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Research Article

On Output Feedback Multiobjective Control for Singularly Perturbed Systems

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A new design procedure for a robust H_2 and H_∞ control of continuous-time singularly perturbed systems via dynamic output feedback is presented. By formulating all objectives in terms of a common Lyapunov function, the controller will be designed through solving a set of inequalities. Therefore, a dynamic output feedback controller is developed such that H_∞ and H_2 performance of the resulting closed-loop system is less than or equal to some prescribed value. Also, H_∞ and H_2 performance for a given upperbound of singular perturbation parameter $\varepsilon \in (0, \varepsilon^*]$ are guaranteed. It is shown that the ε-dependent controller is well defined for any $\varepsilon \in (0, \varepsilon^*]$ and can be reduced to an ε-independent one so long as ε is sufficiently small. Finally, numerical simulations are provided to validate the proposed controller. Numerical simulations coincide with the theoretical analysis.

1. Introduction

It is well known that the multiple time-scale systems, otherwise known as singularly perturbed systems, often raise serious numerical problems in the control engineering field. In the past three decades, singularly perturbed systems have been intensively studied by many researchers [1–8].

In practice, many systems involve dynamics operating on two or more time-scales [3]. In this case, standard control techniques lead to ill-conditioning problems. Singular perturbation methods can be used to avoid such numerical problems [1]. By utilizing the time scale properties, the system is decomposed into several reduced order subsystems. Reduced order controllers are designed for each subsystems.

In the framework of singularly perturbed systems, H_∞ control has been investigated by many researchers, and various approaches have been proposed in this field [9–14].

However, to the best of our knowledge, the problem of multiobjective control for linear singular perturbed systems is still an open problem. By multiobjective control, we refer to synthesis problems with a mix of time- and frequency-domain specifications ranging from H_2 and H_∞ performances to regional pole placement, asymptotic tracking or regulation, and settling time or saturation constraints.

In [12], the H_∞ control problem is concerned via state feedback for fast sampling discrete-time singularly perturbed systems. A new H_∞ controller design method is given in terms of solutions to linear matrix inequalities (LMIs), which eliminate the regularity restrictions attached to the Riccati-based solution. In [13], the H_∞ control problem via state feedback for fast sampling discrete-time singularly perturbed systems is investigated. In fact, a new sufficient condition which ensures the existence of state feedback controllers is presented such that the resulting closed-loop system is asymptotically stable. In addition, the results were extended to robust controller design for fast sampling discrete-time singularly perturbed systems with polytopic uncertainties. Presented condition, in terms of a linear matrix inequality (LMI), is independent of the singular perturbation parameter.

Undoubtedly, output feedback stabilization is one of the most important problems in control theory and applications. In many real systems, the state vector is not always accessible and only partial information is available via measured output. Furthermore, the reliability of systems and the simplicity of implementation are other reasons of interest in output control which is adopted to stabilize a system.

LMI's have emerged as a powerful formulation and design technique for a variety of linear control problems. Since solving LMI's is a convex optimization problem, such formulations offer a numerically tractable means of attacking problems that lack an analytical solution. Consequently, reducing a control design problem to an LMI can be considered as a practical solution to this. Since the nonconvex formulation of the output feedback control, its conditions are restrictive or not numerically tractable [15]. It has been an open question how to make these conditions tractable by means of the existing software. Many research results on such a question have been reported in [15–20].

In [1, 2], the dynamic output feedback control of singular perturbation systems has been investigated. However, to the best of our knowledge, the design of dynamic output feedback for robust controller via LMI optimization is still an open problem. The main contribution of this paper is to solve the problem of multiobjective control for linear singularly perturbed systems. Considered problem consists of H_∞ control, H_2 performance, and singular perturbation bound design. For given an H_∞ performance bound γ or H_2 performance bound υ, and an upperbound ε^* for the singular perturbation, an ε-dependent dynamic output feedback controller will be constructed, such that for all $\varepsilon \in (0, \varepsilon^*]$. Due to this, the closed-loop system is admissible and the H_∞ norm (H_2 norm) of the closed-loop system is less than a prescribed γ (prescribed υ). Sufficient conditions for such a controller are obtained in form of strict LMIs. A mixed control H_2/H_∞ problem for singular systems is also considered in this paper. It is shown that the designed controller is well-defined for for all $\varepsilon \in (0, \varepsilon^*]$. It is shown that if ε is sufficiently small, the controller can be reduced to an ε-independent one. Numerical examples are given to illustrate the main results.

The paper is organized as follows. Section 2 gives the problem statement and motivations. Section 3 presents the main results. The theorems for H_2, H_∞ and multiobjective H_2/H_∞ design via output feedback control are presented in this section. Due to proposed theorems, robust H_2, H_∞, and multiobjective performance of continuous-time singularly perturbed systems via dynamic output feedback for a linear systems will be accessible.

Section 4 illustrates numerical simulations for the proposed theorems. Finally, conclusions in Section 5 close the paper.

Notation. A star $(*)$ in a matrix indicates a transpose quantity. For example: $(*)+A>0$ stands for $A^T+A>0$, or in a symmetric matrix $\left[\begin{smallmatrix} A & * \\ B & C \end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix} A & B^T \\ B & C \end{smallmatrix}\right]$.

2. Problem Statement

Consider the following singularly perturbed system with slow and fast dynamics described in the standard "singularly perturbed" form:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + A_2 x_2 + B_1 u + B_{w1} w, \\ \varepsilon \dot{x}_2 &= A_3 x_1 + A_4 x_2 + B_2 u + B_{w2} w, \\ z_1 &= C_{z11} x_1 + C_{z12} x_2 + D_{zu1} u + D_{zw1} w, \\ z_2 &= C_{z21} x_1 + C_{z22} x_2 + D_{zu2} u, \\ y &= C_{y1} x_1 + C_{y2} x_2 + D_{yw} w, \end{aligned} \tag{2.1}$$

where $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ form the state vector, $u(t) \in \Re^p$ is the control input vector, $y(t) \in \Re^{m_1}$ is the output, $w(t)$ is a vector of exogenous inputs (such as reference signals, disturbance signals, sensor noise) and z_1, z_2 are regulated outputs.

By introducing the following notations:

$$\begin{gathered} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C_y = \begin{bmatrix} C_{y1} & C_{y2} \end{bmatrix}, \\ C_{z1} = \begin{bmatrix} C_{z11} & C_{z12} \end{bmatrix}, \qquad C_{z2} = \begin{bmatrix} C_{z21} & C_{z22} \end{bmatrix}, \\ B_w = \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix}. \end{gathered} \tag{2.2}$$

The system (2.1) can be rewritten into the following compact form:

$$\begin{aligned} E_\varepsilon \dot{x} &= Ax + Bu + B_w w, \\ z_1 &= C_{z1} x + D_{zu1} u + D_{zw1} w, \\ z_2 &= C_{z2} x + D_{zu2} u, \\ y &= C_y x + D_{yw} w. \end{aligned} \tag{2.3}$$

By applying dynamic output feedback controller in the following form:

$$\begin{aligned}\dot{x}_{c1} &= A_{c1}x_{c1} + A_{c2}x_{c2} + B_{c1}y,\\ \varepsilon\dot{x}_{c2} &= A_{c3}x_{c1} + A_{c4}x_{c2} + B_{c2}y,\\ u &= \begin{bmatrix} C_{c1} & C_{c2}\end{bmatrix}\begin{bmatrix} x_{c1}\\ x_{c2}\end{bmatrix} + D_c y.\end{aligned} \tag{2.4}$$

The controller (2.4) can be rewritten into the following compact form:

$$\begin{aligned} E_\varepsilon \dot{x}_c &= A_c x_c + B_c y,\\ u &= C_c x_c + D_c y,\end{aligned} \tag{2.5}$$

where

$$A_c = \begin{bmatrix} A_{c1} & A_{c2}\\ A_{c3} & A_{c4}\end{bmatrix}, \qquad B_c = \begin{bmatrix} B_{c1}\\ B_{c2}\end{bmatrix}, \qquad C_c = \begin{bmatrix} C_{c1} & C_{c2}\end{bmatrix}. \tag{2.6}$$

