TWO GENERALIZED CONSTANTS RELATED TO ZERO-SUM PROBLEMS FOR TWO SPECIAL SETS

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Abstract

Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{Z}_n$ be such that A is non–empty and does not contain 0. Adhikari et al proposed two generalized constants related to the zero-sum problem. One is $D_A(n)$, which denotes the least natural number k such that for any sequence $(x_1, \cdots, x_k) \in \mathbb{Z}^k$, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $(a_1, \cdots, a_l) \in A^l$ such that $\sum_{i=1}^{l} a_i x_{j_i} \equiv 0 \pmod{n}$. The other is $E_A(n)$, defined as the smallest $t \in \mathbb{N}$ such that for all sequences $(x_1, \ldots, x_t) \in \mathbb{Z}^t$, there exist indices $j_1, \ldots, j_n \in \mathbb{N}, 1 \le j_1 < \cdots < j_n \le t$ and $(\vartheta_1, \cdots, \vartheta_n) \in A^n$ with $\sum_{i=1}^{n} \vartheta_i x_{j_i} \equiv 0 \pmod{n}$. S. D. Adhikari et al proposed characterizing any other sets for which $E_A(n) = n + 1$ or even those for which $E_A(n) = n + j$ for specific small values of j. In this paper we give two kinds of sets, calculate $D_A(n)$ and $E_A(n)$ for these sets, and partially solve Adhikari's problem.

1. Introduction

Let G be an additive finite abelian group. A finite sequence $S = (g_1, g_2, \cdots, g_l) = g_1 g_2 \cdots g_l$ of elements of G, where repetition of elements is allowed and their order is disregarded, is called a zero-sum sequence if $g_1 + g_2 + \cdots + g_l = 0$.

For a finite abelian group G of cardinality n, the Davenport constant $D(G)$ is the smallest natural number t such that any sequence of t elements in G has a non-empty zero-sum subsequence. Another interesting constant, $E(G)$, is the smallest natural number k such that any sequence of k elements in G has a zero-sum subsequence of length n.

For the particular group $\mathbb{Z}_n$, the following generalization of $E(G)$ has been considered in [1] and [2] recently. Let $n \in \mathbb{N}$ and assume $A \subseteq \mathbb{Z}_n$. Then $E_A(n)$ is the least $t \in \mathbb{N}$ such that for all sequences $(x_1, \ldots, x_t) \in \mathbb{Z}^t$, there exist indices $j_1, \ldots, j_n \in \mathbb{N}, 1 \le j_1 <$

$\cdots < j_n \leq t$ and $(\vartheta_1, \cdots, \vartheta_n) \in A^n$ with

$$\sum_{i=1}^{n} \vartheta_i x_{j_i} \equiv 0 \pmod{n}.$$

To avoid trivial cases, one assumes that the weight set A does not contain 0 and it is non–empty.

Similarly, for any such set $A \subseteq \mathbb{Z}_n \setminus \{0\}$ of weights, we define the Davenport constant of $\mathbb{Z}_n$ with weight A, denoted by $D_A(n)$, as the least natural number k such that for any sequence $(x_1, \cdots, x_k) \in \mathbb{Z}^k$, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $(a_1, \cdots, a_l) \in A^l$ such that

$$\sum_{i=1}^{l} a_i x_{j_i} \equiv 0 \pmod{n}.$$

Thus, for the group $G = \mathbb{Z}_n$, if we take $A = \{1\}$, then $E_A(n)$ and $D_A(n)$ are, respectively, $E(G)$ and $D(G)$ as defined earlier.

$E_A(n)$ and $D_A(n)$ were studied in [1], [2] and [3].

It is not difficult to observe the following result.

Lemma 1 *$D_A(n) + n - 1 \leq E_A(n) \leq 2n - 1$ for any $A \subseteq \mathbb{Z}_n \setminus \{0\}$.*

Lemma 2 ([2]) *Let $A = \mathbb{Z}_n \backslash \{0\}$. Then $E_A(n) = n + 1$.*

In [2] Adhikari et al proposed characterizing any other sets for which $E_A(n) = n + 1$ or even those for which $E_A(n) = n + j$ for specific small values of j. It is easy to see that if $A \subseteq B$, then $D_A(n) \geq D_B(n)$.

In this paper we prove the following results:

Theorem 1 *Let n be a positive integer and p be a prime satisfying $p^k \| n$. If $A = \{a \mid a \not\equiv 0 \pmod{p}\}$, then $D_A(n) = k + 1$ and $E_A(n) = n + k$.*

Theorem 2 *If A is an arithmetic progression with length $l = \lceil \frac{n}{2} \rceil$, where for any real number x, $\lceil x \rceil$ denotes the smallest integer $\geq x$, and common difference 1, that is, A is the set of the form $\{a + i | i = 1, 2, \cdots, l\}$ where $1 \leq a < a + l \leq n - 1$, then $D_A(n) = 2$, $E_A(n) = n + 1$.*

2. Proofs of Theorem 1 and Theorem 2

In order to prove the theorems, we need the following result.

