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Research Article

First Integrals for Two Linearly Coupled Nonlinear Duffing Oscillators

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We investigate Noether and partial Noether operators of point type corresponding to a Lagrangian and a partial Lagrangian for a system of two linearly coupled nonlinear Duffing oscillators. Then, the first integrals with respect to Noether and partial Noether operators of point type are obtained explicitly by utilizing Noether and partial Noether theorems for the system under consideration. Moreover, if the partial Euler-Lagrange equations are independent of derivatives, then the partial Noether operators become Noether point symmetry generators for such equations. The difference arises in the gauge terms due to Lagrangians being different for respective approaches. This study points to new ways of constructing first integrals for nonlinear equations without regard to a Lagrangian. We have illustrated it here for nonlinear Duffing oscillators.

1. Introduction

In this paper, we study a system of two linearly coupled nonlinear Duffing oscillators

$$\begin{aligned} y'' + \Omega^2 y &= -\beta(y-z) - \alpha y^3, \\ z'' + \Omega^2 z &= -\beta(z-y) - \alpha z^3, \end{aligned} \tag{1.1}$$

where α and β are nonlinearity linear coupling parameters, and prime denotes differentiation with respect to x. The system of two second-order ordinary differential equations (ODEs) frequently arises in nonlinear oscillations, nonlinear dynamics, relativity, fluid mechanics, and so forth. These oscillators, in general, describe different mechanical systems of practical importance and have two types of characteristics (see, e.g., [1, 2]). The hardening characteristic approaches to linear for small amplitudes and it raises towards infinity when the amplitude tends to certain limits. The softening characteristic performs in a non-monotonic way, and for large amplitude the vibration shape changes to the rectangular shape.

There is great interest in studying the system of two linearly coupled nonlinear Duffing oscillators, especially for first integrals, which are important from a physical point of view and for reduction purposes as well. Of course, a Lagrangian exists for the system of two equations under study. Our purpose is two fold: one is how first integrals can be constructed for ODEs; secondly, we want to investigate the effectiveness of the partial Lagrangian approach.

The Noether symmetries and first integrals are important due to their wide range of applications for Euler-Lagrange equations (see, e.g., Noether [3] and later works [4, 5]). This theorem is based on the existence of a Lagrangian and variational symmetries. It gives the relationship between equivalence classes of point symmetries and first integrals. There are equations that arise in application that do not admit standard Lagrangians, for example, the simple system of two second-order equations $y'' = y^2 + z^2$, $z'' = y$; the curve family is nonextremal for this system. The scalar evolution equations also do not admit Lagrangians. Similarly, for the system of two coupled Van der pol oscillators with linear diffusive coupling

$$\begin{aligned} y'' + \epsilon(y^2 - 1)y' + y &= A(z - y) + B(z' - y'), \\ z'' + \epsilon(z^2 - 1)z' + z &= A(y - z) + B(y' - z'), \end{aligned} \tag{1.2}$$

where A and B are constants and ϵ is a small parameter and no variational problem exists which can be verified from Douglas [5]. The interested reader is referred to the interesting paper cited above for the classification of Lagrangians in which Douglas [5] has provided the complete solution to the inverse problem for three-dimensional space (system of two second-order ODEs). Now, we raise the question as to how one can construct first integrals for equations which are not variational as mentioned above. It is well known that there are other methods which provide first integrals without making use of a Lagrangian. The most elementary method is the direct method [6, 7] which is often used for constructing first integrals without the variational principle. There are some other approaches as well (see, e.g., [4, 8, 9]) in which the equations can be expressed in characteristic form.

The Noether symmetries and the corresponding first integrals have been subjects of rigorous investigations and play an important role in the reduction of differential equations. Some important works have been done relating to first integrals. In [10, 11], the authors deduce a relation between "symmetries" and first integrals without a variational principle. The classification of Noether point symmetries for a three degrees of freedom Lagrangian system was given by Damianou and Sophocleous in [12], and the results for one and two degrees of freedom were also reported in their paper. The first integrals associated with Noether and partial Noether operators of point type for a linear system of two second-order ODEs with variable coefficients were constructed by the authors in [13]. They showed that the first integrals resulting from Noether and partial Noether approaches are equivalent. The difference occurs in the gauge terms due to Lagrangians being different for the respective approaches. The classification of partial Noether operators and first integrals for a system with two degrees of freedom was also discussed by Naeem and Mahomed in [14]. Furthermore, a new perturbation method based on integrating vectors and multiple scales for perturbed systems of ODEs is presented in [15]. By using this perturbation method a strongly nonlinear forced oscillator based on integrating factors is studied which yields the approximate first integrals [16]. For an account of this theory, the reader is referred to an interesting book [17].