The closed loop system is

$$\begin{aligned}\begin{bmatrix} E_\varepsilon & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{x}_c\end{bmatrix} &= \begin{bmatrix} A + BD_cC_y & BC_c\\ E_\varepsilon^{-1}B_cC_y & E_\varepsilon^{-1}A_c\end{bmatrix}\begin{bmatrix} x\\ x_c\end{bmatrix} + \begin{bmatrix} B_w + BD_cD_{yw}\\ E_\varepsilon^{-1}B_cD_{yw}\end{bmatrix}w(t),\\ z_i &= \begin{bmatrix} C_{zi} + D_{zui}D_cC_y & D_{zui}C_c\end{bmatrix}\begin{bmatrix} x\\ x_c\end{bmatrix} + \left(D_{zwi} + D_{zui}D_cD_{yw}\right)w(t), \quad \text{for } i = 1,2,\end{aligned} \tag{2.7}$$

where $D_{\mathrm{cl}2} = 0$ and

$$\begin{aligned} E_e = \operatorname{diag}(E_\varepsilon, I), \qquad E_\varepsilon &= \operatorname{diag}(I, \varepsilon I), \qquad x_{\mathrm{cl}} = \begin{bmatrix} x^T & x_c^T\end{bmatrix}^T,\\ A_{\mathrm{cl}} &= \begin{bmatrix} A + BD_cC_y & B_wC_c\\ E_\varepsilon^{-1}B_cC_y & E_\varepsilon^{-1}A_c\end{bmatrix},\\ B_{\mathrm{cl}} &= \begin{bmatrix} B_w + BD_cD_{yw}\\ E_\varepsilon^{-1}B_cD_{yw}\end{bmatrix},\\ C_{\mathrm{cl}i} &= \begin{bmatrix} C_{zi} + D_{zui}D_cC_y & D_{zui}C_c\end{bmatrix},\\ D_{\mathrm{cl}1} &= \left(D_{zw1} + D_{zu1}D_cD_{yw}\right).\end{aligned} \tag{2.8}$$

Note, following definitions and lemmas are useful in next sections.

Definition 2.1. For a linear time-invariant operator $G : \omega \in L_2(\Re^+) \rightarrow z \in L_2(\Re^+)$, G is L_2 stable if $\omega \in L_2(\Re)$ implies $z \in L_2(\Re)$. Here, G is said to have L_2 gain less than or equal to $\gamma > 0$ if and only if

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^t \|w(t)\|^2 dt \tag{2.9}$$

for all $T \in \Re^+$.

Lemma 2.2 (see [21]). *For system $G : (A, B, C, D)$, the L_2 gain will be less than $\gamma > 0$ if there exist a positive definite matrix $X = X^T > 0$ such that*

$$\begin{bmatrix} XA + A^T X & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{2.10}$$

Definition 2.3. For a linear time-invariant operator $G : w \rightarrow z$, H_2 norm G is defined by

$$\|G\|_2^2 = \frac{1}{2\pi} \text{trace} \int_{-\infty}^{\infty} G(j\omega) G(j\omega)^* d\omega. \tag{2.11}$$

Lemma 2.4 (see [21]). *For stable system $G : (A, B, C)$ the H_2 performance will be less than v if there exist a matrix Z and a positive definite matrix $X = X^T > 0$ such that the following LMIs are feasible:*

$$\begin{aligned} &\begin{bmatrix} A^T X + XA & XB \\ B^T X & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} X & C^T \\ C & Z \end{bmatrix} > 0, \\ &\text{trace}(Z) < \nu. \end{aligned} \tag{2.12}$$

Lemma 2.5 (see [22]). *For a positive scalar ε^* and symmetric matrices S_1, S_2, and S_3 with appropriate dimensions, inequality*

$$S_1 + \varepsilon S_2 + \varepsilon^2 S_3 > 0 \tag{2.13}$$

holds for all $\varepsilon \in (0, \varepsilon^*]$, if

$$\begin{aligned} &S_1 \geq 0, \\ &S_1 + \varepsilon^* S_2 > 0, \\ &S_1 + \varepsilon^* S_2 + \varepsilon^{*2} S_3 > 0. \end{aligned} \tag{2.14}$$

3. Main Result

Here, we address stability, H_∞ stability, H_2 performance, and multiobjective H_2/H_∞ performance for a singularly perturbed system via dynamic output feedback control.

3.1. Stability Problem

Consider closed loop system (2.7) without disturbance $w(t)$:

$$E_e\dot{x}_{\rm cl} = \begin{bmatrix} A + BD_cC_y & BC_c \\ E_\varepsilon^{-1}B_cC_y & E_\varepsilon^{-1}A_c \end{bmatrix} x_{\rm cl}, \tag{3.1}$$

where A, B, C_y were defined in (2.2). In the following theorem, we propose design procedure for obtaining controllers parameters such that the closed loop system (3.1) becomes asymptotically stable.

Theorem 3.1. *Given an upperbound ε^* for the singular perturbation ε, if there exist matrices $A_k, B_k, C_k, D_k, Y_{11}, Y_{12}, Y_{22}, X_{11}, X_{12}$, and X_{22} satisfying the following LMIs*

$$\Upsilon_{11}(0) \le 0, \tag{3.2}$$

$$\Upsilon_{11}(\varepsilon^*) < 0, \tag{3.3}$$

$$\left[\begin{array}{c|c} \begin{bmatrix} Y_{11} & \varepsilon^* Y_{12} \\ \varepsilon^* Y_{12}^T & \varepsilon^* Y_{22} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \varepsilon^* I \end{bmatrix} \\ \hline \begin{bmatrix} I & 0 \\ 0 & \varepsilon^* I \end{bmatrix} & \begin{bmatrix} X_{11} & \varepsilon^* X_{12} \\ \varepsilon^* X_{12}^T & \varepsilon^* X_{22} \end{bmatrix} \end{array}\right] > 0, \tag{3.4}$$

$$\begin{bmatrix} Y_{11} & I \\ I & X_{11} \end{bmatrix} > 0, \tag{3.5}$$

where

$$\Upsilon_{11}(\varepsilon^*) = \left[\begin{array}{c|c} A\begin{bmatrix} Y_{11} & \varepsilon^* Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} + BC_k + (*) & A + BD_kC_y + A_k^T \\ \hline (*) & \begin{bmatrix} X_{11} & X_{12}^T \\ \varepsilon^* X_{12} & X_{22} \end{bmatrix} A + B_kC_y + (*) \end{array}\right]. \tag{3.6}$$

Then, for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop singularly perturbed system (3.1) *is asymptotically stable* *via dynamic output controller* (2.4), *also controller parameters are obtained from following equations:*

$$\begin{aligned} D_c &= D_k, \\ C_c &= (C_k - D_c C Q_{11}), \\ B_c &= \vartheta_\varepsilon^{-1}\left(B_k - P_{11}^T B D_c\right), \\ A_c &= \vartheta_\varepsilon^{-1}\left(A_k - \left(P_{11}^T (A + BD_c C_y)Q_{11} + \vartheta_\varepsilon B_c C_y Q_{11} + P_{11}^T B C_c\right)\right), \end{aligned} \tag{3.7}$$

where

$$\begin{gathered} P_{11} = \begin{bmatrix} X_{11} & \varepsilon X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \qquad Q_{11} = \begin{bmatrix} Y_{11} & \varepsilon Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}, \\ \vartheta_\varepsilon = \begin{bmatrix} I - \vartheta_1 - \varepsilon\vartheta_2 & -\vartheta_3 \\ -\varepsilon\vartheta_4 & I - \varepsilon\vartheta_2^T - \vartheta_5 \end{bmatrix}, \\ \vartheta_1 = X_{11}Y_{11}, \qquad \vartheta_2 = X_{12}Y_{12}^T, \qquad \vartheta_3 = X_{11}Y_{12} + X_{12}Y_{22}, \\ \vartheta_4 = X_{12}^T Y_{11} + X_{22}Y_{12}^T, \qquad \vartheta_5 = X_{22}Y_{22}. \end{gathered} \tag{3.8}$$

Proof. Choose the Lyapunov function as

$$V(t) = x_{\text{cl}}^T(t) E_e P_\varepsilon x_{\text{cl}}(t), \tag{3.9}$$

where

$$E_e P_\varepsilon = P_\varepsilon^T E_e > 0, \tag{3.10}$$

$$P_\varepsilon = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T E_\varepsilon & P_{22} \end{bmatrix}. \tag{3.11}$$

From (3.10), it is concluded that

$$E_e P_\varepsilon = P_\varepsilon^T E_e \Longrightarrow \begin{bmatrix} E_\varepsilon P_{11} & E_\varepsilon P_{12} \\ P_{12}^T E_\varepsilon & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11}^T E_\varepsilon & E_\varepsilon P_{12} \\ P_{12}^T E_\varepsilon & P_{22}^T \end{bmatrix} \tag{3.12}$$

and from (3.12), P_{11} has following structure:

$$E_\varepsilon P_{11} = P_{11}^T E_\varepsilon \Longrightarrow P_{11} = \begin{bmatrix} X_{11} & \varepsilon X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}. \tag{3.13}$$