Lemma 3 ([4]) *Let A, B be subsets of a finite group G such that $|A| + |B| \geq |G| + 1$. Then $A + B = G$.*

Proof of Theorem 1. (1) We will prove that $D_A(n) = k + 1$.

First, we prove that $D_A(n) > k$. We assert that $0 \notin \sum_{i\in I} \frac{n}{p^i} A$ with $I \subseteq \{1, 2, \cdots, k\}$. We proceed by induction on the cardinality of I. Note that for $|I| = 1$, the result follows trivially. Inductively, assume the result holds true for $1 \leq |I| < k$. Now consider $|I| = k$. If $0 \in \sum_{i=1}^{k} \frac{n}{p^i} A$, then there must exist $a_i \in A$ for $i = 1, 2, \cdots, k$ such that

$$\frac{n}{p}a_1 + \frac{n}{p^2}a_2 + \cdots + \frac{n}{p^k}a_k \equiv 0 \pmod{n}.$$

Multiplying the above equation by p, we get

$$\frac{n}{p}a_2 + \frac{n}{p^2}a_3 + \cdots + \frac{n}{p^{k-1}}a_k \equiv 0 \pmod{n}.$$

Hence $0 \in \frac{n}{p}A + \frac{n}{p^2}A + \cdots + \frac{n}{p^{k-1}}A$, which contradicts to the inductive hypothesis.

Next we prove that $D_A(n) \leq k + 1$. Let $S = (s_1, \cdots, s_N)$ be a sequence of elements in $\mathbb{Z}_n$ of length $N = k+1$. We will prove that S has a zero-sum subsequence with weight A. We distinguish two cases:

Case 1. If there exist two elements s_1 and s_2 such that $p^i \parallel s_1$, $p^i \parallel s_2$ for some $i = 0, 1, \cdots, k-1$, then $\frac{s_2}{p^i},\ n - \frac{s_1}{p^i} \in A$. Hence,

$$s_1\frac{s_2}{p^i} + s_2(n - \frac{s_1}{p^i}) \equiv 0 \pmod{n}.$$

Case 2. If Case 1 does not hold, then there must exist one element, say s_{i_0}, satisfying $p^k \| s_{i_0}$. Since $\frac{n}{p^k} \in A$, we have

$$s_{i_0}\frac{n}{p^k} \equiv 0 \pmod{n}.$$

Thus, we have proved that $D_A(n) = k + 1$.

(2) We will prove that $E_A(n) = n + k$. Assume that $S = (s_1, \cdots, s_{N'})$ is a sequence of elements in $\mathbb{Z}_n$ of length $N' = n + k$. To prove $E_A(n) = n + k$, because of Lemma 1 it suffices to prove that S has a zero-sum subsequence of length n with weight A. We partition S into the following multisets (sets with repetitions allowed).

$$M_i = \left\{s_j |\ p^i \parallel s_j,\ s_j \in S\right\}, \text{ for } i = 0, 1, 2, \cdots, k.$$

Note that every pair of elements $s_i^{(1)}, s_i^{(2)}$ in M_i constitutes a zero-sum subsequence of S with weight A since

$$s_i^{(1)} \cdot \frac{s_i^{(2)}}{p^i} + s_i^{(2)}\big(n - \frac{s_i^{(1)}}{p^i}\big) \equiv 0 \pmod{n},$$

where $\frac{s_i^{(2)}}{p^i},\ n - \frac{s_i^{(1)}}{p^i} \in A$, for $i = 0, 1, \cdots, k-1$.

Since every element $s_k^{'}$ in M_k produces a zero-sum subsequence of S of length 1 with weight A since $s_k^{'}\frac{n}{p^k} \equiv 0 \pmod{n}$, $\frac{n}{p^k} \in A$. We consider two cases:

Case 1. n is even. We can choose m $(0 \le m \le k)$ integers $l_1, l_2, \cdots, l_m$ satisfying $l_1 + l_2 + \cdots + l_m = \frac{n-t}{2}$, where $t = |M_k|$, $l_1 = [\frac{|M_{i_1}|}{2}], l_2 = [\frac{|M_{i_2}|}{2}], \cdots, l_m = [\frac{|M_{i_m}|}{2}]$, $0 \le i_1 \le i_2 \le i_m \le k-1$. Hence, we can obtain $\frac{n-t}{2}$ pairs of disjoint zero-sum subsequences of S with weight A and t disjoint zero-sum subsequences of S of length 1 with weight A, and it follows that we can get a zero-sum subsequence of S of length n with weight A.

