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A Qualitative Theory for Parabolic Problems under Dynamical Boundary Conditions

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For nonlinear parabolic problems in a bounded domain under dynamical boundary conditions, general comparison techniques are established similar to the ones under Neumann or Dirichlet boundary conditions. In particular, maximum principles and basic *a priori* estimates are derived, as well as lower and upper solution techniques that lead to functional band type estimates for classical solutions. Finally, attractivity properties of equilibria are discussed that also illustrate the damping effect of the dissipative dynamical boundary condition.

Keywords: Parabolic problems; Dynamical boundary conditions; Maximum and comparison principles; Upper and lower solutions; Convergence to equilibria

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1 INTRODUCTION

The aim of this paper is to develop a qualitative theory for parabolic problems in a bounded domain under dynamical boundary conditions, i.e. conditions of the form

$$\sigma\partial_t u + c\partial_\nu u = \rho u + h$$

on a part of the time lateral boundary. Throughout we deal with upper and lower solutions or with pairs of functions with separating parabolic

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defect. First, we derive comparison techniques and monotonicity properties of the flow similar to those in the nondynamical case. Then we establish the existence of particular solutions notably the infimal and supremal solutions obtained for a comparable pair of lower and upper solutions. These techniques are applied in order to obtain attractivity results for equilibria for reaction–diffusion-equations, that, in turn, illustrate the damping effect of the dissipative dynamical boundary condition on the convergence behaviour.

Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain whose boundary is decomposed into two disjoint parts

$$\partial\Omega = \partial_1\Omega \uplus \partial_2\Omega,$$

where $\partial_2\Omega$ is of class C^2 and relatively open in $\partial\Omega$. Let $\nu : \partial_2\Omega \to \mathbb{R}^n$ denote the outer normal unit vector field on $\partial_2\Omega$ and ∂_ν the outer normal derivative. For $T > 0$ we set $\bar{Q}_T = \bar{\Omega} \times [0, T]$ and introduce the *parabolic interior* Q_T and the *parabolic boundary* q_T as

$$Q_T = (\Omega \cup \partial_2\Omega) \times (0, T] \text{ and } q_T = \bar{Q}_T \setminus Q_T.$$

This terminology will be justified by the results below. We consider general parabolic equations of the form

$$\partial_t u = F(x, t, u, \nabla u, D^2u) =: F[u]$$

and inequalities associated to them, where throughout we suppose that

$$F : \bar{Q}_T \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \longrightarrow \mathbb{R} \text{ is increasing with respect to } q = D^2u. \tag{1}$$

Here the order $A \leq B$ between symmetric matrices means that the matrix $B - A$ is positively semidefinite. Unless otherwise stated, we do not require strict monotonicity. Thus, many of the results below include possible degeneracies of the principal part as e.g. the porous medium equation.

On $\partial_1\Omega \times (0, T]$ we prescribe an inhomogeneous Dirichlet condition, while on $\partial_2\Omega \times (0, T]$ we consider the dynamical boundary condition

$\mathcal{B}(u) = 0$ with

$$\mathcal{B}(u) := \sigma(x,t)\partial_t u + c(x,t)\partial_\nu u - \rho(x,t)u. \tag{2}$$

Throughout we will assume the dissipativity condition

$$c > 0, \ \ \sigma \geq 0 \quad \text{on } \partial_2\Omega \times (0,T]. \tag{3}$$

Without condition (3) blow up and nonuniqueness phenomena can occur. Take e.g. the function $u(x,t) = (T + x_1 - t)^{-1}$ defined on the open unit ball $\Omega = \{x \in \mathbb{R}^n \mid \|x\|_2 < 1\}$. Then u satisfies

$$\partial_t u = \tfrac{1}{2}(T + x_1 - t + 2)\Delta u - 2(T + x_1 - t)\|\nabla u\|_2^2$$

in $\Omega \times [0,T)$ and $x_1\partial_t u + \partial_\nu u = 0$ on $\partial\Omega \times [0,T)$, while u blows up for $t = T$ in Ω and on $\partial\Omega$.

2 COMPARISON AND MAXIMUM PRINCIPLES

The basic tool for comparing classical solutions is given by the following lemma that generalizes the techniques developed in [11] and that can also be used in more general cases [5].

LEMMA 2.1 *Let* $\varphi, \psi \in \mathcal{C}(\bar{Q}_T) \cap \mathcal{C}^{2,1}(Q_T)$ *satisfy*

$$\mathcal{B}(\varphi) \leq \mathcal{B}(\psi) \quad \text{on } \partial_2\Omega \times (0,T] \tag{4}$$

and the test point implication

$$\begin{aligned} &\varphi = \psi, \ \ \nabla\varphi = \nabla\psi, \ \ D^2\varphi \leq D^2\psi \quad \text{at } (x,t) \in Q_T \\ &\quad \Longrightarrow \partial_t\varphi < \partial_t\psi \text{ at } (x,t). \end{aligned} \tag{5}$$

Then $\varphi < \psi$ *on* q_T *implies* $\varphi < \psi$ *in* Q_T.