Now, define invert of $E_e P_\varepsilon$ as follow

$$(E_e P_\varepsilon)^{-1} = P_\varepsilon^{-1} E_e^{-1},$$
$$P_\varepsilon^{-1} = Q_\varepsilon = \begin{bmatrix} Q_{11} & E_\varepsilon^{-1} Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{3.14}$$

and Q_{11} has following structure:

$$Q_\varepsilon (E_e)^{-1} = E_e^{-1} Q_{11}^T \Longrightarrow \begin{bmatrix} Q_{11} E_\varepsilon^{-1} & E_\varepsilon^{-1} Q_{12} \\ Q_{12}^T E_\varepsilon^{-1} & Q_{22} \end{bmatrix} = \begin{bmatrix} E_\varepsilon^{-1} Q_{11}^T & E_\varepsilon^{-1} Q_{12} \\ Q_{12}^T E_\varepsilon^{-1} & Q_{22} \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} Y_{11} & \varepsilon Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}. \tag{3.15}$$

Using the following equality:

$$\begin{bmatrix} E_\varepsilon P_{11} & E_\varepsilon P_{12} \\ P_{12}^T E_\varepsilon & P_{22} \end{bmatrix} \begin{bmatrix} Q_{11} E_\varepsilon^{-1} & E_\varepsilon^{-1} Q_{12} \\ Q_{12}^T E_\varepsilon^{-1} & Q_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{3.16}$$

we also have the constraint that

$$E_\varepsilon \left(P_{11} Q_{11} + P_{12} Q_{12}^T \right) E_\varepsilon^{-1} = I \Longrightarrow P_{11} Q_{11} + P_{12} Q_{12}^T = I. \tag{3.17}$$

Here, we define new matrices Π_1 and Π_2 as follows:

$$\Pi_1 = \begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix}, \tag{3.18}$$

$$P_\varepsilon \Pi_1 = \Pi_2 \tag{3.19}$$

and Π_2 is obtained from (3.17) as follows:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T E_\varepsilon & P_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix} = \begin{bmatrix} P_{11} Q_{11} + P_{12} Q_{12}^T & P_{11} \\ P_{12}^T E_\varepsilon Q_{11} + P_{22} Q_{12}^T & P_{12}^T E_\varepsilon \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} I & P_{11} \\ 0 & P_{12}^T E_\varepsilon \end{bmatrix}. \tag{3.20}$$

The derivative of the Lyapunov function (3.9) is

$$\dot V = \dot x_{\mathrm{cl}}^T E_e P_\varepsilon x_{\mathrm{cl}} + x_{\mathrm{cl}}^T E_e P_\varepsilon \dot x_{\mathrm{cl}},$$
$$\dot V < 0 \Longrightarrow x_{\mathrm{cl}}^T \left(A_{\mathrm{cl}}^T P_\varepsilon + P_\varepsilon^T A_{\mathrm{cl}} \right) x_{\mathrm{cl}} < 0, \tag{3.21}$$

where A_{cl} is defined in (2.8).

Sufficient condition to satisfy (3.21) is

$$A_{\mathrm{cl}}^T P_\varepsilon + P_\varepsilon^T A_{\mathrm{cl}} < 0. \tag{3.22}$$

Equation (3.22) will hold if and only if

$$\Pi_1^T A_{\mathrm{cl}}^T P_\varepsilon \Pi_1 + \Pi_1^T P_\varepsilon^T A_{\mathrm{cl}} \Pi_1 < 0, \tag{3.23}$$

where Π_1 is defined in (3.18).

From (3.19), equation (3.23) can be rewritten as

$$\Pi_1^T A_{\mathrm{cl}}^T \Pi_2 + \Pi_2^T A_{\mathrm{cl}} \Pi_1 < 0. \tag{3.24}$$

With substituting Π_1 and A_{cl} from (3.18) and (2.8), respectively, we have

$$\begin{bmatrix} I & 0 \\ P_{11}^T & E_\varepsilon P_{12} \end{bmatrix} \begin{bmatrix} A + BD_cC & BC_c \\ E_\varepsilon^{-1} B_c C & E_\varepsilon^{-1} A_c \end{bmatrix} \begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix} + (*) < 0. \tag{3.25}$$

Now, we define the new variables:

$$\begin{aligned} C_k &= D_c C Q_{11} + C_c Q_{12}^T, \\ B_k &= P_{11}^T B D_c + E_\varepsilon P_{12} E_\varepsilon^{-1} B_c, \\ A_k &= P_{11}^T \left(A + BD_c C_y\right) Q_{11} + E_\varepsilon P_{12} E_\varepsilon^{-1} B_c C_y Q_{11} + P_{11}^T B C_c Q_{12}^T + E_\varepsilon P_{12} E_\varepsilon^{-1} A_c Q_{12}^T. \end{aligned} \tag{3.26}$$

Then, from (3.13), (3.15), and (3.26), equation (3.25) can be rewritten as

$$\Upsilon_{11}(\varepsilon) = \left[\begin{array}{c|c} A \begin{bmatrix} Y_{11} & \varepsilon Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} + BC_k + (*) & A + BD_k C_y + A_k^T \\ \hline (*) & \begin{bmatrix} X_{11} & X_{12}^T \\ \varepsilon X_{12} & X_{22} \end{bmatrix} A + B_k C_y + (*) \end{array}\right] < 0. \tag{3.27}$$

Also, the condition (3.10) holds if and only if

$$\Pi_1^T E_e P_\varepsilon \Pi_1 > 0 \Longrightarrow \left[\begin{array}{c|c} \begin{bmatrix} Y_{11} & \varepsilon Y_{12} \\ \varepsilon Y_{12}^T & \varepsilon Y_{22} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix} \\ \hline \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix} & \begin{bmatrix} X_{11} & \varepsilon X_{12} \\ \varepsilon X_{12}^T & \varepsilon X_{22} \end{bmatrix} \end{array}\right] > 0. \tag{3.28}$$

According to Lemma 2.5, the conditions (3.27) and (3.28) are valid for all $\varepsilon \in (0, \varepsilon^*]$, If (3.2), (3.3), (3.4), and (3.5) are satisfied.

For computing controller parameters we obtain P_{11} and Q_{11} from solving LMIs in (3.2), (3.3), (3.5), and (3.6). Then from constraint (3.17) with assumption $Q_{12} = I$ we have

$$P_{12} = I - P_{11}Q_{11} = I - \begin{bmatrix} X_{11} & \varepsilon X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}\begin{bmatrix} Y_{11} & \varepsilon Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} = I - \begin{bmatrix} \vartheta_1 + \varepsilon\vartheta_2 & \varepsilon\vartheta_3 \\ \vartheta_4 & \varepsilon\vartheta_2^T + \vartheta_5 \end{bmatrix},$$
$$E_\varepsilon P_{12} E_\varepsilon^{-1} = \begin{bmatrix} I - \vartheta_1 - \varepsilon\vartheta_2 & -\vartheta_3 \\ -\varepsilon\vartheta_4 & I - \varepsilon\vartheta_2^T - \vartheta_5 \end{bmatrix}, \tag{3.29}$$

where

$$\vartheta_1 = X_{11}Y_{11}, \qquad \vartheta_2 = X_{12}Y_{12}^T,$$
$$\vartheta_3 = X_{11}Y_{12} + X_{12}Y_{22}, \qquad \vartheta_4 = X_{12}^T Y_{11} + X_{22}Y_{12}^T, \tag{3.30}$$
$$\vartheta_5 = X_{22}Y_{22}.$$

Also From (3.29) and (3.26) we can obtain controller parameters from (3.7). Also from (3.7) controllers parameters are always well defined for all $\varepsilon \in (0, \varepsilon^*]$ and $\lim_{\varepsilon \to 0^+} \left[\begin{smallmatrix} A_c & B_c \\ C_c & D_c \end{smallmatrix}\right]$ become

$$D_{c0} = D_k,$$
$$C_{c0} = (C_k - D_c C Q_{110}),$$
$$B_{c0} = \vartheta_{\varepsilon 0}^{-1}\left(B_k - P_{110}^T B D_c\right), \tag{3.31}$$
$$A_{c0} = \vartheta_{\varepsilon 0}^{-1}\left(A_k - \left(P_{110}^T (A + BD_c C_y) Q_{11} + \vartheta_{\varepsilon 0} B_c C_y Q_{110} + P_{110}^T B C_{c0}\right)\right),$$

where $\vartheta_1, \vartheta_3, \vartheta_5$ are defined in (3.30) and

$$P_{110} = \begin{bmatrix} X_{11} & 0 \\ X_{12}^T & X_{22} \end{bmatrix}, \qquad Q_{110} = \begin{bmatrix} Y_{11} & 0 \\ Y_{12}^T & Y_{22} \end{bmatrix},$$
$$\vartheta_{\varepsilon 0} = \begin{bmatrix} I - \vartheta_1 & -\vartheta_3 \\ 0 & I - \vartheta_5 \end{bmatrix}. \tag{3.32}$$

This completes the proof. □

Remark 3.2. Throughout the paper, it is assumed that the singular perturbation parameter ε is available for feedback. Indeed, in many singular perturbation systems, the singular perturbation parameter ε can be measured. In these cases, ε is available for feedback, which has attracted much attention. For example, ε-dependent controllers were designed for singular perturbation systems in [22–24]. Since ε is usually very small, an ε-dependent controller may be ill-conditioning as ε tends to zero. Thus, it is a key task to ensure the obtained controller to be well defined. This problem will be discussed later.

