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Research Article

Positive Almost Periodic Solution on a Nonlinear Differential Equation

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We study the following nonlinear equation $dx(t)/dt = x(t)[a(t) - b(t)x^{\alpha}(t) - f(t, x(t))] + g(t)$, by using fixed point theorem, the sufficient conditions of the existence of a unique positive almost periodic solution for above system are obtained, by using the theories of stability, the sufficient conditions which guarantee the stability of the unique positive almost periodic solution are derived.

1. Introduction

Let us consider the following Logistic-type equation

$$\frac{dx(t)}{dt} = a(t)x(t) - b(t)x^2(t) + e(t). \tag{1.1}$$

When the external perturbation $e(t) \equiv 0$ and $a(t), b(t)$ are positive constants, (1.1) is the typical Logistic equation. It was firstly introduced as a mathematical model for studying population dynamics and has become a classic topic in the textbooks on ordinary differential equations and its quality theory (see [1, 2]). By using the method of separation of variables and integration by partial fractions, we can get explicitly all the solutions of the typical Logistic equation and completely analyze the behavior of all the solutions. But when $a(t), b(t)$ are no longer constants and there exists some perturbation, the problem is not so simple as no explicit solutions can be found in general. In [2–6], the time-periodic case was considered. In [7], the existence of positive almost periodic solutions was considered when $a(t), b(t)$ are almost periodic and there is no perturbation. When $a(t), b(t)$ satisfy the assumption $\alpha \le a(t) \le A$, $\beta \le b(t) \le B$, and there is no external perturbation ($e(t) \equiv 0$), but there is a nonlinear perturbation $q(t,x)x$, Nkashama [8] obtained the existence of bounded and positive almost periodic solutions. In [9], Zhu et al. considered the case that $a(t), b(t)$ satisfy the assumption $\alpha \leq a(t) \leq A$, $\beta \leq b(t) \leq B$ and there exists an external perturbation $e(t)$, and they got the existence and uniqueness of positive periodic solutions and almost periodic solutions of (1.1).

In this paper, we consider the following more complex system:

$$\frac{dx(t)}{dt} = x(t)\left[a(t) - b(t)x^{\alpha}(t) - f(t, x(t))\right] + g(t), \tag{1.2}$$

where $\alpha > 0$, $t \in R$, $a(t), b(t), g(t)$ are all continuous almost periodic functions, and $f(t,x)$ is almost periodic in t and uniformly with respect to $x \in R$.

In this paper, we use the fixed point theorem and get the existence and uniqueness of positive almost periodic solution for (1.2), the stability of the unique positive almost periodic solution of (1.2) is also discussed, and some new results are obtained.

2. The Existence and Uniqueness of Positive Almost Periodic Solution

Before we start with our main results, for the sake of convenience, suppose that f is a continuous bounded function, and we denote $f_M = \sup_{t\in R} f(t)$ and $f_L = \inf_{t \in R} f(t)$; first, we introduce some lemmas.

Lemma 2.1 (see [10]). *Consider the following equation:*

$$\frac{dx}{dt} = a(t)x(t) + b(t), \tag{2.1}$$

where $a(t), b(t)$ are continuous almost periodic functions; if $\operatorname{Re} m(a(t)) \neq 0$, then (2.1) *exists a unique almost periodic solution $\eta(t)$, and $\eta(t)$ can be written as follows:*

$$\eta(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\int_s^t a(\tau)d\tau} b(s)ds, & \operatorname{Re} m(a(t)) < 0, \\ \displaystyle -\int_{t}^{+\infty} e^{\int_s^t a(\tau)d\tau} b(s)ds, & \operatorname{Re} m(a(t)) > 0, \end{cases} \tag{2.2}$$

where $m(a(t)) = \lim_{T\to+\infty}(1/T)\int_0^T a(t)dt$, $\operatorname{Re} m(a(t))$ is the real part of $m(a(t))$.

Lemma 2.2. *Consider* (1.2)*; if $g(t) \geq 0$; then the domain $R_+ = \{x \mid x > 0\}$ is a positive invariant with respect to* (1.2).