In this paper, we show that the first integrals corresponding to Noether and partial Noether operators of point type for a system of two linearly coupled nonlinear Duffing oscillators are equivalent, and the algebras for both cases are isomorphic. We give an elegant way to construct first integrals for such equations with respect to point symmetries. Firstly, we obtain the partial Noether operators of point type and then first integrals are constructed by utilizing a partial Noether's theorem. This approach gives new ways of constructing first integrals for the equations without regard to a Lagrangian as partial Lagrangians do exist for such equations. It would definitely be of interest to obtain the first integrals of the Duffing oscillator system with respect to first-order Lie-Bäcklund operators. This will provide a more complete picture for such systems. Further work can then be done on reductions using these symmetries.

The outline of the work is as follows. In Section 2, the basic definitions are adapted from the literature. The Noether point symmetries and the corresponding first integrals are presented in Section 3. In Section 4, we construct the partial Noether operators of point type and the relating first integrals. Concluding remarks are summarized in Section 5.

2. Preliminaries

Suppose x is considered to be the independent variable and $u = (u^1, u^2) = (y, z)$, the dependent variable with coordinates y and z. The derivatives of u^α with respect to x are denoted by

$$u_x^\alpha = D_x(u^\alpha), \quad \alpha = 1, 2, \tag{2.1}$$

in which

$$D_x = \frac{\partial}{\partial x} + u_x^\alpha \frac{\partial}{\partial u^\alpha} + u_{xx}^\alpha \frac{\partial}{\partial u_x^\alpha} + \cdots, \tag{2.2}$$

is the total derivative operator with respect to x. Note that the summation convention is assumed for repeated indices throughout. The collection of first derivative u_x^α is denoted by $u' = (u^1, u^2) = (y', z')$, and all higher-order derivatives are denoted by $u', \ u'', \ u''', \ldots$.

The following definitions and results are easily adaptable from the literature (see, e.g., [10–12, 18–20]).

(1) The Lie-Bäcklund operator is

$$X = \xi \frac{\partial}{\partial x} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_x^\alpha \frac{\partial}{\partial u_x^\alpha}, \tag{2.3}$$

where ζ_x^α are defined by

$$\zeta_x^\alpha = D_x(\eta^\alpha) - u_x^\alpha D_x(\xi), \quad \alpha = 1, 2. \tag{2.4}$$

(2) A Lie-Bäcklund operator X defined in (2.3) is a Noether symmetry generator corresponding to a Lagrangian L if there exists a function $B(x,u)$ such that

$$XL + (D_x\xi)L = D_x(B), \tag{2.5}$$

where D_x is the total derivative operator given in (2.2). The function $B(x,u)$ is known as the gauge term.

(3) The Lie-Bäcklund operator X is known to be a partial Noether operator corresponding to a partial Lagrangian L if it is determined from

$$XL + (D_x\xi)L = \left(\eta^\alpha - \xi u_x^\alpha\right)\frac{\delta L}{\delta u^\alpha} + D_x(B), \tag{2.6}$$

for a suitable function $B(x,u)$, where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_x\frac{\partial}{\partial u^\alpha} + D_x^2\frac{\partial}{\partial u_x^\alpha} - D_x^3\frac{\partial}{\partial u_{xx}^\alpha} + \cdots, \qquad \alpha = 1,2 \tag{2.7}$$

is the Euler operator.

(4) If the Lie-Bäcklund operator X is a Noether symmetry generator corresponding to a Lagrangian $L \in \mathcal{A}$ of Euler-Lagrange differential equations, then

$$I = B - \left[\xi L + \left(\eta^\alpha - \xi u_x^\alpha\right)\frac{\partial L}{\partial u_x^\alpha}\right], \qquad \alpha = 1,2 \tag{2.8}$$

is known as the first integral of Euler-Lagrange differential equations.