3.2. H_∞ *Performance*

Consider the closed loop system (2.7) with regulated output z_1. In following theorem, we proposed a procedure for obtaining controller parameters such that the closed loop singular perturbed system (2.7) with regulated output z_1 becomes asymptotically stable and guarantees the H_∞ performance with attenuation parameter γ.

Theorem 3.3. *Given an H_∞ performance bound γ and an upperbound ε^* for the singular perturbation ε, if there exist matrices A_k, B_k, C_k, D_k, Y_{11}, Y_{12}, Y_{22}, X_{11}, X_{12}, and X_{22} satisfying the following LMIs*

$$\psi_{11}(0) = \left[\begin{array}{c|cc} \Upsilon_{11}(0) & \multicolumn{2}{c}{\Upsilon_{12}(0)} \\ \hline (*) & -\gamma I & D_{zw1} + D_{zu1} D_k D_{yw} \\ & (*) & -\gamma I \end{array}\right] \leq 0,$$
$$\psi_{11}(\varepsilon^*) = \left[\begin{array}{c|cc} \Upsilon_{11}(\varepsilon^*) & \multicolumn{2}{c}{\Upsilon_{12}(\varepsilon^*)} \\ \hline (*) & -\gamma I & D_{zw1} + D_{zu1} D_k D_{yw} \\ & (*) & -\gamma I \end{array}\right] < 0, \tag{3.33}$$

where $\Upsilon_{11}(\varepsilon^*)$ defined in (3.6) $\Upsilon_{12}(\varepsilon^*)$ *is*

$$\Upsilon_{12}(\varepsilon^*) = \begin{bmatrix} B_w + B D_k D_{yw} & \left[\begin{array}{c|c} Y_{11} & Y_{12}^T \\ \hline \varepsilon^* Y_{12} & Y_{22} \end{array}\right] C_z^T + C_k^T D_{zu1}^T \\ \left[\begin{array}{c|c} X_{11} & X_{12}^T \\ \hline \varepsilon^* X_{12} & X_{22} \end{array}\right] B_w + B_k D_{yw} & C_{z1}^T + C_y^T D_k^T D_{zu1}^T \end{bmatrix}. \tag{3.34}$$

Then, for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop singularly perturbed system (2.7) *is asymptotically stable and with an H_∞-norm less than or equal to γ, also parameters controller are obtained from* (3.7).

Proof. According to Lemma 2.2, the closed-loop singularly perturbed system (2.7) is asymptotically stable and the L_2 gain will be less or equal γ if (3.10) and following inequality are satisfied:

$$\begin{bmatrix} A_{\text{cl}}^T P_\varepsilon + P_\varepsilon^T A_{\text{cl}} & P_\varepsilon^T B_{\text{cl}} & C_{\text{cl}1}^T \\ (*) & -\gamma I & D_{\text{cl}1}^T \\ (*) & (*) & -\gamma I \end{bmatrix} < 0. \tag{3.35}$$

Also A_{cl}, B_{cl}, $C_{\text{cl}1}$, and $D_{\text{cl}1}$ are defined in (2.8).

By pre- and postmultiplying with matrices $\operatorname{diag}(\Pi_1, I, I)$ and $\operatorname{diag}(\Pi_1^T, I, I)$, respectively, it is concluded that:

$$\begin{bmatrix} \Pi_1^T A_{\rm cl}^T \Pi_2 + \Pi_2^T A_{\rm cl} \Pi_1 & \Pi_2^T B_{\rm cl} & \Pi_1^T C_{\rm cl\,1}^T \\ (*) & -\gamma I & D_{\rm cl\,1}^T \\ (*) & (*) & -\gamma I \end{bmatrix} < 0, \tag{3.36}$$

where Π_1 is defined in (3.18). According to proof of Theorem 3.1, $\Pi_1^T A_{\rm cl}^T \Pi_2 + \Pi_2^T A_{\rm cl} \Pi_1^T$ is obtained. Also $\Pi_2^T B_{\rm cl}$ and $\Pi_1^T C_{\rm cl\,1}^T$ are presented as follows:

$$\begin{aligned} \Pi_2^T B_{\rm cl} &= \begin{bmatrix} I & 0 \\ P_{11}^T & E_\varepsilon P_{12} \end{bmatrix} \begin{bmatrix} B_w + BD_c D_{yw} \\ E_\varepsilon^{-1} B_c D_{yw} \end{bmatrix} = \begin{bmatrix} B_w + BD_k D_{yw} \\ P_{11}^T B_w + P_{11}^T BD_k D_{yw} + E_\varepsilon P_{12} E_\varepsilon^{-1} B_c D_{yw} \end{bmatrix} \\ &= \begin{bmatrix} B_w + BD_k D_{yw} \\ P_{11}^T B_w + B_k D_{yw} \end{bmatrix}, \\ \Pi_1^T C_{\rm cl\,1}^T &= \begin{bmatrix} Q_{11}^T & Q_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} C_{z1}^T + C_y^T D_c^T D_{zu1}^T \\ C_c^T D_{zu1}^T \end{bmatrix} = \begin{bmatrix} Q_{11} C_{z1}^T + Q_{11} C_y^T D_c^T D_{zu1}^T + Q_{12} C_c^T D_{zu1}^T \\ C_{z1}^T + C_y^T D_c^T D_{zu1}^T \end{bmatrix} \\ &= \begin{bmatrix} Q_{11} C_{z1}^T + C_k^T D_{zu1}^T \\ C_{z1}^T + C_y^T D_k^T D_{zu1}^T \end{bmatrix}. \end{aligned} \tag{3.37}$$

From (3.36), and (3.37), we have

$$\psi_{11}(\varepsilon) = \left[\begin{array}{c|cc} \Upsilon_{11}(\varepsilon) & \multicolumn{2}{c}{\Upsilon_{12}(\varepsilon)} \\ \hline \Upsilon_{12}^T(\varepsilon) & -\gamma I & D_{zw1} + D_{zu1} D_k D_{yw} \\ & (*) & -\gamma I \end{array}\right] < 0, \tag{3.38}$$

where $\Upsilon_{11}(\varepsilon)$ defined in (3.27) and

$$\Upsilon_{12}(\varepsilon) = \begin{bmatrix} B_w + BD_k D_{yw} & \left[\begin{array}{c|c} Y_{11} & Y_{12}^T \\ \hline \varepsilon Y_{12} & Y_{22} \end{array}\right] C_{z1}^T + C_k^T D_{zu1}^T \\ \left[\begin{array}{c|c} X_{11} & X_{12}^T \\ \hline \varepsilon X_{12} & X_{22} \end{array}\right] B_w + B_k D_{yw} & C_{z1}^T + C_y^T D_k^T D_{zu1}^T \end{bmatrix}. \tag{3.39}$$

According to Lemma 2.5, the condition (3.38) is satisfied for all $\varepsilon \in (0, \varepsilon^*]$, if (3.33) is satisfied. (3.4) and (3.5) compute from procedure similar to proof of Theorem 3.1 and this completes the proof. □

3.3. H_2 *Performance*

Consider the closed loop system (2.7) with regulated output z_2, assume $A_{\rm cl}$ is stable, $D_{zw2} = 0$ and $D_{yw} = 0$. In following theorem, we proposed a procedure for obtaining controller parameters such that the closed loop singularly perturbed system (2.7) with regulated output z_2 guarantees the H_2 performance with attenuation parameter v. First we propose the following lemma that is effective in proof of Theorem 3.5.