Proof Suppose that $\varphi < \psi$ on q_T. Set

$$t^* = \sup\{\tau \in [0,T] \mid \varphi < \psi \quad \text{on } (\Omega \cup \partial_2\Omega) \times (0,\tau)\}$$

and $H = (\Omega \cup \partial_2\Omega) \times \{t^*\}$. Hence $\delta := \psi - \varphi \geq 0$ on H. Suppose that $\delta(p) = 0$ for $p = (\tilde{x}, t^*) \in H$. If $\tilde{x} \in \Omega$, then $\partial_t\delta(p) \leq 0$ which is excluded by (5). If $\tilde{x} \in \partial_2\Omega$, then $\partial_t\delta(p) \leq 0$, $\partial_\nu\delta(p) \leq 0$ and (4) imply $\sigma(p)\partial_t\delta(p) = 0$ and $\nabla\delta(p) = 0$, which leads to $D^2\delta(p) \geq 0$. Thus $\partial_t\delta(p) > 0$ by (5), which contradicts $\partial_t\delta(p) \leq 0$. We conclude $\varphi < \psi$ on H, and finally, a compactness argument yields $t^* = T$.

Lemma 2.1 yields the following comparison principles and estimates with respect to the parabolic boundary q_T.

THEOREM 2.2 *Suppose that F satisfies a one-sided Lipschitz condition*

$$w \geq u \Rightarrow F(x,t,w,p,q) - F(x,t,u,p,q) \leq L(w-u) \tag{6}$$

in $Q_T \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ for some constant $L > 0$, and there exists $b \in \mathbb{R}_0^+$ such that $\rho \leq b\sigma$ on $\partial_2\Omega \times (0,T]$. Let $u, v \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ satisfy

$$\begin{aligned} &\partial_t u - F[u] \leq \partial_t v - F[v] && \text{in } Q_T, \\ &\mathcal{B}(u) \leq \mathcal{B}(v) && \text{on } \partial_2\Omega \times (0,T]. \end{aligned}$$

Then $u \leq v$ on q_T implies $u \leq v$ in Q_T.

Proof We may assume $L \geq 1$, $b \geq 1$. For $\varepsilon > 0$ set $\varphi = u$ and $\psi = v + \varepsilon L^{-1}e^{2Lbt}$. Then

$$\mathcal{B}(\psi) = \mathcal{B}(v) + \varepsilon e^{2Lbt}(2b\sigma - L^{-1}\rho) \geq \mathcal{B}(\varphi) \quad \text{on } \partial_2\Omega \times (0,T]$$

and at a test point with the hypotheses from (5) we conclude, using (1),

$$\begin{aligned} 0 &\leq \partial_t v - \partial_t u - F[v] + F[u] \\ &\leq \partial_t v - \partial_t u - F[\psi] + F[u] + \varepsilon \exp(2Lbt) \\ &\leq \partial_t v - \partial_t u + \varepsilon \exp(2Lbt) < \partial_t\psi - \partial_t\varphi. \end{aligned}$$

Lemma 2.1 implies $\varphi < \psi$, and, since $\varepsilon > 0$ was arbitrary, $u \leq v$ in Q_T.

COROLLARY 2.3 *Under the assumptions of Theorem 2.2 the initial boundary value problem (7) admits at most one solution in $C(\bar{Q}_T) \cap C^{2,1}(Q_T)$:*

$$\begin{aligned} &\partial_t u = F(x,t,u,\nabla u, D^2u) && \text{in } Q_T, \\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u - \rho(x,t)u = 0 && \text{on } \partial_2\Omega \times (0,T], \\ &u\,|_{q_T} = \psi \in C(q_T). \end{aligned} \tag{7}$$

As usual, the comparison principle assures the positivity of the flow, if 0 has nonnegative parabolic defect.

COROLLARY 2.4 *Under the assumptions of Theorem* 2.2 *and the additional hypothesis* $F(\cdot,\cdot,0,0,0)\geq 0$, *a solution* $u\in C(\bar{Q}_T)\cap C^{2,1}(Q_T)$ *of*

$$\begin{aligned} &\partial_t u - F[u]\geq 0 && \text{in } Q_T,\\ &\mathcal{B}(u)\geq 0 && \text{on } \partial_2\Omega\times(0,T],\\ &u\geq 0 && \text{on } q_T,\end{aligned}$$

satisfies $u\geq 0$ *in* Q_T.

Next, we deduce a weak maximum principle.

THEOREM 2.5 *Suppose* $F(\cdot,\cdot,\cdot,0,0)\leq 0$ $\langle F(\cdot,\cdot,\cdot,0,0)\geq 0\rangle$ *and* $\rho\leq 0$ *on* $\partial_2\Omega\times(0,T]$. *Let* $u\in C(\bar{Q}_T)\cap C^{2,1}(Q_T)$ *satisfy*

$$\begin{aligned} &\partial_t u\leq F[u]\ \ \langle\partial_t u\geq F[u]\rangle && \text{in } Q_T,\\ &\mathcal{B}(u)\leq 0\ \ \langle\mathcal{B}(u)\geq 0\rangle && \text{on } \partial_2\Omega\times(0,T].\end{aligned}$$

Then

$$\max_{q_T}\{u,0\}=\max_{\bar{Q}_T}\{u,0\}\quad \Big\langle \min_{q_T}\{u,0\}=\min_{\bar{Q}_T}\{u,0\}\Big\rangle.$$

Proof For $\varepsilon>0$, apply Lemma 2.1 to $\varphi=u$ and $\psi=\varepsilon t+\varepsilon+\max_{q_T}\{u,0\}$ in the maximum case and to $\varphi=\min_{q_T}\{u,0\}-\varepsilon-\varepsilon t$ and $\psi=u$ in the minimum case.