Proof. Since $g(t) \geq 0$, it follows that

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left[a(t) - b(t)x^{\alpha}(t) - f(t,x(t))\right] + g(t) \\ &\geq x(t)\left[a(t) - b(t)x^{\alpha}(t) - f(t,x(t))\right], \end{aligned} \tag{2.3}$$

thus we have

$$x(t) \geq x(t_0) \exp\left\{ \int_{t_0}^{t} \left[a(s) - b(s)x^{\alpha}(s) - f(s, x(s))\right] ds \right\}, \tag{2.4}$$

the assertion is valid for all $x(t_0) > 0$. The proof is completed. □

Theorem 2.3. *Consider* (1.2)*; α is a constant and $0 < \alpha < b_L/(b_M - b_L)$, $a(t), b(t), g(t)$ are continuous almost periodic functions, $f(t, x)$ is a continuous almost periodic function in $t \in R$ and uniformly with respect to $x \in R$, $f(t, 0) \equiv 0$; if the following conditions hold:*

(1) $a_L > 0$, $b_L > 0$, $g_L \geq 0$,

(2) $|f(t, x) - f(t, y)| \leq L|x - y|$, *for all* $t, x, y \in R$, *where* L *is a positive number,*

(3) $L\gamma/\alpha a_L \beta^{(1+\alpha)/\alpha} + L/a_L \beta^{1/\alpha} + g_M \gamma^{1/\alpha}(1+\alpha)/\alpha a_L < 1$, *where* $\beta = ((1+\alpha)b_L - \alpha b_M)/((1+\alpha)(a_M + \alpha a_L) + \alpha^2 a_L)$, $\gamma = ((1+\alpha)[b_M(a_M + \alpha a_L) + \alpha a_L b_L])/(a_L[(1+\alpha)(a_M + \alpha a_L) + \alpha^2 a_L])$,

then (1.2) *exists a unique positive almost periodic solution $\phi^*(t)$, and $1/\gamma^{1/\alpha} \leq \phi^*(t) \leq 1/\beta^{1/\alpha}$.*

Proof. Let $u(t) = 1/x^{\alpha}(t)$, since we only consider positive solutions of (1.2), by Lemma 2.2, it follows that $x(t) = u^{-1/\alpha}(t)$, then (1.2) can be written as follows:

$$\frac{du}{dt} = -\alpha a(t)u(t) + \alpha b(t) + \alpha u(t) f\left(t, u^{-1/\alpha}(t)\right) - \alpha u^{(1+\alpha)/\alpha}(t) g(t). \tag{2.5}$$

Define

$$B = \left\{\varphi(t); \varphi(t) \in AP(R), \beta \leq \varphi(t) \leq \gamma \right\}, \tag{2.6}$$

where $AP(R)$ denotes the set of almost periodic functions in R, the norm is defined as $\|\varphi\| = \sup_{t\in R}|\varphi(t)|$, thus $(B, \|\cdot\|)$ is a Banach space, given any $\varphi(t) \in B$, consider the following equation:

$$\frac{du}{dt} = -\alpha a(t)u(t) + \alpha b(t) + \alpha \varphi(t) f\left(t, \varphi^{-1/\alpha}(t)\right) - \alpha \varphi^{(1+\alpha)/\alpha}(t) g(t), \tag{2.7}$$

since $a_L > 0$, it follows that $m(a(t)) > 0$, thus $m[-\alpha a(t)] < 0$, from (2.7), by virtue of Lemma 2.1, we can know that (2.7) has a unique almost periodic solution $\eta(t)$, it can be written as follows:

$$\eta(t) = \alpha \int_{-\infty}^{t} e^{-\alpha \int_s^t a(\tau)d\tau} \left[b(s) + \varphi(s) f\left(s, \varphi^{-1/\alpha}(s)\right) - \varphi^{(1+\alpha)/\alpha}(s) g(s)\right] ds. \tag{2.8}$$

Now, define an operator T as follows:

$$T\varphi(t) = \alpha \int_{-\infty}^{t} e^{-\alpha \int_s^t a(\tau)d\tau} \left[b(s) + \varphi(s) f\left(s, \varphi^{-1/\alpha}(s)\right) - \varphi^{(1+\alpha)/\alpha}(s) g(s) \right] ds, \tag{2.9}$$

note that

$$\begin{aligned} T\varphi(t) &\le \alpha \int_{-\infty}^{t} e^{-\alpha a_L (t-s)} \left[b(s) + \varphi(s) f\left(s, \varphi^{-1/\alpha}(s)\right) \right] ds \\ &\le \alpha \int_{-\infty}^{t} e^{-\alpha a_L (t-s)} \left[|b(s)| + |\varphi(s)| \left| f\left(s, \varphi^{-1/\alpha}(s)\right) - f(s,0) \right| \right] ds \\ &\le \alpha \int_{-\infty}^{t} e^{-\alpha a_L (t-s)} \left[|b(s)| + L|\varphi(s)| \left|\varphi^{-1/\alpha}(s)\right| \right] ds \\ &\le \frac{1}{a_L} \left(b_M + \gamma L \beta^{-1/\alpha} \right). \end{aligned} \tag{2.10}$$