Lemma 3.4. *The closed-loop singularly perturbed* $G : (A_{\mathrm{cl}}, B_{\mathrm{cl}}, C_{\mathrm{cl}})$ *has the* H_2 *performance with attenuation parameter* v *if following LMIs hold*

$$\begin{bmatrix} A_{\mathrm{cl}}^T P_\varepsilon + P_\varepsilon^T A_{\mathrm{cl}} & P_\varepsilon^T B_{\mathrm{cl}} \\ (*) & -I \end{bmatrix} < 0, \tag{3.40}$$

$$\begin{bmatrix} P_\varepsilon^T E_\varepsilon & P_\varepsilon^T C_{\mathrm{cl}}^T \\ (*) & Z \end{bmatrix} > 0, \tag{3.41}$$

$$\operatorname{trace}(Z) < v. \tag{3.42}$$

Proof. Consider the following closed-loop singular perturbed system:

$$\begin{gathered} \dot{x}(t) = E_e^{-1} A_{\mathrm{cl}} x(t) + E_e^{-1} B_{\mathrm{cl}} v(t), \\ z = C_{\mathrm{cl}} x(t), \end{gathered} \tag{3.43}$$

where E_e is defined in (2.8). From Definition 2.3 we have the following equality:

$$\|G\|_{H_2}^2 = \int_0^\infty \operatorname{trace}\Big(C_{\mathrm{cl}} e^{E_e^{-1} A_{\mathrm{cl}} t} E_e^{-1} B_{\mathrm{cl}} B_{\mathrm{cl}}^T E_e^{-1} e^{A_{\mathrm{cl}}^T E_e^{-1} t} C_{\mathrm{cl}}^T \Big) dt. \tag{3.44}$$

Now we define symmetric matrix W_ε as follows:

$$W_\varepsilon = \int_0^\infty \Big(e^{E_e^{-1} A_{\mathrm{cl}} t} E_e^{-1} B_{\mathrm{cl}} B_{\mathrm{cl}}^T E_e^{-1} e^{A_{\mathrm{cl}}^T E_e^{-1} t} \Big) dt. \tag{3.45}$$

$\|G\|_{H_2}^2$ is obtained from the following equality:

$$\|G\|_{H_2}^2 = \operatorname{trace}\Big(C_{\mathrm{cl}} W_\varepsilon C_{\mathrm{cl}}^T \Big). \tag{3.46}$$

W_e can be obtained from the following equation:

$$E_e^{-1} A_{\mathrm{cl}} W_\varepsilon + W_\varepsilon A_{\mathrm{cl}}^T E_e^{-1} + E_e^{-1} B_{\mathrm{cl}} B_{\mathrm{cl}}^T E_e^{-1} = 0. \tag{3.47}$$

Suppose that there exist the matrix X_ε with following structure:

$$X_\varepsilon = \begin{bmatrix} X'_{11} & E_\varepsilon^{-1} X'_{12} \\ X'^T_{12} & X'_{22} \end{bmatrix} \tag{3.48}$$

that satisfies the following inequality:

$$W_\varepsilon < X_\varepsilon E_e^{-1} \tag{3.49}$$

From (3.49), equation (3.47) can be rewritten as follow:

$$E_e^{-1} A_{\mathrm{cl}} X_\varepsilon E_e^{-1} + E_e^{-1} X_\varepsilon^T A_{\mathrm{cl}}^T E_e^{-1} + E_e^{-1} B_{\mathrm{cl}} B_{\mathrm{cl}}^T E_e^{-1} < 0. \tag{3.50}$$

Now, pre- postmultiplying (3.50) with E_e we have

$$A_{\mathrm{cl}} X_\varepsilon + X_\varepsilon^T A_{\mathrm{cl}}^T + B_{\mathrm{cl}} B_{\mathrm{cl}}^T < 0. \tag{3.51}$$

Also, from (3.46) and (3.49) we have

$$\|G\|_{H_2}^2 = \operatorname{trace}\left(C_{\mathrm{cl}} W_\varepsilon C_{\mathrm{cl}}^T\right) < \operatorname{trace}\left(C_{\mathrm{cl}} X_\varepsilon E_e^{-1} C_{\mathrm{cl}}^T\right) < v. \tag{3.52}$$

This is equivalent to the existence of Z such that

$$\begin{gathered} C_{\mathrm{cl}} X_\varepsilon E_e^{-1} C_{\mathrm{cl}}^T < Z, \\ \operatorname{trace}(Z) < v \end{gathered} \tag{3.53}$$

by using of Schur complement on (3.51) and (3.53) we can conclude

$$\begin{bmatrix} X_\varepsilon^T A_{\mathrm{cl}}^T + A_{\mathrm{cl}} X_\varepsilon & B_{\mathrm{cl}} \\ (*) & -I \end{bmatrix} < 0, \tag{3.54}$$

$$\begin{bmatrix} E_e P_\varepsilon & C_{\mathrm{cl}}^T \\ (*) & Z \end{bmatrix} > 0 \tag{3.55}$$

with assumption $X_\varepsilon^{-1} = P_\varepsilon$ and pre- and postmultiplying (3.54) by $\operatorname{diag}(P_\varepsilon^T, I)$ and $\operatorname{diag}(P_\varepsilon, I)$, respectively, we have

$$\begin{gathered} \begin{bmatrix} A_{\mathrm{cl}}^T P_\varepsilon + P_\varepsilon^T A_{\mathrm{cl}} & P_\varepsilon B_{\mathrm{cl}} \\ (*) & -I \end{bmatrix} < 0, \\ \begin{bmatrix} E_e P_\varepsilon & P_\varepsilon^T C_{\mathrm{cl}}^T \\ (*) & Z \end{bmatrix} > 0, \\ \operatorname{trace}(Z) < v\ . \end{gathered} \tag{3.56}$$

This completes the proof. □

Theorem 3.5. *Given an H_2 performance bound v and an upperbound ε^* for the singular perturbation ε, if there exist matrices A_k, B_k, C_k, D_k, Y_{11}, Y_{12}, Y_{22}, X_{11}, X_{12}, and X_{22} such that* $\operatorname{trace}(Z) < v$ *andsatisfying the following LMIs:*

$$\tilde{\varphi}_{11}(0) = \begin{bmatrix} \Upsilon_{11}(0) & \tilde{\Upsilon}_{12}(0) \\ (*) & -I \end{bmatrix} \leq 0,$$

$$\tilde{\varphi}_{11}(\varepsilon^*) = \begin{bmatrix} \Upsilon_{11}(\varepsilon^*) & \tilde{\Upsilon}_{12}(\varepsilon^*) \\ (*) & -I \end{bmatrix} < 0,$$

$$\left[\begin{array}{c|c} \begin{bmatrix} Y_{11} & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & X_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix} C_{z2}^T + C_k^T D_{zu2}^T \\ C_{z2}^T + C_y^T D_k^T D_{zu2}^T \end{array} \\ \hline * & Z \end{array}\right] \geq 0, \tag{3.57}$$

$$\left[\begin{array}{c|c} \left[\begin{array}{c|c} \begin{bmatrix} Y_{11} & \varepsilon^* Y_{12} \\ \varepsilon^* Y_{12}^T & \varepsilon^* Y_{22} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \varepsilon^* I \end{bmatrix} \\ \hline \begin{bmatrix} I & 0 \\ 0 & \varepsilon^* I \end{bmatrix} & \begin{bmatrix} X_{11} & \varepsilon^* X_{12} \\ \varepsilon^* X_{12}^T & \varepsilon^* X_{22} \end{bmatrix} \end{array}\right] & \begin{array}{c} \begin{bmatrix} Y_{11} & Y_{12} \\ \varepsilon^* Y_{12}^T & Y_{22} \end{bmatrix} C_{z2}^T + C_k^T D_{zu2}^T \\ C_{z2}^T + C_y^T D_k^T D_{zu2}^T \end{array} \\ \hline (*) & Z \end{array}\right] > 0,$$

where $\Upsilon_{11}(\varepsilon^*)$ *defined in* (3.6) *and* $\tilde{\Upsilon}_{12}(\varepsilon^*)$ *is*

$$\tilde{\Upsilon}_{12}(\varepsilon^*) = \begin{bmatrix} B_w \\ \left[\begin{array}{c|c} X_{11} & X_{12}^T \\ \hline \varepsilon^* X_{12} & X_{22} \end{array}\right] B_w \end{bmatrix}. \tag{3.58}$$

Then, for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop singularly perturbed system (2.7) *is asymptotically stable and with an H_2-norm less than or equal to v, also parameters controller are obtained from* (3.7).

Proof. The proof is similar to proof of Theorem 3.3. From Lemma 3.4, the closed-loop singularly perturbed system (2.7) has H_2 performance less than or equal v if (3.40), (3.41) and (3.42) are satisfied.