Of course, the weak maximum principle contains a positivity conclusion similar to the one of Corollary 2.4 under the stronger condition $F(\cdot,\cdot,\cdot,0,0)\geq 0$, but without (6).

Moreover, for a homogeneous boundary operator we deduce the

COROLLARY 2.6 *Under the conditions of Theorem* 2.5 *and, in addition,* $\rho=0$ *on* $\partial_2\Omega\times(0,T]$, *u satisfies*

$$\max_{q_T} u=\max_{\bar{Q}_T} u\quad \Big\langle \min_{q_T} u=\min_{\bar{Q}_T} u\Big\rangle.$$

Another classical *a priori* estimate [9] can be carried over to the dynamical case.

THEOREM 2.7 *Suppose $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ is a solution of the IBVP* (7), *where F fulfills an Osgood type sign condition*

$$\exists b_1, b_2 \geq 0, \ \ \forall (x,t) \in Q_T, \forall z \in \mathbb{R}: \quad zF(x,t,z,0,0) \leq b_1 z^2 + b_2 \tag{8}$$

and $\rho \leq b\sigma$ on $\partial_2\Omega \times (0,T]$ for some constant $b \geq 0$. Then

$$\max_{\bar{Q}_T} |u| \leq \inf_{\lambda > b_1, \lambda \geq b} \left(e^{\lambda T} \max\left\{ \max_{q_T} |\psi|, \sqrt{\frac{b_2}{\lambda - b_1}} \right\} \right). \tag{9}$$

Proof Apply Lemma 2.1 to $\varphi = u$, $\psi = (1+\varepsilon)e^{-\lambda t}\max\{\max_{q_T}|u|, \sqrt{b_2/(\lambda - b_1)}\}$ with $\lambda > b_1$, $\lambda \geq b$ and $\varepsilon > 0$. We have $\mathcal{B}(\psi) \geq 0$ on $\partial_2\Omega \times (0,T]$ and with (8) we conclude that at a test point (x_0, t_0) as in (5),

$$\begin{aligned}
\partial_t\psi - \partial_t\varphi &= \lambda\psi - F(x_0, t_0, \varphi, 0, D^2\varphi) \\
&\geq \lambda\psi - F(x_0, t_0, \varphi, 0, 0) \\
&\geq \psi\left(\lambda - b_1 - \frac{b_2}{\psi^2}\right) \geq \psi(1 - e^{-2\lambda t_0})(\lambda - b_1) > 0.
\end{aligned}$$

Thus $u \leq \psi$. In order to show $-u \leq \psi$ we apply Lemma 2.1 to $\varphi = -u$ and proceed similarly.

Next, we derive a strong maximum principle for strongly parabolic quasilinear operators of the form

$$D[u] := a^{ik}(x,t,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_k} + f(x,t,u,\nabla u)$$

using tensor notation and with positive constants μ_1 and μ_2 such that

$$0 \leq \mu_1 \xi^*\xi \leq a^{jm}(\cdot,\cdot,\cdot,\cdot)\xi_j\xi_m \leq \mu_2 \xi^*\xi$$

for all $\xi \in \mathbb{R}^n$. Moreover, we have to assume $\rho = 0$ in (2), thus we define

$$\mathcal{B}_0(u) := \sigma(x,t)\partial_t u + c(x,t)\partial_\nu u.$$

THEOREM 2.8 *Suppose there exists a positive constant C such that*

$$f(\cdot,\cdot,\cdot,p) \leq C|p| \quad \langle f(\cdot,\cdot,\cdot,p) \geq -C|p| \rangle \tag{10}$$

in $Q_T \times \mathbb{R} \times \mathbb{R}^n$. *Let* $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ *be a solution of*

$$\partial_t u \leq D[u] \;\; \langle \partial_t u \geq D[u] \rangle \quad \textit{in } \Omega \times (0, T], \tag{11}$$

and

$$\mathcal{B}_0(u) \leq 0 \;\; \langle \mathcal{B}_0(u) \geq 0 \rangle \quad \textit{in } \partial_2\Omega \times (0, T]. \tag{12}$$

Then

$$\max_{\bar{Q}_T} u = \max_{q_T} u \quad \Big\langle \min_{\bar{Q}_T} u = \min_{q_T} u \Big\rangle,$$

and if u *attains its maximum* M $\langle$*its minimum* $m\rangle$ *at some point* $(x_0, t_0) \in Q_T$, *then* $u = M$ $\langle u = m \rangle$ *in* $\bar{Q}_{t_0}$.

Proof If suffices to show the assertion in the maximum case, in the minimum case we proceed similarly.

If u attains its maximum M at (x_0, t_0) with $x_0 \in \Omega$ then we conclude $u = M$ for $t \leq t_0$ using the classical strong maximum principle for domains, see e.g. [11, IV.26]. By continuity, this shows $\max_{\bar{Q}_T} u = \max_{q_T} u$ and the strong assertion in the case $x_0 \in \Omega$.

