give a very short proof of Theorem 1 by using the q-Lucas theorem (see [4, Eq.(1.2.4)] or [1, 2]).

Theorem 2 (q-Lucas). *Let n, k, d be positive integers, and write $n = ad + b$ and $k = rd + s$, where $0 \le b, s \le d - 1$. Let ω be a primitive d-th root of unity. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_\omega .$$

[1]This work was sponsored by Shanghai Educational Development Foundation under the Chenguang Project 2007CG29.

Proof of Theorem 1. Let $\omega = e^{-2i\pi m/n}$. Suppose that ω is a primitive d-th root of unity. Then

$$d = \frac{n}{(m,n)}.$$

By the q-Lucas theorem, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega = \binom{(m,n)}{r} \begin{bmatrix} 0 \\ s \end{bmatrix}_\omega,$$

where $k = rd + s$ and $0 \le s \le d-1$.

If $n \mid km$ then $n \mid k(m,n)$ and $d = n/(m,n) \mid k$, so

$$r = \frac{k}{d} = \frac{(m,n)k}{n}, \quad \text{and} \quad s = 0.$$

Otherwise, $d \nmid k$ and $s > 0$. Since $\left[\begin{smallmatrix} 0 \\ s \end{smallmatrix}\right]_\omega$ is equal to 1 if $s = 0$ and 0 if $s > 0$, this completes the proof. □

References

[1] J. Désarménien, Un analogue des congruences de Kummer pour les q-nombres d'Euler, European J. Combin. **3** (1982), 19–28.

[2] V.J.W. Guo, J. Zeng, Some arithmetic properties of the q-Euler numbers and q-Salie numbers, European. J. Combin **27** (2006), 884–895.

[3] K.R. Slavin, q-binomials and the greatest common divisor, Integers **8** (2008), #A05.

[4] G. Olive, Generalized powers, Amer. Math. Monthly **72** (1965), 619–627.