From (3.18) and (3.37), multiplying inequalities (3.40) and (3.41), by the matrices $\operatorname{diag}(\Pi_1, I, I)$ and $\operatorname{diag}(\Pi_1^T, I, I)$, gives

$$\widetilde{\varphi}_{11}(\varepsilon) = \begin{bmatrix} \Upsilon_{11}(\varepsilon) & \widetilde{\Upsilon}_{12}(\varepsilon) \\ (*) & -I \end{bmatrix} < 0, \tag{3.59}$$

$$\begin{bmatrix} \left[\begin{array}{c|c} \begin{bmatrix} Y_{11} & \varepsilon Y_{12} \\ \varepsilon Y_{12}^T & \varepsilon Y_{22} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix} \\ \hline \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix} & \begin{bmatrix} X_{11} & \varepsilon X_{12} \\ \varepsilon X_{12}^T & \varepsilon X_{22} \end{bmatrix} \end{array}\right] & \begin{array}{c} \begin{bmatrix} Y_{11} & Y_{12} \\ \varepsilon Y_{12}^T & Y_{22} \end{bmatrix} C_{z2}^T + C_k^T D_{zu2}^T \\ C_{z2}^T + C_y^T D_k^T D_{zu2}^T \end{array} \\ \hline (*) & Z \end{bmatrix} > 0, \tag{3.60}$$

where:

$$\widetilde{\Upsilon}_{12}(\varepsilon) = \begin{bmatrix} B_w + BD_k D_{yw} \\ \left[\begin{array}{c|c} X_{11} & X_{12}^T \\ \hline \varepsilon X_{12} & X_{22} \end{array}\right] B_w + B_k D_{yw} \end{bmatrix} \tag{3.61}$$

According to Lemma 2.5, the inequalities (3.59) and (3.60) are valid for all $\varepsilon \in (0, \varepsilon^*]$, if (3.57) is satisfied and proof is complete. □

3.4. Multiobjective H_2/H_∞ Performance

Now we have got the LMIs in Theorems 3.3 and 3.5 thus we can solve the multiobjective H_2/H_∞ synthesis problem easily for closed-loop singularly perturbed system (2.7) with regulated outputs z_1 and z_2.

Theorem 3.6. *Given an H_∞ performance bound γ, H_2 performance bound v and an upperbound ε^* for the singular perturbation ε, if there exist matrices A_k, B_k, C_k, D_k, Y_{11}, Y_{12}, Y_{22}, X_{11}, X_{12}, and X_{22} such that* trace$(Z) < v$ *satisfying the LMIs* (3.4), (3.5), (3.33), *and* (3.57). *Then, for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop singularly perturbed system* (2.7) *is asymptotically stable and with an H_∞-norm less than or equal to γ, an H_2-norm less than or equal to v, also parameters controller are obtained from* (3.7).

Proof. It is from the proof of Theorems 3.3 and 3.5 and omitted for save of brevity. □

4. Numerical Example

In this section, we present a numerical results to validate the designed dynamic output feedback controller for singularly perturbed systems with H_∞ or H_2 performance.

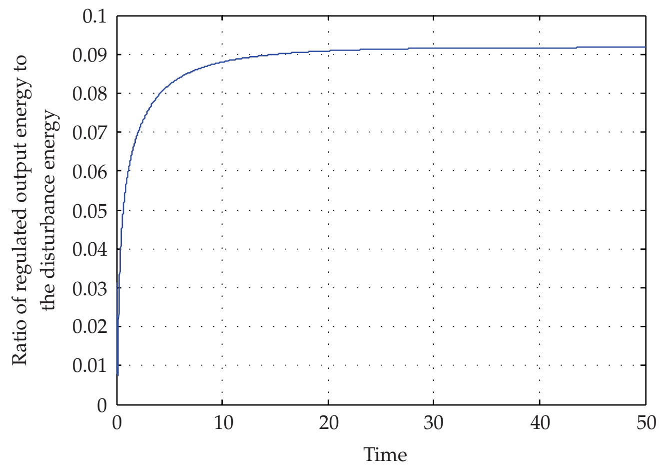

Figure 1: Simulation for full-order system (4.1) with $\varepsilon^* = 0.1$.

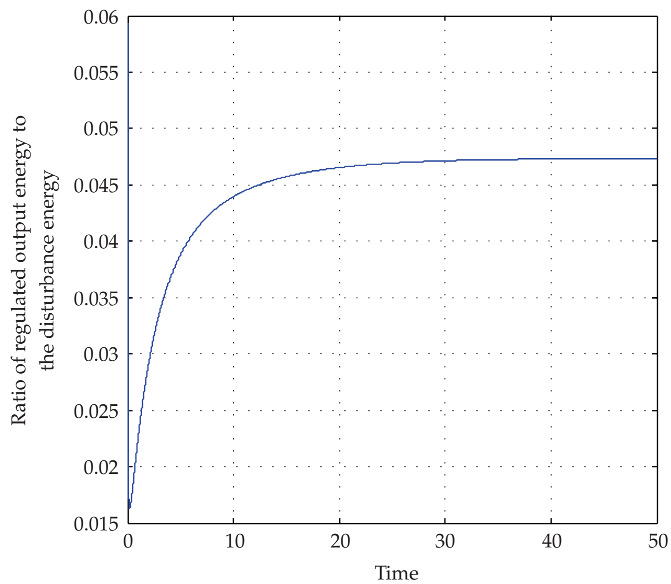

Figure 2: Simulation for full-order system (4.1) with $\varepsilon^* = 0.001$.

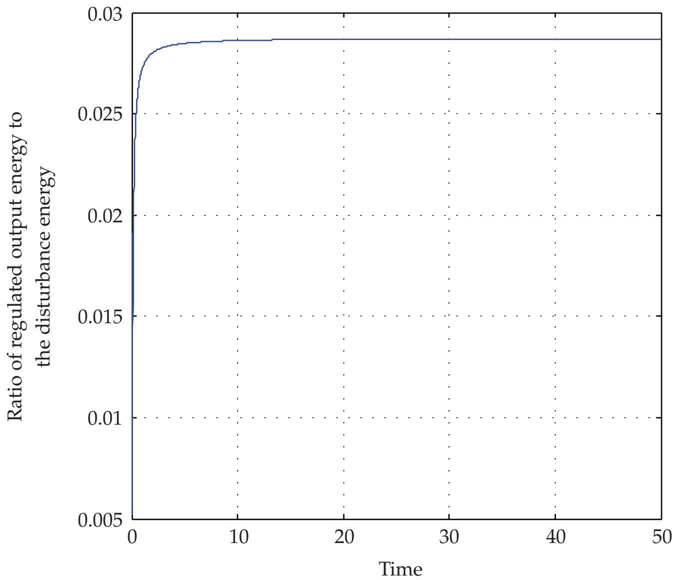

Figure 3: Simulation for reduced order system (4.3).

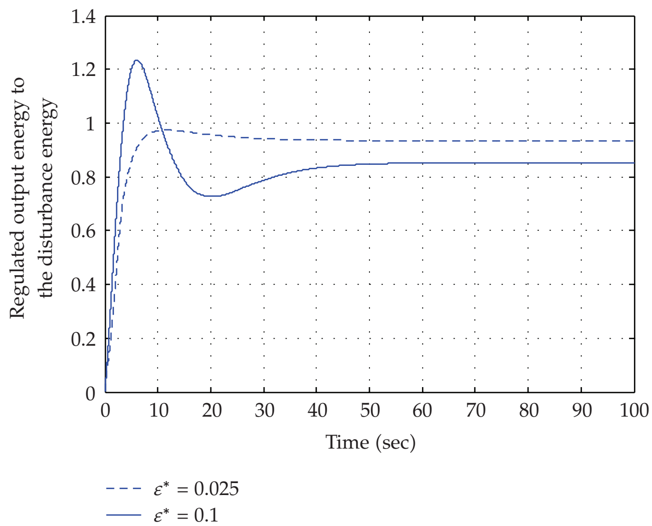

Figure 4: Simulation for full-order system (4.4) for $\varepsilon^* = 0.1,\ 0.025$.