Next, suppose that u attains M at (x_0, t_0) with $x_0 \in \partial_2\Omega$ and $u < M$ in $\Omega \times (0, t_0]$. Then, as $\partial_2\Omega$ is a relatively open C^2-part of the boundary of Ω, we find an open ball $B = \{y \in \Omega \mid \|y - y_0\|_2 < \varepsilon\} \subseteq \Omega$ of radius $\varepsilon > 0$ with $x_0 \in \partial B$. By (10) and (11), u satisfies a linear inequality with bounded coefficients

$$\partial_t u \leq \tilde{a}^{jm}(x, t) \frac{\partial^2 u}{\partial x_j \partial x_m} + \tilde{b}^j(x, t) \frac{\partial u}{\partial x_j},$$

where we have set $\tilde{a}^{jm}(x, t) = a^{jm}(x, t, u(x, t), \nabla u(x, t))$, $\tilde{b}^j(x, t) = C \operatorname{sign}((\partial/\partial x_j) u(x, t))$. This allows the application of the Friedman–Viborni-Theorem 3.4.6 of [10] in order to conclude that

$$\partial_\nu u(x_0, t_0) > 0.$$

But this is impossible by (12) and (3) and so u has to attain its maximum in $\Omega \times (0, t_0]$. With the case already shown above we conclude $u = M$ in $\bar{Q}_{t_0}$.

Note that in Corollary 2.6, we had to suppose the differential inequality also in $\partial_2\Omega \times (0, T]$. We note in passing that for simplicity reasons, the same assumption has been made in [6, Theorem 2.7]. But, following the arguments in the network case [2,4], the strong maximum principle for quasilinear operators holds also in ramified spaces without the assumption of differential inequality on the interfaces.

3 LOWER AND UPPER SOLUTIONS FOR TIME-PERIODIC PROBLEMS

We consider the periodic problem

$$\begin{aligned}
&\partial_t u = a(x)\Delta u + f(x,t,u) && \text{in } \Omega \times (0,T],\\
&\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u - \rho(x,t)u = 0 && \text{on } \partial_2\Omega \times (0,T],\\
&u = 0 && \text{on } \partial_1\Omega \times (0,T],\\
&u(x,0) = u(x,T) && \text{in } \Omega.
\end{aligned} \tag{13}$$

We assume

$$\begin{aligned}
&a \in \mathcal{C}(\Omega, (0,\infty));\\
&f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is continuous and } T\text{-periodic in } t;\\
&c \in \mathcal{C}(\partial_2\Omega \times [0,T], (0,\infty)), \sigma \in \mathcal{C}(\partial_2\Omega \times [0,T], [0,\infty)) \text{ and}\\
&\rho \in \mathcal{C}(\partial_2\Omega \times [0,T]) \text{ are } T\text{-periodic in } t.
\end{aligned} \tag{14}$$

We extend all the coefficients by periodicity to $\Omega \times [0,\infty)$. Moreover, for the sake of simplicity and as we do not want to treat existence results here, we assume to have enough regularity such that

for every $\tau > 0$, the operator $N : \mathcal{C}(\bar{Q}_\tau) \times \mathcal{C}_0(\Omega) \to \mathcal{C}(\bar{Q}_\tau) \times \mathcal{C}_0(\Omega)$, is continuous, compact, and satisfies (15)

$$\mathrm{Range}(N) \subset (\mathcal{C}(\bar{Q}_\tau) \cap \mathcal{C}^{2,1}(Q_\tau)) \times \mathcal{C}_0(\Omega),$$

where $C_0(\Omega) = \{u \in C(\Omega) \mid u = 0 \text{ on } \partial_1\Omega\}$ and $(u, u(\cdot, 0)) = N(v, u_0)$ is the solution of

$$\begin{aligned} &\partial_t u = a(x)\Delta u + f(x,t,v) && \text{in } \Omega \times (0,\tau],\\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = \rho(x,t)v && \text{on } \partial_2\Omega \times (0,\tau],\\ &u = 0 && \text{on } \partial_1\Omega \times (0,\tau],\\ &u(x,0) = u_0(x) && \text{in } \Omega. \end{aligned}$$

Note that the operator N is well defined due to Corollary 2.3. Moreover, in view of known existence results, e.g. [3,4 (Chap. 12), 6 or 8], the hypothesis (15) is reasonable.

DEFINITION 3.1 *A function $\alpha \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ is a lower solution of* (13) *if*

$$\begin{aligned} &\partial_t \alpha \le a(x)\Delta\alpha + f(x,t,\alpha) && \textit{in } \Omega \times (0,T],\\ &\sigma(x,t)\partial_t \alpha + c(x,t)\partial_\nu \alpha \le \rho(x,t)\alpha && \textit{on } \partial_2\Omega \times (0,T],\\ &\alpha \le 0 && \textit{on } \partial_1\Omega \times (0,T],\\ &\alpha(x,0) \le \alpha(x,T) && \textit{in } \Omega. \end{aligned}$$

An upper solution $\beta \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ *of* (13) *is defined in a similar way by reversing all the above inequalities.*

We also consider the Cauchy problem

$$\begin{aligned} &\partial_t u = a(x)\Delta u + f(x,t,u) && \text{in } \Omega \times (0,\tau],\\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = \rho(x,t)u && \text{on } \partial_2\Omega \times (0,\tau],\\ &u = 0 && \text{on } \partial_1\Omega \times (0,\tau],\\ &u(x,0) = u_0(x) && \text{in } \Omega. \end{aligned} \tag{16}$$