(5) If the Lie-Bäcklund operator X is a partial Noether operator with respect to a partial Lagrangian $L(x,u,u')$ of partial Euler-Lagrange equations, then the constant of motion or first integral I can be constructed from (2.8).

3. Noether's Approach

In this section, we derive the Noether point symmetry operators for the coupled nonlinear Duffing oscillators (1.1) corresponding to a standard Lagrangian and construct the first integrals by Noether's theorem.

3.1. Noether Point Symmetry Generators for Coupled Nonlinear Duffing Oscillators

A Lagrangian for system (1.1) satisfying the Euler-Lagrange equations $\delta L/\delta y = 0$ and $\delta L/\delta z = 0$ is

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{\Omega^2}{2}\left(y^2 + z^2\right) - \frac{\beta}{2}\left(y^2 + z^2\right) - \frac{\alpha}{4}\left(y^4 + z^4\right) + \beta yz. \tag{3.1}$$

The Noether point symmetry operator X corresponding to a Lagrangian for a system of two linearly coupled nonlinear Duffing oscillators (1.1) is calculated from the formula (2.5) with respect to some function $B(x, y, z)$. The Noether point symmetry determining equations are

$$\xi_y = 0, \quad \xi_z = 0, \tag{3.2}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0, \quad \eta^2_z - \frac{1}{2}\xi_x = 0, \quad \eta^1_z + \eta^2_y = 0, \tag{3.3}$$

$$\eta^1_x = B_y, \quad \eta^2_x = B_z, \tag{3.4}$$

$$\eta^1\left(-\Omega^2 y - \beta y - \alpha y^3 + \beta z\right) + \eta^2\left(-\Omega^2 z - \beta z - \alpha z^3 + \beta y\right) \tag{3.5}$$

$$+\xi_x\left[-\frac{1}{2}\left(y^2 + z^2\right)\left(\beta + \Omega^2\right) - \frac{\alpha}{4}\left(y^4 + z^4\right) + \beta yz\right] = B_x. \tag{3.6}$$

After some simple calculations, (3.2)–(3.4) give rise to

$$\begin{aligned} \xi &= f(x), \\ \eta^1 &= \frac{1}{2}f' y - z b_1 + g(x), \\ \eta^2 &= \frac{1}{2}f' z + y b_1 + h(x), \\ B &= \frac{1}{4}f''\left(y^2 + z^2\right) + y g'(x) + z h'(x) + T(x), \end{aligned} \tag{3.7}$$

where b_1 is an arbitrary constant and f, g, h, and T are arbitrary functions of x. The insertion of (3.7) in (3.6) and separation with respect to powers of y and z reduce to the following system:

$$\alpha b_1 = 0, \quad \beta f' = 0, \tag{3.8}$$

$$\alpha f'(x) = 0, \quad \alpha g(x) = 0, \quad \alpha h(x) = 0, \tag{3.9}$$

$$\frac{1}{4}f''' + \beta f' + \Omega^2 f' - \beta b_1 = 0, \tag{3.10}$$

$$\frac{1}{4}f''' + \beta f' + \Omega^2 f' + \beta b_1 = 0, \tag{3.11}$$

$$g''(x) + \left(\beta + \Omega^2\right)g(x) - \beta h(x) = 0, \tag{3.12}$$

$$h''(x) + \left(\beta + \Omega^2\right)h(x) - \beta g(x) = 0, \tag{3.13}$$

$$T'(x) = 0. \tag{3.14}$$

Equation (3.14) results in

$$T(x) = c_0, \tag{3.15}$$

where c_o is a constant.

In order to solve system (3.8)–(3.13), the following cases arise.

Case 1. $\alpha = 0$, $\beta = 0$.