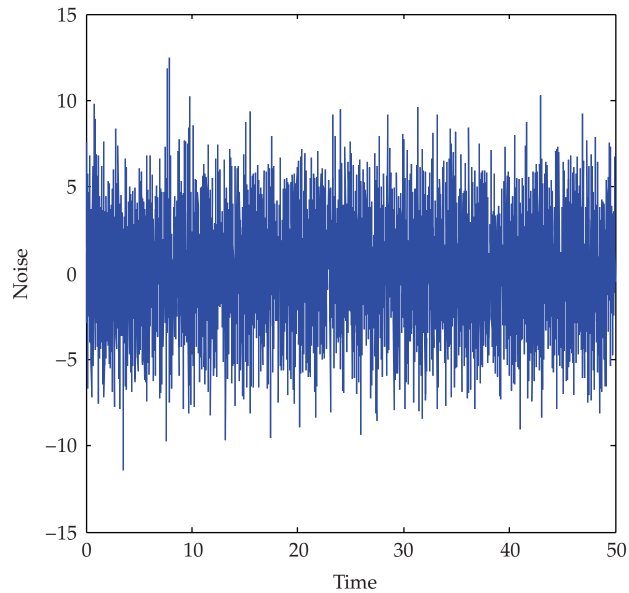

Figure 5: Band-limited white noise.

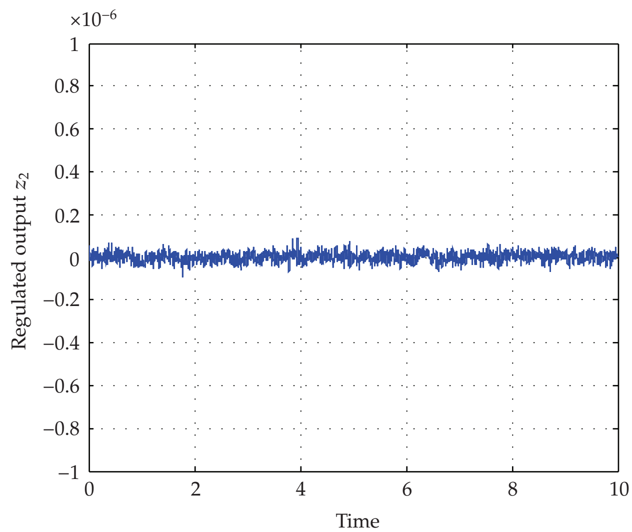

Figure 6: Regulated output z_2 for $\varepsilon^* = 0.001$.

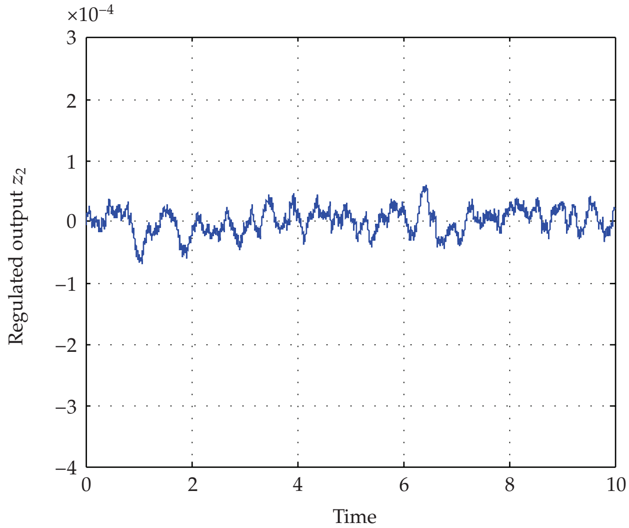

Figure 7: Regulated output z_2 for $\varepsilon^* = 0.1$.

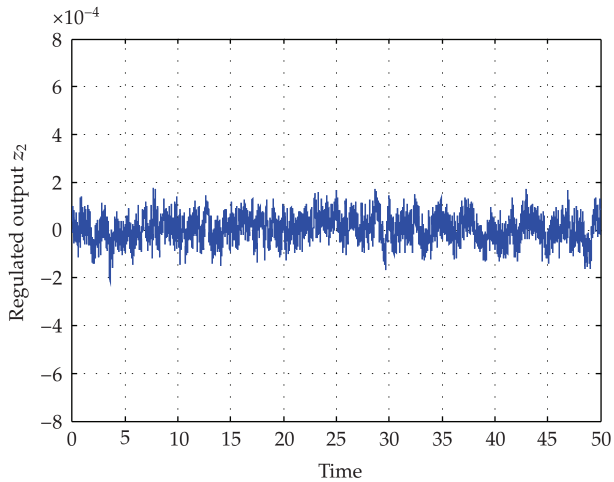

Figure 8: Regulated output z_2 for $\varepsilon^* = 0.001$.

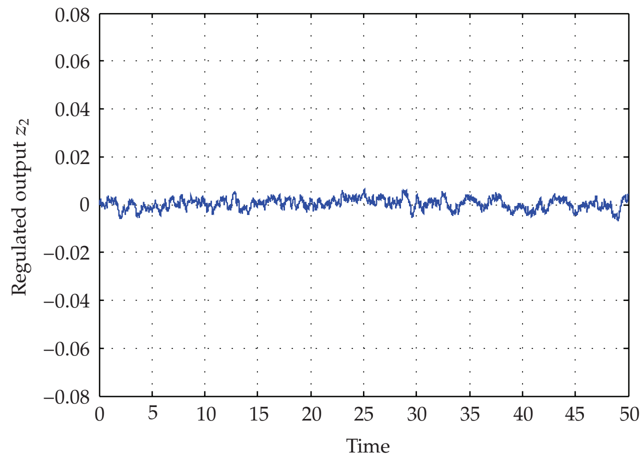

Figure 9: Regulated output z_2 for $\varepsilon^* = 0.1$.

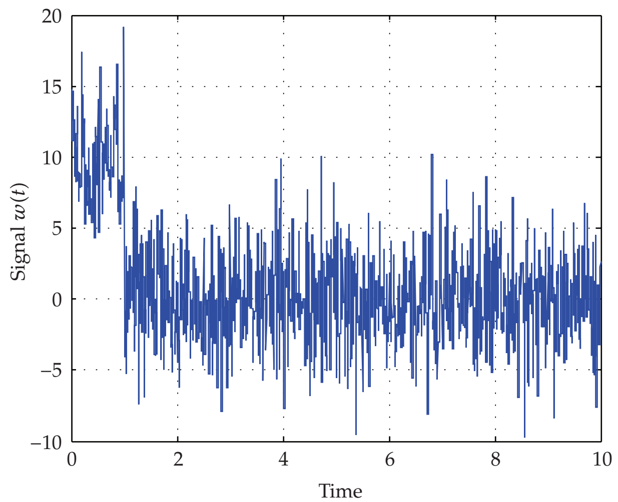

Figure 10: Disturbance signal $w(t)$.

4.1. H_∞ ***Performance***

Consider the following two-dimensional system and performance index [10]

$$\begin{gathered}\begin{bmatrix}\dot{x}_1(t)\\ \varepsilon\dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}2 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}2\\ 1\end{bmatrix}u(t)+\begin{bmatrix}1\\ 3\end{bmatrix}w(t),\\ z_1=\begin{bmatrix}2 & 0.1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+0.1w(t),\\ y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+w(t).\end{gathered}\tag{4.1}$$

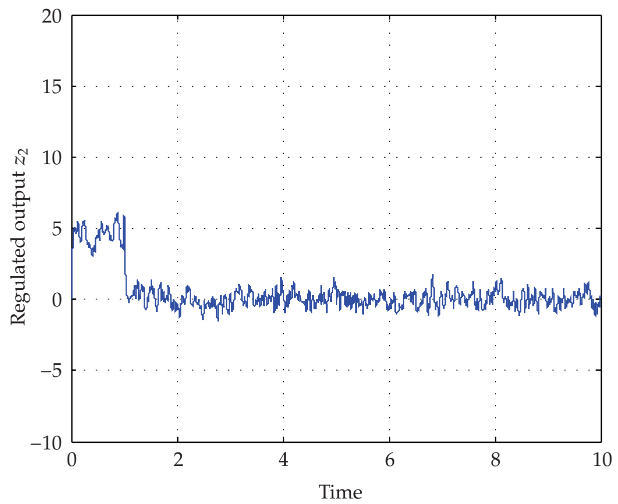

Figure 11: Regulated output z_2 with $\varepsilon^* = 0.1$.

