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OBSERVATIONS ON THE PARITY OF THE TOTAL NUMBER OF PARTS IN ODD–PART PARTITIONS

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Abstract

In recent years, numerous functions which count the number of parts of various types of partitions have been studied. In this brief note, we consider the function $pt_o(n)$ which counts the total number of parts in all odd–part partitions of n (or what Chen and Ji recently called the number of rooted partitions of n into odd parts). In particular, we prove a number of results on the parity of $pt_o(n)$, including infinitely many Ramanujan–like congruences satisfied by the function.

1. Introduction and Motivation

In recent years, numerous functions which count the number of parts of various types of partitions have been studied. For example, Knopfmacher and Robbins [6] considered a variety of functions which count the total number of parts in partitions of n based on the types of partitions in question (unrestricted partitions, partitions with distinct parts, partitions into distinct and odd parts, and self–conjugate partitions). They obtained generating functions for, and numerous identities involving, all of these functions.

In an unrelated vein, Chen and Ji [4, section 3] recently coined the term "rooted partitions" and used these objects in their pursuit of proofs of weighted forms of Euler's Theorem. Chen and Ji also note that the enumerating functions for rooted partitions are identical to functions which count the total number of parts in all partitions in question. As with Knopfmacher and Robbins, Chen and Ji considered numerous types of partitions, including partitions into odd parts, partitions into distinct parts, and a number of other variants.

Quite recently, Andrews [2] has considered arithmetic properties of the function $spt(n)$ which counts the number of **smallest** parts in all the unrestricted partitions of n. So, for example, $spt(5) = 14$ since the partitions of 5 are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$$

and the number of smallest parts in these partitions is

$$1+1+1+2+1+3+5 = 14.$$

In the process, Andrews proved that, for all $n \geq 0$,

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod 5 \\ spt(7n+5) &\equiv 0 \pmod 7 \\ spt(13n+6) &\equiv 0 \pmod{13} \end{aligned}$$

which are reminiscent of Ramanujan's first three congruences for the partition function $p(n)$. Garvan [5] has pursued this topic even further and has proven additional congruences satisfied by $spt(n)$ for larger prime moduli.

In this brief note, we consider the function $pt_o(n)$, the total number of parts in all odd–part partitions of n (or what Chen and Ji [4] called the number of rooted partitions of n into odd parts). For example, $pt_o(7) = 19$ since the odd–part partitions of 7 are

$$7,\ 5+1+1,\ 3+3+1,\ 3+1+1+1+1,\ 1+1+1+1+1+1+1$$

and the total number of parts in these partitions is

$$1+3+3+5+7 = 19.$$

See [7, http://www.research.att.com/cgi-bin/access.cgi/as/ njas/sequences/eisA.cgi?Anum=A067588] for the first several values of $pt_o(n)$.

Unfortunately, $pt_o(n)$ does not appear to satisfy any congruences modulo small odd primes p such as those mentioned above for $spt(n)$. However, $pt_o(n)$ does have a rich structure modulo 2, which is hinted at by the sparseness of the values in [7, http://www.research.att.com/cgi-bin/access.cgi/as/ njas/sequences/eisA.cgi?Anum=A067589]. This is the focus of the work below.

2. Parity Results

We first share an almost trivial observation.

Theorem 1. *For all $n \geq 1$, $pt_o(2n) \equiv 0 \pmod 2$.*

Proof. It is clear that every partition of $2n$ into odd parts must contain an even number of parts. Thus, $pt_o(2n)$ must be even because it is the sum of even integers. □

Before stating our main theorem, we quote a recent result from Berndt and Yee [3] which will prove pivotal below.

Theorem 2. *For $n \geq 1$, let $\sigma(n)$ be the sum of the divisors of n and define $\sigma(0) = -\frac{1}{24}$. For nonnegative integers n,*

$$-24 \sum_{\substack{j+k(3k\pm 1)/2=n \\ j,k\geq 0}} (-1)^k \sigma(j) = \begin{cases} (-1)^r(6r-1)^2, & \text{if } n = r(3r-1)/2, \\ (-1)^r(6r+1)^2, & \text{if } n = r(3r+1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

We now state and prove the main theorem of this note. As a corollary, we then prove infinitely many congruences mod 2 satisfied by $pt_o(n)$ in the spirit of Ramanujan's results for $p(n)$.