DEFINITION 3.2 *A function $\alpha \in C(\bar{Q}_\tau) \cap C^{2,1}(Q_\tau)$ is a* lower solution *of* (16) *if*

$$\begin{aligned} &\partial_t \alpha \le a(x)\Delta\alpha + f(x,t,\alpha) && \textit{in } \Omega \times (0,\tau],\\ &\sigma(x,t)\partial_t \alpha + c(x,t)\partial_\nu \alpha \le \rho(x,t)\alpha && \textit{on } \partial_2\Omega \times (0,\tau],\\ &\alpha \le 0 && \textit{on } \partial_1\Omega \times (0,\tau],\\ &\alpha(x,0) \le u_0(x) && \textit{in } \Omega. \end{aligned}$$

An upper solution $\beta \in \mathcal{C}(\bar{Q}_\tau) \cap \mathcal{C}^{2,1}(Q_\tau)$ *of* (16) *is defined a similar way by reversing all the above inequalities.*

PROPOSITION 3.3 *Assume that* (14) *and* (15) *are satisfied and* $u_0 \in \mathcal{C}_0(\Omega)$. *Let* α *and* β *be lower and upper solutions of* (16) *such that* $\alpha \le \beta$ *on* Q_τ. *Then the problem* (16) *has at least one solution* $u \in \mathcal{C}(\bar{Q}_\tau) \cap \mathcal{C}^{2,1}(Q_\tau)$ *with*

$$\alpha \le u \le \beta \quad on\ Q_\tau.$$

Proof Consider the modified problem

$$\begin{aligned} &\partial_t u = a(x)\Delta u + f(x,t,\gamma(x,t,u)) && \text{in } \Omega \times (0,\tau], \\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = \rho(x,t)\gamma(x,t,u) && \text{on } \partial_2\Omega \times (0,\tau], \\ &u = 0 && \text{on } \partial_1\Omega \times (0,\tau], \\ &u(x,0) = u_0(x) && \text{in } \Omega, \end{aligned} \tag{17}$$

where $\gamma(x,t,u) = \alpha(x,t) + (u - \alpha(x,t))^+ - (u - \beta(x,t))^+$. By assumption (15), we can apply Schauder's Fixed Point Theorem to prove that (17) has at least one solution $u \in \mathcal{C}(\bar{Q}_\tau) \cap \mathcal{C}^{2,1}(Q_\tau)$. Let us show that $\alpha \le u$ on Q_τ. Similarly one shows that $u \le \beta$ on Q_τ.

Set $v = u - \alpha$ and assume that $\min_{\bar{Q}_\tau} v < 0$. As $v \ge 0$ on q_τ, there exists $(x_0, t_0) \in Q_\tau$ such that $v(x_0,t_0) = \min_{\bar{Q}_\tau} v$. Then the strong minimum principle Theorem 2.8 applied locally to

$$\partial_t v - a(x)\Delta v \ge 0,$$

and the Friedman–Viborni-Theorem 1.c. yield a contradiction.

Remark 3.4 The assumption $\alpha, \beta \in \mathcal{C}(\bar{Q}_\tau) \cap \mathcal{C}^{2,1}(Q_\tau)$ can be relaxed to: for some $0 = t_0 < t_1 < \cdots < t_n = \tau$, $\alpha|_{\bar{\Omega}\times(t_i,t_{i+1}]}, \beta|_{\bar{\Omega}\times(t_i,t_{i+1}]} \in \mathcal{C}(\bar{\Omega} \times (t_i, t_{i+1}]) \cap \mathcal{C}^{2,1}((\Omega \cup \partial_2\Omega) \times (t_i, t_{i+1}])$ and for each $x \in \bar{\Omega}$, $i = 1, \ldots, n-1$,

$$\alpha(x,t_i) = \lim_{t \to t_i^-} \alpha(x,t) \ge \lim_{t \to t_i^+} \alpha(x,t)$$

and

$$\beta(x,t_i) = \lim_{t \to t_i^-} \beta(x,t) \le \lim_{t \to t_i^+} \beta(x,t).$$

PROPOSITION 3.5 *Assume* (14) *and* (15) *are satisfied and* $u_0 \in C_0(\Omega)$. *Then, the following holds:*

(i) *if* α_1, α_2 *are lower solutions and* β *is an upper solution of* (16) *satisfying* $\alpha_1 \leq \beta$ *and* $\alpha_2 \leq \beta$ *then there exists a solution* u *of* (16) *satisfying* $\max\{\alpha_1, \alpha_2\} \leq u \leq \beta$;
(ii) *if* α *is a lower solutions and* β_1, β_2 *are upper solutions of* (16) *satisfying* $\alpha \leq \beta_1$ *and* $\alpha \leq \beta_2$ *then there exists a solution* u *of* (16) *satisfying* $\alpha \leq u \leq \min\{\beta_1, \beta_2\}$.

Proof The proof of (i) is exactly the same as in Proposition 3.3 with $\alpha = \max\{\alpha_1, \alpha_2\}$ if we observe that $v(x_0, t_0)$ is either $u(x_0, t_0) - \alpha_1(x_0, t_0)$ or $u(x_0, t_0) - \alpha_2(x_0, t_0)$. Part (ii) is similar.