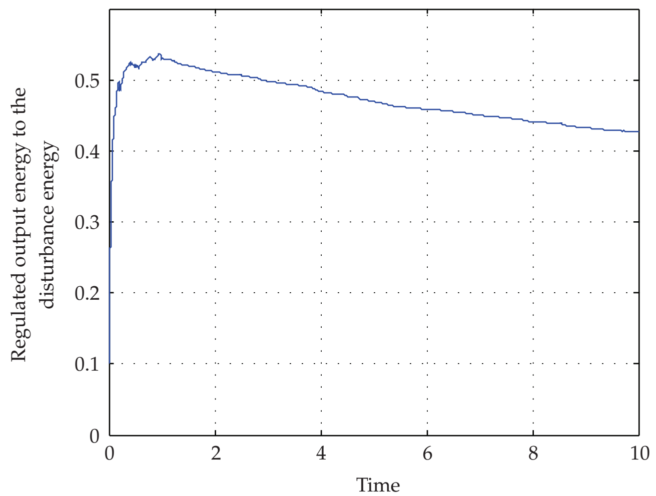

Figure 12: Simulation for $\sqrt{\int_0^t z_1^T(s)z_1(s)/\int_0^t w^T(s)w(s)}$ with $\varepsilon^* = 0.1$.

From Theorem 3.3, for $\varepsilon^* = 0.1$ minimum of γ is obtained as 0.11942. Controller parameters are as follows:

$$A_c = \begin{bmatrix} -108.89 & -31.42 \\ 91.35 & 5.201 \end{bmatrix},$$
$$B_c = \begin{bmatrix} -.22 \\ 0.05 \end{bmatrix}, \tag{4.2}$$
$$C_c = \begin{bmatrix} -1793.2 & -536.1 \end{bmatrix},$$
$$D_c = -4.7658.$$

For upperband $\varepsilon^* = 0.001$, minimum γ is calculated as 0.1. As we expect $\gamma_{\varepsilon^*=0.001} \leq \gamma_{\varepsilon^*=0.1}$.

From $\varepsilon = 0$, we obtain reduced order dynamic as follows:

$$\dot{x}_1(t) = 1.5x_1(t) + 2.5u(t) + 2.5w(t),$$
$$z_1 = 1.95x_1(t) + 0.05u(t) + 0.15w(t), \tag{4.3}$$
$$y(t) = x_1(t) + w(t).$$

Here, minimum value γ is calculated as 0.029. For $w = e^{-0.1t}\sin(10t)$, Figures 1, 2, and 3 exhibit $\sqrt{\int_0^t z_1^T(s)z_1(s)ds / \int_0^t w^T(s)w(s)ds}$, respectively.

As a new example for H_∞ performance, now consider an F-8 aircraft model [25]:

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon\dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) u + \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix} w,$$
$$z = \begin{bmatrix} C_{w1} & C_{w2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{4.4}$$
$$y = \begin{bmatrix} C_{y1} & C_{y2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w$$

with

$$A_1 = \begin{bmatrix} -0.01357 & -0.0644 \\ 0.06 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.003087 & 0 \\ 0.040467 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.0453775 & 0 \\ 0.07125 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -0.03055 & 0.075 \\ -0.075083 & -0.01674 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.0004333 \\ 0.0697 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.052275 \\ 0.019712 \end{bmatrix}, \tag{4.5}$$
$$B_{w1} = \begin{bmatrix} -0.463 \\ 6.07 \end{bmatrix}, \qquad B_{w2} = \begin{bmatrix} -4.5525 \\ -11.262499 \end{bmatrix},$$
$$C_{y1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad C_{y2} = \begin{bmatrix} 0 & 0.02 \\ 0 & 0 \end{bmatrix}.$$

From Theorem 3.3, for $\varepsilon^* = 0.025$ minimum of γ is obtained as 1.908. Controller parameters are as follows:

$$A_c = \begin{bmatrix} -17.0078 & 41.1022 & -62.0967 & 0.15894 \\ 800.6192 & -2017.7 & 3070.05 & -5.964 \\ -1964.9 & 4954.9 & -7541 & 15.033 \\ 39387 & -109682.3 & 165444.1 & -407.4100 \end{bmatrix},$$
$$B_c = \begin{bmatrix} -0.5966 & 0.5893 \\ 29.39 & -28.98 \\ -72.249 & 70.9783 \\ 1586.7 & -1406.2 \end{bmatrix}, \tag{4.6}$$
$$C_c = \begin{bmatrix} -3643.8 & 7991 & -6482 & 336.66 \end{bmatrix},$$
$$D_c = \begin{bmatrix} -76.567 & -4.3010 \end{bmatrix}.$$

For upperband $\varepsilon^* = 0.1$, minimum γ is calculated as 3.96. As we expect $\gamma_{\varepsilon^*=0.025} \le \gamma_{\varepsilon^*=0.1}$.

For $w = e^{-0.1t}\sin(10t)$, Figure 4 exhibits $\sqrt{\int_0^t z_1^T(s) z_1(s)ds / \int_0^t w^T(s)w(s)ds}$.

4.2. H_2 *Performance*

Consider singular perturbed system (3.60) as

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} w(t),$$
$$z_2 = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 2u(t), \tag{4.7}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

By utilizing Theorem 3.5, minimum value of ν for $\varepsilon^* = 0.1$ is obtained as 0.074. The controller parameters are as follows:

$$A_c = \begin{bmatrix} -2558.6 & -4.289 \\ 2566620 & -195.16 \end{bmatrix},$$
$$B_c = \begin{bmatrix} 1.62 \\ -163.03 \end{bmatrix}, \tag{4.8}$$
$$C_c = \begin{bmatrix} -0.034 & 0.011 \end{bmatrix},$$
$$D_c = -4.1.$$

For upperband $\varepsilon^* = 0.001$, minimum v is calculated as 0.0002. As we expect $v_{\varepsilon^*=0.001} \le v_{\varepsilon^*=0.1}$. Figures 5, 6, and 7 show input noise and regulated output z_2, respectively.

As a new example for H_2 performance, now consider singular perturbed system as

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0.5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} w(t),$$
$$z_2 = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 2u(t), \tag{4.9}$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

By utilizing Theorem 3.5, minimum value of ν for $\varepsilon^* = 0.1$ is obtained as 0.003. The controller parameters are as follows:

$$A_c = \begin{bmatrix} 5243.9 & -14292.8 \\ 29566.58 & -80500.28 \end{bmatrix},$$
$$B_c = \begin{bmatrix} -0.167 \\ -9.4145 \end{bmatrix}, \quad (4.10)$$
$$C_c = \begin{bmatrix} -77312.6 & 6829.2 \end{bmatrix},$$
$$D_c = -3.8 \times 10^{-9}.$$

For upperband $\varepsilon^* = 0.001$, minimum v is calculated as 0.0004. As we expect $v_{\varepsilon^*=0.001} \le v_{\varepsilon^*=0.1}$. Input noise is shown in Figures 5, 8, and 9 show the regulated output z_2, respectively.

4.3. H_∞/H_2 Performance

Now, consider singular perturbed system (3.60) as

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 3 \end{bmatrix} w,$$
$$z_1 = \begin{bmatrix} 2 & 0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0.1w,$$
$$z_2 = \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2u, \quad (4.11)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w.$$

According to Theorem 3.6, due to minimization of $\nu+\gamma$ for $\varepsilon^* = 0.1$, the values of $\nu = 0.14$ and $\gamma = 0.233$ are calculated. By similar calculation, due to minimization of $\nu + \gamma$ for $\varepsilon^* = 0.001$, the values of ν and γ are calculated as 0.088 and 0.2018, respectively.
Here, controller parameters are as the follows when $\varepsilon^* = 0.1$:

$$A_c = \begin{bmatrix} -7.4 & -9.17 \\ 817.41 & -0.5 \end{bmatrix},$$
$$B_c = \begin{bmatrix} 4.16 \\ -52.18 \end{bmatrix}, \quad (4.12)$$
$$C_c = \begin{bmatrix} -0.15 \times 10^{-4} & 0 \end{bmatrix},$$
$$D_c = -3.206.$$

Figures 10 and 11 show input signal $w(t)$ and regulated output z_2. Based on the H_∞/H_2, considered controller is designed to minimize effect of signal w on regulated output z_2. Also the ratio of the regulated output energy to the disturbance energy is shown in Figure 12.

5. Conclusions

In this paper, we addressed robust H_2 and H_∞ control via dynamic output feedback control for continuous-time singularly perturbed systems. By formulating all objectives in terms of a common Lyapunov function, the controller was designed through solving a set of inequalities. A dynamic output feedback controller was developed such that first, the H_∞ and H_2 performances of the resulting closed-loop system is less than or equal to some prescribed values, and furthermore, these performances are satisfied for all $\varepsilon \in (0, \varepsilon^*]$. Apart from our main results, Theorems 3.1 to 3.6 show that the ε-dependent controller is well defined for all $\varepsilon \in (0, \varepsilon^*]$ and can be reduced to an ε-independent providing ε is sufficiently small. Finally, numerical simulations were provided to verify the proposed controller.

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