The solution of system (3.8)–(3.13) with $\alpha = 0$ and $\beta = 0$ yields

$$\begin{aligned} f(x) &= b_2 + b_3 \cos 2\Omega x + b_4 \sin 2\Omega x, \\ g(x) &= b_5 \cos \Omega x + b_6 \sin \Omega x, \\ h(x) &= b_7 \cos \Omega x + b_8 \sin \Omega x, \end{aligned} \tag{3.16}$$

where $b_1, \ldots, b_8$ are constants.

Now, (3.7) together with (3.16) gives

$$\xi = b_2 + b_3 \cos 2\Omega x + b_4 \sin 2\Omega x, \tag{3.17}$$

$$\eta^1 = (-b_3 \sin 2\Omega x + b_4 \cos 2\Omega x)\Omega y - b_1 z + b_5 \cos \Omega x + b_6 \sin \Omega x, \tag{3.18}$$

$$\eta^2 = (-b_3 \sin 2\Omega x + b_4 \cos 2\Omega x)\Omega z + b_1 y + b_7 \cos \Omega x + b_8 \sin \Omega x, \tag{3.19}$$

$$\begin{aligned} B = {} & -\Omega^2 [b_3 \cos 2\Omega x + b_4 \sin 2\Omega x]\left(y^2 + z^2\right) + y\Omega[-b_5 \sin \Omega x + b_6 \cos \Omega x] \\ & + z\Omega[-b_7 \sin \Omega x + b_8 \cos \Omega x] + c_o. \end{aligned} \tag{3.20}$$

Setting one of the constants equal to one and the rest to zero, we find the following Noether point symmetry operators and gauge terms:

$$X_1 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \qquad B = 0,$$

$$X_2 = \frac{\partial}{\partial x}, \qquad B = 0,$$

$$X_3 = \cos 2\Omega x\frac{\partial}{\partial x} - y\Omega \sin 2\Omega x\frac{\partial}{\partial y} - z\Omega \sin 2\Omega x\frac{\partial}{\partial z}, \qquad B = -\Omega^2\left(y^2 + z^2\right)\cos 2\Omega x,$$

$$X_4 = \sin 2\Omega x\frac{\partial}{\partial x} + y\Omega \cos 2\Omega x\frac{\partial}{\partial y} + z\Omega \cos 2\Omega x\frac{\partial}{\partial z}, \qquad B = -\Omega^2\left(y^2 + z^2\right)\sin 2\Omega x,$$

$$X_5 = \cos \Omega x\frac{\partial}{\partial y}, \qquad B = -\Omega y \sin \Omega x,$$

$$X_6 = \sin \Omega x \frac{\partial}{\partial y}, \qquad B = \Omega y \cos \Omega x,$$

$$X_7 = \cos \Omega x \frac{\partial}{\partial z}, \qquad B = -\Omega z \sin \Omega x,$$

$$X_8 = \sin \Omega x \frac{\partial}{\partial z}, \qquad B = \Omega z \cos \Omega x \ . \tag{3.21}$$

Case 2. $\alpha = 0, \beta \neq 0$.

The utilization of solution of (3.8)–(3.13) together with (3.7) results in

$$b_1 = 0, \qquad \xi = f(x) = c_1, \tag{3.22}$$

$$\eta^1 = g(x) = \frac{c_2}{2} \cos \Omega x + \frac{c_3}{2} \sin \Omega x + c_4 \cos\left(\sqrt{2\beta + \Omega^2}\right) x + c_5 \sin\left(\sqrt{2\beta + \Omega^2}\right) x, \tag{3.23}$$

$$\eta^2 = h(x) = \frac{c_2}{2} \cos \Omega x + \frac{c_3}{2} \sin \Omega x - c_4 \cos\left(\sqrt{2\beta + \Omega^2}\right) x - c_5 \sin\left(\sqrt{2\beta + \Omega^2}\right) x, \tag{3.24}$$

$$\begin{aligned} B \ &= \frac{\Omega}{2}(y+z)(-c_2 \sin \Omega x + c_3 \cos \Omega x) + (y-z)\sqrt{2\beta + \Omega^2} \\ &\times \left[-c_4 \sin\left(\sqrt{2\beta + \Omega^2}\right) x + c_5 \cos\left(\sqrt{2\beta + \Omega^2}\right) x \right] + c_o, \end{aligned} \tag{3.25}$$

where $c_1, \ldots, c_5$ are constants and $2\beta + \Omega^2 > 0$. If $2\beta + \Omega^2 < 0$ then $\cos(\sqrt{2\beta + \Omega^2})x$ is replaced with $\cosh(\sqrt{2\beta + \Omega^2})x$ and $\sin(\sqrt{2\beta + \Omega^2})x$ with $\sinh(\sqrt{2\beta + \Omega^2})x$.