Since the condition $L\gamma/\alpha a_L \beta^{(1+\alpha)/\alpha} + L/a_L \beta^{1/\alpha} + g_M \gamma^{1/\alpha}(1+\alpha)/\alpha a_L < 1$ holds, it follows that $L\gamma < \alpha a_L \beta^{(1+\alpha)/\alpha}$; thus, we have

$$T\varphi(t) \le \frac{1}{a_L} \left(b_M + \alpha a_L \beta \right) = \gamma. \tag{2.11}$$

Also

$$\begin{aligned} T\varphi(t) &= \alpha \int_{-\infty}^{t} e^{-\alpha \int_s^t a(\tau)d\tau} \left[b(s) + \varphi(s) f\left(s, \varphi^{-1/\alpha}(s)\right) - \varphi^{(1+\alpha)/\alpha}(s) g(s) \right] ds \\ &\ge \alpha \int_{-\infty}^{t} e^{-\alpha a_M (t-s)} \left[b_L - \gamma L \beta^{-1/\alpha} - \gamma^{(1+\alpha)/\alpha} g(s) \right] ds. \end{aligned} \tag{2.12}$$

Since $L\gamma/\alpha a_L \beta^{(1+\alpha)/\alpha} + L/a_L \beta^{1/\alpha} + g_M \gamma^{1/\alpha}(1+\alpha)/\alpha a_L < 1$, it follows that $g(t) \le \alpha a_L/(1+\alpha)\gamma^{1/\alpha}$; thus, we have

$$\begin{aligned} T\varphi(t) &\ge \alpha \int_{-\infty}^{t} e^{-\alpha a_M (t-s)} \left[b_L - \gamma L \beta^{-1/\alpha} - \gamma^{(1+\alpha)/\alpha} g(s) \right] ds \\ &\ge \frac{1}{a_M} \left(b_L - \alpha \beta a_L - \frac{\alpha a_L \gamma}{1+\alpha} \right) = \beta. \end{aligned} \tag{2.13}$$

Hence $T\varphi(t) \in B$; therefore, $T : B \to B$. For any $\varphi(t), \psi(t) \in B$, it follows that

$$\begin{aligned}
\left|T\varphi(t) - T\psi(t)\right| &= \left|\alpha \int_{-\infty}^{t} e^{-\alpha \int_s^t a(\tau)d\tau} \left[\varphi(s) f\left(s, \varphi^{-1/\alpha}(s)\right) - \varphi^{(1+\alpha)/\alpha}(s) g(s) \right.\right. \\
&\qquad \left.\left. -\psi(s) f\left(s, \psi^{-1/\alpha}(s)\right) + \psi^{(1+\alpha)/\alpha}(s) g(s)\right] ds\right| \\
&= \left|\alpha \int_{-\infty}^{t} e^{-\alpha \int_s^t a(\tau)d\tau} \left[\varphi(s) f\left(s, \varphi^{-1/\alpha}(s)\right) - \varphi(s) f\left(s, \psi^{-1/\alpha}(s)\right) \right.\right. \\
&\qquad + \varphi(s) f\left(s, \psi^{-1/\alpha}(s)\right) - \psi(s) f\left(s, \psi^{-1/\alpha}(s)\right) \\
&\qquad \left.\left. + g(s)\left(\psi^{(1+\alpha)/\alpha}(s) - \varphi^{(1+\alpha)/\alpha}(s)\right)\right] ds\right| \\
&\le \alpha \int_{-\infty}^{t} e^{-\alpha \int_s^t a(\tau)d\tau} \left[\left|\varphi(s)\right| \left|f\left(s, \varphi^{-1/\alpha}(s)\right) - f\left(s, \psi^{-1/\alpha}(s)\right)\right| \right. \\
&\qquad + \left|\varphi(s) - \psi(s)\right| \left|f\left(s, \psi^{-1/\alpha}(s)\right)\right| \\
&\qquad \left. + \left|g(s)\right| \left|\varphi^{(1+\alpha)/\alpha}(s) - \psi^{(1+\alpha)/\alpha}(s)\right|\right] ds \\
&\le \alpha \int_{-\infty}^{t} e^{-\alpha a_L (t-s)} \left[L\left|\varphi(s)\right| \left|\varphi^{-1/\alpha}(s) - \psi^{-1/\alpha}(s)\right| + L\left|\psi^{-1/\alpha}(s)\right| \left|\varphi(s) - \psi(s)\right| \right. \\
&\qquad \left. + \left|g(s)\right| \left|\varphi^{(1+\alpha)/\alpha}(s) - \psi^{(1+\alpha)/\alpha}(s)\right|\right] ds.
\end{aligned} \tag{2.14}$$