PROPOSITION 3.6 *Under the assumptions of Proposition* 3.3, *the problem* (16) *has an infimal and a supremal solution* $u_{\inf}$ *and* $u_{\sup}$ *in* $[\alpha, \beta]$ *i.e.* $u_{\inf}, u_{\sup} \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ *are solutions of* (16) *with* $\alpha \leq u_{\inf} \leq u_{\sup} \leq \beta$ *and every solution* $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ *of* (16) *such that* $\alpha \leq u \leq \beta$ *satisfies*

$$u_{\inf} \leq u \leq u_{\sup}.$$

Proof Let $\mathcal{L} = \{\mu : \bar{Q}_T \to \mathbb{R}: \alpha \leq \mu \leq \beta, \mu$ is a lower solution of (16)$\}$ and $\mathcal{U} = \{\nu : \bar{Q}_T \to \mathbb{R}: \alpha \leq \nu \leq \beta, \nu$ is an upper solution of (16)$\}$. Define

$$u_{\inf}(x, t) = \inf\{\nu(x, t): \nu \in \mathcal{U}\},$$
$$u_{\sup}(x, t) = \sup\{\mu(x, t): \mu \in \mathcal{L}\}.$$

We will show that $u_{\inf}$ is an infimal solution. The proof that $u_{\sup}$ is a supremal solution is similar.

Let $\{(x_N, t_N)\}_{N=1}^{\infty}$ be a dense subset of $\bar{Q}_T$ and for $N = 1, 2, \ldots$, let $\{\nu_{N,m}\}_{m=1}^{\infty}$ be a sequence of upper solutions such that

$$\lim_{m \to \infty} \nu_{N,m}(x_N, t_N) = u_{\inf}(x_N, t_N).$$

Let $\beta_1(x, t) = \nu_{1,1}(x, t)$. It follows from Proposition 3.3 that there exists a solution u_1 of (16) such that $\alpha \leq u_1 \leq \beta_1$. Let β_2 be defined by

$$\beta_2(t) = \min\{u_1(x, t), \nu_{1,2}(x, t), \nu_{2,2}(x, t)\}$$

then, by Proposition 3.5, there exists a solution u_2 of (16) such that $\alpha \le u_2 \le \beta_2$. Let us define inductively

$$\beta_{i+1}(x,t) = \min\{u_i(x,t), \nu_{1,i+1}(x,t), \dots, \nu_{i+1,i+1}(x,t)\},$$

then there exists a solution u_{i+1} of (16) such that $\alpha \le u_{i+1} \le \beta_{i+1}$. Hence, we have a sequence $\{u_i\}_{i=1}^{\infty}$ of solutions of (16) such that

$$\alpha \le \cdots \le u_i \le \beta_i \le \cdots \le u_2 \le \beta_2 \le u_1 \le \beta.$$

By assumption (15) and monotonicity, we deduce that the sequence $\{u_i\}_i$ converges in $C(\bar{Q}_T)$ to a solution u of (16). Furthermore, it is clear that, for every $N = 1, \dots,$

$$\lim_{i\to\infty} u_i(x_N, t_N) = u_{\inf}(x_N, t_N).$$

Hence $u(x_N, t_N) = u_{\inf}(x_N, t_N)$ for all $N \in \mathbb{N}$. As $\{(x_N, t_N)\}_N$ is dense in $\bar{Q}_T$, it follows that $u = u_{\inf}$ on $\bar{Q}_T$. In fact, assume by contradiction that for some $(\bar{x}, \bar{t}) \in \bar{Q}_T$, $u(\bar{x}, \bar{t}) > u_{\inf}(\bar{x}, \bar{t})$. By definition of $u_{\inf}$, we can find $\nu \in \mathcal{U}$ so that $u_{\inf}(\bar{x}, \bar{t}) \le \nu(\bar{x}, \bar{t}) < u(\bar{x}, \bar{t})$ and for (x,t) near enough $(\bar{x}, \bar{t})$, $\nu(x,t) < u(x,t)$. This is a contradiction if we choose (x,t) as an element of the set $\{(x_N, t_N)\}_{N=1}^{\infty}$. This concludes the proof if we observe that every solution u with $\alpha \le u \le \beta$ satisfies $u \in \mathcal{U}$ and hence $u \ge u_{\inf}$.

Note that several authors prefer the terminology *maximal* and *minimal* solution for *supremal* and *infimal* solution. But, in order to avoid confusion with maximality in the sense of existence, we prefer the notion adopted here.

THEOREM 3.7 *Assume that* (14) *and* (15) *are satisfied. Let α and β be lower and upper solutions of* (13) *such that $\alpha \le \beta$ on $\bar{Q}_T$ and $\alpha(\cdot, 0)$, $\beta(\cdot, 0) \in C_0(\Omega)$. Then, the following holds:*

(i) *there exist $u_{\inf}$ and $u_{\sup}$ infimal and supremal solutions of* (13) *in* $[\alpha, \beta]$ *i.e. $u_{\inf}$ and $u_{\sup}$ are solutions of* (13) *in* $[\alpha, \beta]$ *such that every solution u of* (13) *with $\alpha \le u \le \beta$ satisfies*

$$u_{\inf} \le u \le u_{\sup};$$

(ii) *there exist* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *solutions of* (16) *with* $\tau = \infty$ *and respectively* $u_0(\cdot) = \alpha(\cdot, 0)$, $u_0(\cdot) = \beta(\cdot, 0)$ *such that*

$$\alpha \leq \tilde{\alpha} \leq u_{\inf} \leq u_{\sup} \leq \tilde{\beta} \leq \beta$$

and

$$\lim_{t \to \infty} \|\tilde{\alpha}(\cdot, t) - u_{\inf}(\cdot, t)\|_{C^0(\bar{\Omega})} = 0,$$
$$\lim_{t \to \infty} \|\tilde{\beta}(\cdot, t) - u_{\sup}(\cdot, t)\|_{C^0(\bar{\Omega})} = 0;$$

(iii) *every solution* u *of* (16) *such that* $\alpha \leq u \leq \beta$ *on* Q_∞ *satisfies* $\tilde{\alpha} \leq u \leq \tilde{\beta}$ *on* Q_∞.