Case 2. n is odd. If n is a prime, the result follows because of Lemma 2. For $n > 3$ and composite, since $2(k+1) < n$ we know that there must exist some M_i $(0 \le i \le k)$ satisfying $|M_i| \ge 3$. Let $s_i^{(1)}, s_i^{(2)}, s_i^{(3)} \in M_i$. We conclude that there must exist $x, y, z \in A$ satisfying $xs_i^{(1)} + ys_i^{(2)} + zs_i^{(3)} \equiv 0 \pmod{n}$.

Indeed, choose $x = 1,\ y = 1$ if $\frac{s_i^{(1)}}{p^i} + \frac{s_i^{(2)}}{p^i} \not\equiv 0 \pmod{p}$, and $x = 1,\ y = n-1$ if $\frac{s_i^{(1)}}{p^i} + \frac{s_i^{(2)}}{p^i} \equiv 0 \pmod{p}$. Then the equation $\frac{s_i^{(3)}}{p_i}z = -(x\frac{s_i^{(1)}}{p_i} + y\frac{s_i^{(2)}}{p_i}) \pmod{\frac{n}{p^i}}$ has a solution in A, and the result follows as before. □

Corollary 1 *Let n be a positive integer and p be a prime satisfying $p \| n$. If $A = \{a\ |a \not\equiv 0 \pmod{p}\}$, then $D_A(n) = 2$ and $E_A(n) = n+1$.*

Proof of Theorem 2. (1) We prove that $D_A(n) = 2$. Let $S = (s_1, s_2)$ be a sequence of elements in $\mathbb{Z}_n$ of length 2. It suffices to show that S has a zero-sum subsequence with weight A. We distinguish two cases:

Case 1. n is even. We see that $\frac{n}{2} \in A$ since $|A| = \lceil \frac{n}{2} \rceil$. So if $2|s_1$ or $2|s_2$, then $s_1\frac{n}{2} \equiv 0 \pmod{n}$ or $s_2\frac{n}{2} \equiv 0 \pmod{n}$. If s_1 and s_2 are both odd, then $\frac{n}{2} \in s_1A, s_2A$. Therefore, $0 \in s_1A + s_2A$.

Case 2. n is odd. If $\gcd(s_1, n) = \gcd(s_2, n) = 1$, then $|s_1A| = |s_2A| = \frac{n+1}{2}$. Hence $|s_1A| + |s_2A| = 2\frac{n+1}{2} = n+1 > n$. From Lemma 3 it follows that $s_1A + s_2A = \mathbb{Z}_n$. Therefore, $0 \in s_1A + s_2A$. If $\gcd(s_1, n) = d \ge 1$, that is $3 \le d \le \frac{n}{3}$, then there must exist $i(1 \le i \le d-1)$ such that $\frac{in}{d} \in A$ since $|A| = \frac{n+1}{2}$. It follows that $s_1\frac{in}{d} \equiv 0 \pmod{n}$.

(2). We now prove that $E_A(n) = n+1$. Assume that $S = (s_1, \cdots, s_N)$ is a sequence of elements in $\mathbb{Z}_n$ of length $N = n+1$. To prove $E_A(n) = n+1$, because of Lemma 1 it suffices to prove that S has a zero-sum subsequence of length n with weight A.

Partition S into the two multi-sets M_1, M_2 where $M_1 = \{s_i | \gcd(s_i, n) = 1, s_i \in S\}$ and $M_2 = \{s_i | \mathrm{gcd(s_i, n)} \neq 1, \mathrm{s_i \in S}\}$. We note the two following facts.

Fact 1. If $s_1, s_2 \in M_1$, then $0 \notin s_1A, s_2A$. Thus, from $D_A(n) = 2$, we conclude that there exist two elements $a_1, a_2 \in A$ such that $a_1s_1 + a_2s_2 \equiv 0 \pmod{n}$.

Fact 2. If $s_{i_0} \in M_2$, then there must exist one element $a_0 \in A$ such that $a_0s_i \equiv 0 \pmod{n}$. We distinguish two cases:

Case 1. n is even.

(1) $|M_1| \geq n$. Using Fact 1, it is easy to see that we can get a zero-sum subsequence of length n with weight A.

(2) $|M_2| < n$. Using Fact 1 and Fact 2, it is easy to see that we can get a zero-sum subsequence of length n with weight A.

Case 2. n is odd.

(1) $|M_2| \geq 1$. Using Fact 1 and Fact 2, it is easy to see that we can get a zero-sum subsequence of length n with weight A.

(2) $M_2 = \emptyset$. Then $|M_1| = n + 1$ and $\gcd(s_i, n) = 1$ for $i = 1, 2, \cdots, n + 1$. Set $A_i = s_iA$. Therefore, $|A_i| = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. Since $|A_1| + |A_2| > n$, the result follows that $\sum_{i=1}^{n} A_i \supseteq A_1 + A_2 = \mathbb{Z}_n$. Therefore, $0 \in \sum_{i=1}^{n} A_i = \mathbb{Z}_n$. □

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