The Noether point symmetries for $\alpha = 0, \beta \neq 0$ are

$$X_1 = \frac{\partial}{\partial x}, \qquad B = 0, \tag{3.26}$$

$$X_2 = \frac{1}{2} \cos \Omega x \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right), \qquad B = -\frac{\Omega}{2}(y+z) \sin \Omega x, \tag{3.27}$$

$$X_3 = \frac{1}{2} \sin \Omega x \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right), \qquad B = \frac{\Omega}{2}(y+z) \cos \Omega x, \tag{3.28}$$

$$X_4 = \cos\left(\sqrt{2\beta + \Omega^2}\right) x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right), \qquad B = -\sqrt{2\beta + \Omega^2}(y-z) \sin\left(\sqrt{2\beta + \Omega^2}\right) x, \tag{3.29}$$

$$X_5 = \sin\left(\sqrt{2\beta + \Omega^2}\right) x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right), \qquad B = \sqrt{2\beta + \Omega^2}(y-z) \cos\left(\sqrt{2\beta + \Omega^2}\right) x. \tag{3.30}$$

Case 3. If $\alpha \neq 0, \beta = 0$, then from system (3.8)–(3.13), we obtain

$$\begin{aligned} b_1 = 0, \quad & f(x) = d_1, \\ h(x) = 0, \quad & g(x) = 0, \end{aligned} \tag{3.31}$$

where d_1 is a constant. Hence,

$$\xi = d_1, \quad \eta^1 = 0, \quad \eta^2 = 0, \quad B = 0. \tag{3.32}$$

We get only one generator for this case as

$$X_1 = \frac{\partial}{\partial x}, \qquad B = 0. \tag{3.33}$$

Case 4. $\alpha \neq 0$, $\beta \neq 0$.

System (3.8)–(3.13) with $\alpha \neq 0$ and $\beta \neq 0$ yields the following results:

$$\begin{aligned} b_1 &= 0, \quad f(x) = e_1, \\ h(x) &= 0, \quad g(x) = 0, \end{aligned} \tag{3.34}$$

where e_1 is a constant. Thus,

$$\xi = e_1, \quad \eta^1 = 0, \quad \eta^2 = 0, \quad B = 0. \tag{3.35}$$

For this case, the symmetry generator is

$$X_1 = \frac{\partial}{\partial x}, \qquad B = 0. \tag{3.36}$$

The interpretation of the results in case of Noether point symmetries are given below.

Noether point symmetries

Case 1: in this case, we obtain an eight-dimensional Lie algebra.
Case 2: the Lie algebra is five dimensional.
Case 3: we deduce a one-dimensional Lie algebra.
Case 4: the Lie algebra for this case is also one dimensional.

3.2. First Integrals

The first integrals for Noether point symmetry generators for system (1.1) corresponding to the Lagrangian (3.1) are (by (2.8))

$$I = B - \left[\xi L + \left(\eta^1 - \xi y'\right) y' + \left(\eta^2 - \xi z'\right) z'\right], \tag{3.37}$$

where we have used $\partial L/\partial y' = y'$ and $\partial L/\partial z' = z'$ which hold for all the cases. The first integrals for each case computed with the help of (3.37) are summarized below.

Case 1. $\alpha = 0,\ \beta = 0$.