Remark 3.8 By (iii), $\tilde{\alpha}$ and $\tilde{\beta}$ are the infimal and the supremal solutions of (16) in $[\alpha, \beta]$, respectively with $u_0(\cdot) = \alpha(\cdot, 0)$ and $u_0(\cdot) = \beta(\cdot, 0)$.

Proof Let us prove the result for $u_{\inf}$ and $\tilde{\alpha}$. The other part is similar.

Define a sequence $(\tilde{\alpha}_n)_n$ of functions as follows. Take as $\tilde{\alpha}_0$ the infimal solution of

$$\begin{aligned}
&\partial_t \tilde{\alpha}_0 = a(x)\Delta\tilde{\alpha}_0 + f(x, t, \tilde{\alpha}_0) && \text{in } \Omega \times (0, T],\\
&\sigma(x, t)\partial_t \tilde{\alpha}_0 + c(x, t)\partial_\nu \tilde{\alpha}_0 = \rho(x, t)\tilde{\alpha}_0 && \text{on } \partial_2\Omega \times (0, T],\\
&\tilde{\alpha}_0 = 0 && \text{on } \partial_1\Omega \times (0, T],\\
&\tilde{\alpha}_0(x, 0) = \alpha(x, 0) && \text{in } \Omega,
\end{aligned} \tag{18}$$

satisfying $\alpha \leq \tilde{\alpha}_0 \leq \beta$. Such an $\tilde{\alpha}_0$ exists by Proposition 3.6 as α and β are lower and upper solutions of (18). Moreover $\tilde{\alpha}_0$ satisfies $\tilde{\alpha}_0(\cdot, T) \geq \alpha(\cdot, T) \geq \alpha(\cdot, 0) = \tilde{\alpha}_0(\cdot, 0)$. Then, we define recursively $(\tilde{\alpha}_n)_n$ by taking, for $n \geq 1$, as $\tilde{\alpha}_n$ the infimal solution of

$$\begin{aligned}
&\partial_t \tilde{\alpha}_n = a(x)\Delta\tilde{\alpha}_n + f(x, t, \tilde{\alpha}_n) && \text{in } \Omega \times (0, T],\\
&\sigma(x, t)\partial_t \tilde{\alpha}_n + c(x, t)\partial_\nu \tilde{\alpha}_n = \rho(x, t)\tilde{\alpha}_n && \text{on } \partial_2\Omega \times (0, T],\\
&\tilde{\alpha}_n = 0 && \text{on } \partial_1\Omega \times (0, T],\\
&\tilde{\alpha}_n(x, 0) = \tilde{\alpha}_{n-1}(x, T) && \text{in } \Omega,
\end{aligned} \tag{19}$$

satisfying $\tilde{\alpha}_{n-1} \leq \tilde{\alpha}_n \leq \beta$. Again such an $\tilde{\alpha}_n$ exists by Proposition 3.6 as $\tilde{\alpha}_{n-1}$ and β are lower and upper solution of (19) with $\tilde{\alpha}_{n-1} \leq \beta$. Moreover, $\tilde{\alpha}_n$ satisfies $\tilde{\alpha}_n(\cdot, T) \geq \tilde{\alpha}_{n-1}(\cdot, T) = \tilde{\alpha}_n(\cdot, 0)$. Accordingly, we have defined a sequence $(\tilde{\alpha}_n)_n$ of lower solutions of (13) such that, for each $n \geq 1$,

$$\alpha \leq \tilde{\alpha}_{n-1} \leq \tilde{\alpha}_n \leq \beta \tag{20}$$

and

$$\tilde{\alpha}_n(x,0) = \tilde{\alpha}_{n-1}(x,T) \quad \text{in } \Omega. \tag{21}$$

By monotonicity, $(\tilde{\alpha}_n)_n$ converges pointwise in $\bar{Q}_T$ to some function u satisfying $\alpha \leq u \leq \beta$. Moreover, by assumption (15) and monotonicity, we deduce that the sequence $(\tilde{\alpha}_n)_n$ converges in $\mathcal{C}(\bar{Q}_T)$ to a solution u of (13).