Theorem 3. *If n is not a generalized pentagonal number, i.e., if*

$$n \neq \frac{k(3k+1)}{2}$$

for some integer k, then $pt_o(n)$ is even.

Proof. From Chen and Ji [4], we know that the generating function for $pt_o(n)$ is given by

$$P(q) = \prod_{n=0}^{\infty} \frac{1}{1-q^{2n+1}} \sum_{d=0}^{\infty} \frac{q^{2d+1}}{1-q^{2d+1}}.$$

Thanks to Euler's result that the number of odd–part partitions of n equals the number of distinct–part partitions of n, we know that

$$\prod_{n=0}^{\infty} \frac{1}{1-q^{2n+1}} = \prod_{n=1}^{\infty} (1+q^n).$$

This implies that

$$P(q) = \prod_{n=1}^{\infty} (1+q^n) \sum_{d=0}^{\infty} \frac{q^{2d+1}}{1-q^{2d+1}}.$$

Next, we recall Euler's Pentagonal Number Theorem [1, Chapter 1]:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}$$

Obviously, this means

$$\prod_{n=1}^{\infty}(1+q^n) \equiv \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \pmod 2$$

and this implies that

$$P(q) \equiv \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \sum_{d=0}^{\infty} \frac{q^{2d+1}}{1-q^{2d+1}} \pmod 2. \tag{1}$$

Now we focus attention on

$$\sum_{d=0}^{\infty} \frac{q^{2d+1}}{1-q^{2d+1}}. \tag{2}$$

First, note that (2) is the generating function for $d_o(n)$, the number of odd divisors of a positive integer n. We are particularly concerned with when $d_o(n)$ is odd, and this is true exactly when n is a square or twice a square. Therefore, we know that

$$\sum_{d=0}^{\infty} \frac{q^{2d+1}}{1-q^{2d+1}} \equiv \sum_{n=1}^{\infty} q^{n^2} + \sum_{n=1}^{\infty} q^{2n^2} \pmod 2.$$

Hence, from (1), we have

$$P(q) \equiv \left(\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2}\right)\left(\sum_{n=1}^{\infty} q^{n^2} + \sum_{n=1}^{\infty} q^{2n^2}\right) \pmod 2.$$

Note that the right–hand side of this congruence is the generating function for the number of representations of n as a sum of a square or twice a square and a generalized pentagonal number. As Berndt and Yee [3] comment after their statement of Theorem 2, "we see that, unless $n = r(3r \pm 1)/2$, the number of representations of n as a sum of a square or twice a square and a generalized pentagonal number $k(3k \pm 1)/2$ is even." This then implies the result of Theorem 3. □

We close by quickly proving a corollary which yields infinitely many congruences modulo 2 for $pt_o(n)$ in arithmetic progressions.

Corollary 4. *Let $p \geq 5$ be prime and let $1 \leq r \leq p-1$ be an integer such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $pt_o(pn+r) \equiv 0 \pmod 2$.*

Proof. Assume p and r satisfy the hypotheses of the corollary and assume n yields

$$pn + r = \frac{k(3k+1)}{2}$$

for some integer k. Then we know

$$r \equiv \frac{k(3k+1)}{2} \pmod{p}$$

or

$$\begin{aligned} 24r+1 &\equiv 24\left(\frac{k(3k+1)}{2}\right)+1 \pmod{p} \\ &\equiv 36k^2+12k+1 \pmod{p} \\ &= (6k+1)^2. \end{aligned}$$

But this contradicts the assumption that $24r+1$ is a quadratic nonresidue modulo p. Therefore, $pn+r$ can never be represented as a generalized pentagonal number. Thus, the result follows. □

This corollary implies that, for each prime $p \geq 5$, $pt_o(n)$ satisfies $\frac{p-1}{2}$ different congruences modulo 2.

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