According to mean value theorem, we can get

$$\begin{aligned}
\left|T\varphi(t) - T\psi(t)\right| \le \alpha \int_{-\infty}^{t} e^{-\alpha a_L (t-s)} &\left[L\left|\varphi(s)\right| \left|-\frac{1}{\alpha}\xi^{(-1/\alpha)-1}\right| \left|\varphi(s) - \psi(s)\right| + L\left|\psi^{-1/\alpha}\right| \left|\varphi(s) - \psi(s)\right| \right. \\
&\left. + \left|g(s)\right| \left|\frac{1+\alpha}{\alpha}\zeta^{1/\alpha}\right| \left|\varphi(s) - \psi(s)\right|\right] ds,
\end{aligned} \tag{2.15}$$

where $\varphi(s) < \xi, \zeta < \psi(s)$ or $\psi(s) < \xi, \zeta < \varphi(s)$. Notice that $\|\varphi(t)\| \le \gamma$, $\|\zeta^{1/\alpha}\| \le \gamma^{1/\alpha}$, $\|\psi^{-1/\alpha}(t)\| \le \beta^{-1/\alpha}$, $\|\xi^{-(1/\alpha)-1}\| \le \beta^{-(1/\alpha)-1}$; it follows that

$$\begin{aligned}
\left|T\varphi(t) - T\psi(t)\right| &\le \frac{1}{a_L}\left(L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1}\|\varphi - \psi\| + L\beta^{-1/\alpha}\|\varphi - \psi\| + g_M\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\|\varphi - \psi\|\right) \\
&= \frac{1}{a_L}\left(L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1} + L\beta^{-1/\alpha} + g_M\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\right)\|\varphi - \psi\|.
\end{aligned} \tag{2.16}$$

Note that $L\gamma/\alpha a_L\beta^{(1+\alpha)/\alpha} + L/a_L\beta^{1/\alpha} + g_M\gamma^{1/\alpha}(1+\alpha)/\alpha a_L < 1$, and L is a positive number; it follows that

$$0 < \frac{1}{a_L}\left(L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1} + L\beta^{-1/\alpha} + g_M\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\right) < 1, \tag{2.17}$$

therefore, T is a contraction mapping on B, that is to say, T has a unique fixed point on B, the unique fixed point is the unique positive almost periodic solution $\phi(t)$ of (2.5), and $\beta \le \phi(t) \le \gamma$. Notice that $u(t) = 1/x^{\alpha}(t)$, and, by Lemma 2.2, (1.2) has a unique positive almost periodic solution $\phi^*(t) = [\phi(t)]^{-1/\alpha}$, and $1/\gamma^{1/\alpha} \le \phi^*(t) \le 1/\beta^{1/\alpha}$. This completes the proof of Theorem 2.3. □

3. The Uniqueness of the Solution with Initial Value Problem

Consider (2.5); if given initial value $u(t_0) = u_0$, and $\beta \le u_0 \le \gamma$, then we have the following theorem.