If we invoke the Noether point symmetry generators and gauge terms from system (3.21) in (3.37), we obtain the following first integrals:

$$\begin{aligned}
I_1 &= y'z - yz', \\
I_2 &= \frac{1}{2}\left(y'^2 + z'^2\right) + \frac{\Omega^2}{2}\left(y^2 + z^2\right), \\
I_3 &= -\Omega^2\left(y^2 + z^2\right)\cos 2\Omega x - \frac{1}{2}\cos 2\Omega x\left[y'^2 + z'^2 - \Omega^2\left(y^2 + z^2\right)\right] \\
&\quad + (yy' + zz')\Omega \sin 2\Omega x + \left(y'^2 + z'^2\right)\cos 2\Omega x, \\
I_4 &= -\Omega^2\left(y^2 + z^2\right)\sin 2\Omega x - \frac{1}{2}\sin 2\Omega x\left[y'^2 + z'^2 - \Omega^2\left(y^2 + z^2\right)\right] \\
&\quad - (yy' + zz')\Omega \cos 2\Omega x + \left(y'^2 + z'^2\right)\sin 2\Omega x, \\
I_5 &= -\Omega y \sin \Omega x - y' \cos \Omega x, \\
I_6 &= \Omega y \cos \Omega x - y' \sin \Omega x, \\
I_7 &= -\Omega z \sin \Omega x - z' \cos \Omega x, \\
I_8 &= \Omega z \cos \Omega x \ - z' \sin \Omega x.
\end{aligned} \tag{3.38}$$

For a system of two second-order ODEs, there are exactly four functionally independent first integrals. What we obtained are seven first integrals. Out of these, there are only four functionally independent first integrals.

Case 2. $\alpha = 0, \beta \neq 0$.

The first integrals for this case are

$$\begin{aligned}
I_1 &= \frac{1}{2}\left(y'^2 + z'^2\right) + \frac{\Omega^2}{2}\left(y^2 + z^2\right) + \frac{\beta}{2}\left(y^2 + z^2\right) - \beta yz, \\
I_2 &= -\frac{\Omega}{2}(y + z)\sin \Omega x - \frac{1}{2}\cos \Omega x (y' + z'), \\
I_3 &= \frac{\Omega}{2}(y + z)\cos \Omega x - \frac{1}{2}\sin \Omega x (y' + z'), \\
I_4 &= -(y - z)\sqrt{2\beta + \Omega^2}\sin\left(\sqrt{2\beta + \Omega^2}\right)x - (y' - z')\cos\left(\sqrt{2\beta + \Omega^2}\right)x, \\
I_5 &= (y - z)\sqrt{2\beta + \Omega^2}\cos\left(\sqrt{2\beta + \Omega^2}\right)x - (y' - z')\sin\left(\sqrt{2\beta + \Omega^2}\right)x.
\end{aligned} \tag{3.39}$$

Case 3. $\alpha \neq 0$, $\beta = 0$.

The simple calculations lead to the following integral:

$$I_1 = \frac{1}{2}\left(y'^2 + z'^2\right) + \frac{\Omega^2}{2}\left(y^2 + z^2\right) + \frac{\alpha}{4}\left(y^4 + z^4\right). \tag{3.40}$$

Case 4. $\alpha \neq 0$, $\beta \neq 0$.

Straightforward manipulations result in the following first integral:

$$I_1 = \frac{1}{2}\left(y'^2 + z'^2\right) + \frac{1}{2}\left(y^2 + z^2\right)\left(\beta + \Omega^2\right) + \frac{\alpha}{4}\left(y^4 + z^4\right) - \beta y z. \tag{3.41}$$

Note that the partial Lagrangian approach is similar to the Lagrangian approach; when one uses point-type operators one may not obtain all the first integrals. A familiar example is the Laplace-Runge-Lenz vector for the well-known Kepler plroblem which comes from a higher symmetry and not a geometric sysmmetric of point type. Here, too, we expect that the existence of an elliptic type integral will arise from a derivative dependent operator which will be of interest to look at in a future work. We only obtain the kinetic plus potential energy integrals here. Presumably, a higher derivative dependent operator could yield further integrals if they exist. These need to be further investigated.

4. Partial Noether's Approach

In this section, we derive the partial Noether operators of point type associated with a partial Lagrangian of (1.1) and then construct the first integrals for (1.1).