Theorem 3.1. *Consider* (2.5); *α is a constant and $0 < \alpha < b_L/(b_M - b_L)$, $a(t), b(t), g(t)$ are continuous functions, $f(t,x)$ is a continuous function in $t \in R$ and uniformly with respect to $x \in R$, $f(t,0) \equiv 0$; if the following conditions hold:*

(1) $a_L > 0$, $b_L > 0$, $g_L \ge 0$,

(2) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all* $t, x, y \in R$, *where L is a positive number,*

(3) $L\gamma/\alpha a_L\beta^{(1+\alpha)/\alpha} + L/a_L\beta^{1/\alpha} + g_M\gamma^{1/\alpha}(1+\alpha)/\alpha a_L < 1$, *where* $\beta = (1+\alpha)b_L - \alpha b_M/(1+\alpha)(a_M+\alpha a_L) + \alpha^2 a_L$, $\gamma = ((1+\alpha)[b_M(a_M+\alpha a_L) + \alpha a_L b_L])/(a_L[(1+\alpha)(a_M+\alpha a_L) + \alpha^2 a_L])$,

then (2.5) *exists a unique continuous solution $u(t)$ with initial value $u(t_0) = u_0$ and $\beta \le u(t) \le \gamma (t \ge t_0)$.*

Proof. The initial value problem of (2.5) is equivalent to the solution of the following integral equation:

$$u(t) = e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}u_0 + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + u(s)f\left(s, u^{-1/\alpha}(s)\right) - u^{(1+\alpha)/\alpha}(s)g(s)\right]ds. \tag{3.1}$$

Define an operator T as follows:

$$Tu(t) = e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}u_0 + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + u(s)f\left(s, u^{-1/\alpha}(s)\right) - u^{(1+\alpha)/\alpha}(s)g(s)\right]ds. \tag{3.2}$$

The following proof is similar as the proof of Theorem 2.3, so we omit it here. □

4. The Stability of the Positive Almost Periodic Solution

Theorem 4.1. *Consider* (1.2)*; α is a constant and* $0 < \alpha < b_L/(b_M - b_L)$*,$a(t), b(t), g(t)$ are continuous almost periodic functions, $f(t,x)$ is a continuous almost periodic function in $t \in R$ and uniformly with respect to $x \in R$; if the following conditions hold:*

(1) $a_L > 0$, $b_L > 0$, $g_L \geq 0$,

(2) $|f(t,x) - f(t,y)| \leq L|x-y|$, *for all* $t, x, y \in R$, *where L is a positive number,*

(3) $L\gamma/\alpha a_L \beta^{(1+\alpha)/\alpha} + L/a_L\beta^{1/\alpha} + g_M\gamma^{1/\alpha}(1+\alpha)/\alpha a_L < 1$, *where* $\beta = ((1+\alpha)b_L - \alpha b_M)/((1+\alpha)(a_M + \alpha a_L) + \alpha^2 a_L)$, $\gamma = ((1+\alpha)[b_M(a_M+\alpha a_L) + \alpha a_L b_L])/(a_L[(1+\alpha)(a_M+\alpha a_L) + \alpha^2 a_L])$,

then there exists a region $\Omega = \{(t,x); 1/\gamma^{1/\alpha} \leq x \leq 1/\beta^{1/\alpha},\ t \geq t_0\}$*, and, in this region, the unique positive almost periodic solution of* (1.2) *is uniformly asymptotically stable.*

Proof. From Theorem 3.1, we know when $\beta \leq u_0 \leq \gamma$, (2.5) exists a unique solution $u(t)$ with initial value $u(t_0) = u_0$, and $\beta \leq u(t) \leq \gamma$, since the transformation $u(t) = 1/x^{\alpha}(t)$, it follows that $x(t) = 1/u^{1/\alpha}(t)$, so (1.2) exists a unique solution with initial value $x(t_0) = 1/u_0^{1/\alpha}$, and $1/\gamma^{1/\alpha} \leq x(t) \leq 1/\beta^{1/\alpha}$. It is easy to know that, the conditions of Theorem 4.1 are met with Theorem 2.3, so (2.5) exists a unique positive almost periodic solution $\phi(t)$; from (2.9), the unique positive almost periodic solution $\phi(t)$ of (2.5) can be expressed by integral equation as follows:

$$\begin{aligned}
\phi(t) &= \alpha \int_{-\infty}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + \phi(s)f\left(s, \phi^{-1/\alpha}(s)\right) - \phi^{(1+\alpha)/\alpha}(s)g(s)\right]ds \\
&= \alpha \int_{-\infty}^{t_0} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + \phi(s)f\left(s, \phi^{-1/\alpha}(s)\right) - \phi^{(1+\alpha)/\alpha}(s)g(s)\right]ds \\
&\quad + \alpha \int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + \phi(s)f\left(s, \phi^{-1/\alpha}(s)\right) - \phi^{(1+\alpha)/\alpha}(s)g(s)\right]ds \\
&= e^{\int_{t_0}^t(-\alpha a(\tau))d\tau}\phi(t_0) + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + \phi(s)f\left(s, \phi^{-1/\alpha}(s)\right) - \phi^{(1+\alpha)/\alpha}(s)g(s)\right]ds.
\end{aligned} \tag{4.1}$$

The solution of (2.5) with initial value $u(t_0) = u_0$ is given as follows:

$$u(t) = e^{\int_{t_0}^t(-\alpha a(\tau))d\tau}u_0 + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[b(s) + u(s)f\left(s, u^{-1/\alpha}(s)\right) - u^{(1+\alpha)/\alpha}(s)g(s)\right]ds, \tag{4.2}$$

from (4.1) and (4.2), when $t \geq t_0$, we have

$$\begin{aligned}
\left|u(t)-\phi(t)\right| = \Bigg| & e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}\left(u(t_0)-\phi(t_0)\right) \\
& + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\Big[f\left(s,u^{-1/\alpha}(s)\right)\left(u(s)-\phi(s)\right) \\
& \qquad + \phi(s)\left[f\left(s,u^{-1/\alpha}(s)\right)-f\left(s,\phi^{-1/\alpha}(s)\right)\right] \\
& \qquad + g(s)\left(\phi^{(1+\alpha)/\alpha}(s)-u^{(1+\alpha)/\alpha}(s)\right)\Big]ds\Bigg| \\
\leq\; & e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}\left|u(t_0)-\phi(t_0)\right| \\
& + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\Big[\left|f\left(s,u^{-1/\alpha}(s)\right)\right|\left|u(s)-\phi(s)\right| \\
& \qquad + \left|\phi(s)\right|\left|f\left(s,\phi^{-1/\alpha}(s)\right)-f\left(s,u^{-1/\alpha}(s)\right)\right| \\
& \qquad + \left|g(s)\right|\left|\phi^{(1+\alpha)/\alpha}(s)-u^{(1+\alpha)/\alpha}(s)\right|\Big]ds \\
\leq\; & e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}\left|u(t_0)-\phi(t_0)\right| \\
& + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\Big[\left|Lu^{-1/\alpha}(s)\right|\left|u(s)-\phi(s)\right| + L\left|\phi(s)\right|\left|\phi^{-1/\alpha}(s)-u^{-1/\alpha}(s)\right| \\
& \qquad + \left|g(s)\right|\left|\phi^{(1+\alpha)/\alpha}(s)-u^{(1+\alpha)/\alpha}(s)\right|\Big]ds.
\end{aligned} \tag{4.3}$$

According to mean value theorem, we can get

$$\begin{aligned}
\left|u(t)-\phi(t)\right| \leq\; & e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}\left|u(t_0)-\phi(t_0)\right| \\
& + \alpha\int_{t_0}^{t} e^{-\alpha\int_s^t a(\tau)d\tau}\left[\left|Lu^{-1/\alpha}(s)\right|\left|u(s)-\phi(s)\right| + L\left|\phi(s)\right|\left|-\frac{1}{\alpha}\xi^{-(1/\alpha)-1}\left(\phi(s)-u(s)\right)\right|\right. \\
& \qquad \left. + \left|g(s)\right|\left|\frac{1+\alpha}{\alpha}\zeta^{1/\alpha}\left(\phi(s)-u(s)\right)\right|\right]ds,
\end{aligned} \tag{4.4}$$

where $\phi(s) < \xi, \zeta < u(s)$ or $u(s) < \xi, \zeta < \phi(s)$. Notice that $|\phi| \leq \gamma$, $|\xi^{-1-(1/\alpha)}| \leq \beta^{-1-(1/\alpha)}$, $|u^{-1/\alpha}| \leq \beta^{-1/\alpha}$, $|\zeta^{1/\alpha}| \leq \gamma^{1/\alpha}$; it follows that