4.1. Partial Noether Operators of Point Type for Coupled Nonlinear Duffing Oscillators

A partial Lagrangian for system (1.1) is

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2, \tag{4.1}$$

so that system (1.1) can be written as

$$\frac{\delta L}{\delta y} = \Omega^2 y + \beta(y - z) + \alpha y^3, \qquad \frac{\delta L}{\delta z} = \Omega^2 z + \beta(z - y) + \alpha z^3. \tag{4.2}$$

The partial Noether operator X corresponding to a partial Lagrangian for the system (1.1) can be determined from (2.6) with respect to some function $B(x, y, z)$. The partial Noether operator of point-type determining equations are

$$\xi_y = 0, \qquad \xi_z = 0, \tag{4.3}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0, \qquad \eta^2_z - \frac{1}{2}\xi_x = 0, \qquad \eta^1_z + \eta^2_y = 0, \tag{4.4}$$

$$\eta^1_x = B_y - \xi\left(\Omega^2 y + \beta y + \alpha y^3 - \beta z\right), \tag{4.5}$$

$$\eta^2_x = B_z - \xi\left(\Omega^2 z + \beta z + \alpha z^3 - \beta y\right), \tag{4.6}$$

$$B_x + \eta^1\left(\Omega^2 y + \beta y + \alpha y^3 - \beta z\right) + \eta^2\left(\Omega^2 z + \beta z + \alpha z^3 - \beta y\right) = 0. \tag{4.7}$$

The solution of (4.3)–(4.6) yields

$$\begin{aligned} \xi &= f(x), \\ \eta^1 &= \frac{1}{2}f'y - zb_1 + g(x), \\ \eta^2 &= \frac{1}{2}f'z + yb_1 + h(x), \\ B &= \frac{1}{4}f''\left(y^2 + z^2\right) + yg'(x) + zh'(x) \\ &\quad + f\left[\frac{1}{2}\left(\Omega^2 + \beta\right)\left(y^2 + z^2\right) + \frac{\alpha}{4}\left(y^4 + z^4\right) - yz\beta\right] + T(x), \end{aligned} \tag{4.8}$$

where b_1 is an arbitrary constant and f, g, h, and T are arbitrary functions of x. The substitution of (4.8) in (4.7) and separation with respect to powers of y and z reduce to the system of (3.8)–(3.13) that we have obtained in the case of Noether point symmetries and $T(x) = c_o$. In case of partial Noether operators of point type the same cases need to be considered as we have for the Noether point symmetry operators.

Case 1. $\alpha = 0, \beta = 0$.

For this case ξ, η^1, and η^2 are the same as given in (3.17), (3.18), and (3.19) and, therefore, the partial Noether operators of point type are identical to the Noether point symmetry operators but the gauge terms are different. The gauge term for the partial Noether operators of point type is

$$\begin{aligned} B &= \frac{\Omega^2 b_2}{2}\left(y^2 + z^2\right) - \frac{1}{2}\Omega^2[b_3\cos 2\Omega x + b_4\sin 2\Omega x]\left(y^2 + z^2\right) \\ &\quad + y\Omega[-b_5\sin\Omega x + b_6\cos\Omega x] + z\Omega[-b_7\sin\Omega x + b_8\cos\Omega x] + c_o. \end{aligned} \tag{4.9}$$

Notice from (3.20) and (4.9) that only the B's relating to X_2, X_3, and X_4 for the partial Noether's case are different from that for the Noether's approach. The B's relating to X_2, X_3, and X_4 for the partial Noether case are

$$B = \frac{\Omega^2}{2}\left(y^2 + z^2\right),$$
$$B = -\frac{\Omega^2}{2}\left(y^2 + z^2\right)\cos 2\Omega x, \tag{4.10}$$
$$B = -\frac{\Omega^2}{2}\left(y^2 + z^2\right)\sin 2\Omega x.$$

Case 2. $\alpha = 0, \beta \neq 0$.