$$\begin{aligned}
\left|u(t)-\phi(t)\right| \leq\; & e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}\left|u(t_0)-\phi(t_0)\right| \\
& + \alpha\int_{t_0}^{t} e^{-\alpha a_L(t-s)}\left[\left|L\beta^{-1/\alpha}\right|\left|u(s)-\phi(s)\right| + L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1}\left|u(s)-\phi(s)\right|\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left|g(s)\right|\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\left|u(s)-\phi(s)\right|\Big]ds\\
&\le e^{\int_{t_0}^{t}(-\alpha a(\tau))d\tau}\left|u(t_0)-\phi(t_0)\right|\\
&\quad+\alpha\int_{t_0}^{t}e^{-\alpha a_L(t-s)}\left[L\beta^{-1/\alpha}+L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1}+g_M\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\right]\left|u(s)-\phi(s)\right|ds\\
&\le e^{-\alpha a_L(t-t_0)}\left|u(t_0)-\phi(t_0)\right|\\
&\quad+\alpha\int_{t_0}^{t}e^{-\alpha a_L(t-s)}\left[L\beta^{-1/\alpha}+L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1}+g_M\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\right]\left|u(s)-\phi(s)\right|ds.
\end{aligned}
\tag{4.5}
$$

Multiplying both sides of the above inequality by $e^{\alpha a_L(t-t_0)}$, we have

$$
\begin{aligned}
e^{\alpha a_L(t-t_0)}\left|u(t)-\phi(t)\right| \le{}& \left|u(t_0)-\phi(t_0)\right|\\
&+\alpha\int_{t_0}^{t}e^{\alpha a_L(s-t_0)}\left[L\beta^{-1/\alpha}+L\gamma\frac{1}{\alpha}\beta^{-(1/\alpha)-1}+g_M\frac{1+\alpha}{\alpha}\gamma^{1/\alpha}\right]\left|u(s)-\phi(s)\right|ds.
\end{aligned}
\tag{4.6}
$$

According to Bellman's inequality, we can obtain

$$
e^{\alpha a_L(t-t_0)}\left|u(t)-\phi(t)\right| \le \left|u(t_0)-\phi(t_0)\right|e^{\int_{t_0}^{t}[\alpha L\beta^{-1/\alpha}+L\gamma\beta^{-(1/\alpha)-1}+g_M(1+\alpha)\gamma^{1/\alpha}]ds}. \tag{4.7}
$$

Multiplied both sides of the above inequality by $e^{-\alpha a_L(t-t_0)}$, we get

$$
\left|u(t)-\phi(t)\right| \le \left|u(t_0)-\phi(t_0)\right|e^{(-\alpha a_L+\alpha L\beta^{-1/\alpha}+L\gamma\beta^{-(1/\alpha)-1}+g_M(1+\alpha)\gamma^{1/\alpha})(t-t_0)}. \tag{4.8}
$$

Notice that $u(t) = 1/x^{\alpha}(t)$, hence $x(t) = u^{-1/\alpha}(t)$, and $\phi^*(t) = \phi^{-1/\alpha}(t)$ is the unique positive almost periodic solution of (1.2), so we have

$$
\begin{aligned}
\left|x(t)-\phi^*(t)\right| &= \left|u^{-1/\alpha}(t)-\phi^{-1/\alpha}(t)\right|\\
&= \left|-\frac{1}{\alpha}\right|\left|\zeta_1^{-(1/\alpha)-1}\right|\left|u(t)-\phi(t)\right|\\
&\le \frac{1}{\alpha\beta^{(1+\alpha)/\alpha}}\left|u(t_0)-\phi(t_0)\right|e^{(-\alpha a_L+\alpha L\beta^{-1/\alpha}+L\gamma\beta^{-(1/\alpha)-1}+g_M(1+\alpha)\gamma^{1/\alpha})(t-t_0)},
\end{aligned}
\tag{4.9}
$$

where $u(t) < \zeta_1 < \phi(t)$ or $\phi(t) < \zeta_1 < u(t)$, note that the condition (3) holds, it follows that

$$
-\alpha a_L + \alpha L\beta^{-1/\alpha} + L\gamma\beta^{-(1/\alpha)-1} + g_M(1+\alpha)\gamma^{1/\alpha} < 0, \tag{4.10}
$$

therefore, there must be a small positive constant δ such that the following equality holds:

$$\left|x(t) - \phi^*(t)\right| \leq \frac{1}{\alpha\beta^{(1+\alpha)/\alpha}}\left|u(t_0) - \phi(t_0)\right|e^{-\delta(t-t_0)}. \tag{4.11}$$

From (4.11), we know that the unique positive almost periodic solution $\phi^*(t)$ of (1.2) is uniformly asymptotically stable. □

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