For this case, ξ, η^1, and η^2 are the same as given in (3.22), (3.23), and (3.24) and therefore the partial Noether operators of point type are similar to the case of Noether point symmetries (3.30) but the gauge terms are different. The gauge term for partial Noether case is

$$\begin{aligned} B = {} & \frac{\Omega}{2}(y+z)(-c_2 \sin\Omega x + c_3 \cos\Omega x) \\ & + (y-z)\sqrt{2\beta+\Omega^2}\left[-c_4 \sin\left(\sqrt{2\beta+\Omega^2}\right)x + c_5 \cos\left(\sqrt{2\beta+\Omega^2}\right)x\right] \\ & + c_1\left[\frac{1}{2}\left(\Omega^2+\beta\right)\left(y^2+z^2\right) - yz\beta\right] + c_o. \end{aligned} \tag{4.11}$$

Note that in (4.11) all the gauge terms are similar as in the case of Noether symmetries (see (3.25)) except one gauge term relating to X_1 is different which is given as follows:

$$B = \frac{1}{2}\left(\Omega^2 + \beta\right)\left(y^2 + z^2\right) - \beta yz. \tag{4.12}$$

Case 3. $\alpha \neq 0, \beta = 0$.

The partial Noether operator of point type when $\alpha \neq 0$, $\beta = 0$ is $X_1 = \partial/\partial x$ which is identical to the case of Noether point symmetries but the difference occurs in the gauge term as

$$B = \frac{\Omega^2}{2}\left(y^2 + z^2\right) + \frac{\alpha}{4}\left(y^4 + z^4\right). \tag{4.13}$$

Case 4. $\alpha \neq 0, \beta \neq 0$.

The operator for this case is also similar to the case of Noether point symmetries except B in (3.36) is different as follows

$$B = \frac{1}{2}\left(\Omega^2 + \beta\right)\left(y^2 + z^2\right) + \frac{\alpha}{4}\left(y^4 + z^4\right) - \beta yz. \tag{4.14}$$

4.2. First Integrals

For the partial Lagrangian (4.1) $\partial L/\partial y' = y'$ and $\partial L/\partial z' = z'$ also hold and therefore the first integrals can be computed from (3.37). Moreover, since Noether and partial Noether operator of point type are the same for each case, only the term $B-\xi L$ in formula (3.37) will be different from the Noether case.

Case 1. $\alpha = 0, \beta = 0.$

For operators X_1 and X_5 to X_8, we have $\xi = 0$ and the gauge terms are the same as for the case of Noether point symmetries. Therefore, we obtain the same first integrals. The gauge term B's relating to X_2, X_3, and X_4 for the partial Noether case are different from that for the Noether's approach and $\xi \neq 0$ for these operators; hence, formula (3.37) yields the same first integrals as we have constructed for the Noether's case.

Similarly, for other cases, the first integrals for the partial Noether operators of point type are identical to the case of Noether point symmetry generators.

The interpretation of the results in case of partial Noether operators of point type is as follows.

4.3. Partial Noether Operators of Point Type

The partial Noether operators of point type for the coupled nonlinear Duffing oscillators are the same as the Noether point symmetry operators. The difference arises in the "gauge" terms due to the Lagrangians being different. The algebras for both cases are isomorphic due to the fact that $\delta L/\delta y$ and $\delta L/\delta z$ are independent of derivatives. The first integrals due to the partial Noether operators of point type associated with a partial Lagrangian are equivalent to the case of Noether point symmetries.

5. Concluding Remarks

The Noether point symmetry operators corresponding to a Lagrangian of a system of two linearly coupled nonlinear Duffing oscillators have been constructed. We have computed the first integrals associated with Noether point operators by utilization of Noether's theorem. Then the partial Noether operators of point type and first integrals for the system under study are constructed corresponding to a partial Lagrangian. The partial Euler-Lagrange equations obtained herein are independent of derivatives, and therefore the partial Noether operators become point symmetry operators and the algebras are isomorphic for both the Noether and partial Noether cases. The first integrals corresponding to the Noether and partial Noether operators of point type are equivalent and these form one-, five-, and eight-dimensional Lie algebras. It would be of value to extend the results here to more general symmetries of Lie-Bäcklund type. Moreover, it is of interest to obtain reductions for the system using point or Lie-Bäcklund symmetries.

This study also points to new ways of constructing first integrals for systems without use of a Lagrangian. However, a partial Lagrangian can exist for such equations and one can invoke a formula for the construction of the first integrals. Since, in general, no variational problem exists for equations, one can utilize a partial Lagrangian approach which we have shown here to be effective for the coupled Duffing oscillators. The first integrals still can be constructed by using a Noether-like theorem.

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