Now, define a function $\tilde{\alpha} : \bar{Q}_\infty \to \mathbb{R}$ as follows. If, for some $n \in \mathbb{N}$, $(x,t) \in \bar{Q} \times [nT,(n+1)T)$, we set

$$\tilde{\alpha}(x,t) := \tilde{\alpha}_n(x, t - nT).$$

It is easy to see that $\tilde{\alpha}$ is continuous, $\tilde{\alpha}(x,t) = 0$ on $\partial_1\Omega \times (0,\infty)$, $\tilde{\alpha}(\cdot,0) = \alpha(\cdot,0)$ and, for each $n \in \mathbb{N}$, $\tilde{\alpha}|_{\Omega \times [nT,(n+1)T]} \in \mathcal{C}(\bar{\Omega} \times [nT,(n+1)T]) \cap \mathcal{C}^{2,1}((\Omega \cup \partial_2\Omega) \times (nT,(n+1)T])$. By the periodicity of the coefficients, $\tilde{\alpha}$ also satisfies, for each $n \in \mathbb{N}$, the equations

$$\begin{aligned} &\partial_t\tilde{\alpha} = a(x)\Delta\tilde{\alpha} + f(x,t,\tilde{\alpha}) && \text{in } \Omega \times (nT,(n+1)T], \\ &\sigma(x,t)\partial_t\tilde{\alpha} + c(x,t)\partial_\nu\tilde{\alpha} = \rho(x,t)\tilde{\alpha} && \text{on } \partial_2\Omega \times (nT,(n+1)T]. \end{aligned}$$

We prove that, for each $n \in \mathbb{N}^+$, $\tilde{\alpha} \in \mathcal{C}^{2,1}((\Omega \cup \partial_2\Omega) \times (0,nT])$ and therefore $\tilde{\alpha}$ is a solution of

$$\begin{aligned} &\partial_t\tilde{\alpha} = a(x)\Delta\tilde{\alpha} + f(x,t,\tilde{\alpha}) && \text{in } \Omega \times (0,\infty), \\ &\sigma(x,t)\partial_t\tilde{\alpha} + c(x,t)\partial_\nu\tilde{\alpha} = \rho(x,t)\tilde{\alpha} && \text{on } \partial_2\Omega \times (0,\infty), \\ &\tilde{\alpha} = 0 && \text{on } \partial_1\Omega \times (0,\infty), \\ &\tilde{\alpha}(x,0) = \alpha(x,0) && \text{in } \Omega. \end{aligned}$$

Let us show that $\tilde{\alpha} \in \mathcal{C}^{2,1}((\Omega \cup \partial_2\Omega) \times (0,2T])$; then the general conclusion follows by induction. By (15), let $w \in \mathcal{C}(\bar{Q}_{2T}) \cap \mathcal{C}^{2,1}(Q_{2T})$ be the unique solution of the linear initial value problem

$$\begin{aligned} &\partial_t u - a(x)\Delta u = f(x,t,\tilde{\alpha}) && \text{in } \Omega \times (0,2T],\\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = \rho(x,t)\tilde{\alpha} && \text{on } \partial_2\Omega \times (0,2T],\\ &u = 0 && \text{on } \partial_1\Omega \times (0,2T],\\ &u(x,0) = \alpha(x,0) && \text{in } \Omega. \end{aligned}$$

Since both $\tilde{\alpha}|_{\Omega\times(0,T]}$ and $w|_{\Omega\times(0,T]}$ are solutions of the linear initial value problem

$$\begin{aligned} &\partial_t u - a(x)\Delta u = f(x,t,\tilde{\alpha}) && \text{in } \Omega \times (0,T],\\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = \rho(x,t)\tilde{\alpha} && \text{on } \partial_2\Omega \times (0,T],\\ &u = 0 && \text{on } \partial_1\Omega \times (0,T],\\ &u(x,0) = \alpha(x,0) && \text{in } \Omega, \end{aligned}$$

by uniqueness, we get $\tilde{\alpha} = w$ in $\bar{\Omega} \times [0,T]$. Further, both $\tilde{\alpha}|_{\Omega\times(T,2T)}$ and $w|_{\Omega\times(T,2T)}$ are solutions of the linear initial value problem

$$\begin{aligned} &\partial_t u - a(x)\Delta u = f(x,t,\tilde{\alpha}) && \text{in } \Omega \times (T,2T],\\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = \rho(x,t)\tilde{\alpha} && \text{on } \partial_2\Omega \times (T,2T],\\ &u = 0 && \text{on } \partial_1\Omega \times (T,2T],\\ &u(x,T) = \tilde{\alpha}(x,T) = w(x,T) && \text{in } \Omega. \end{aligned}$$

Then, by uniqueness, we get $\tilde{\alpha} = w$ in $\bar{\Omega} \times [T,2T]$. Therefore, we conclude that $\tilde{\alpha} = w$ in $\bar{\Omega} \times [0,2T]$, so that $\tilde{\alpha} \in C(\bar{Q}_{2T}) \cap C^{2,1}(Q_{2T})$.

Moreover, by periodicity and construction, we have

$$\lim_{t\to\infty} \|\tilde{\alpha}(\cdot,t) - u(\cdot,t)\|_{C^0(\bar{\Omega})} = 0.$$

To complete the proof of (i) and (ii), it remains to prove that every solution v of (13) such that $\alpha \le v \le \beta$ satisfies $v \ge \tilde{\alpha}_n$ for every n. This is clear as, if v is such a solution, v is an upper solution of (18) and by Proposition 3.3, there is a solution $\bar{\alpha}_0$ of (18) with $\alpha \le \bar{\alpha}_0 \le v \le \beta$. As $\tilde{\alpha}_0$ is the infimal solution of (18) in $[\alpha,\beta]$, we have $\alpha \le \tilde{\alpha}_0 \le \bar{\alpha}_0 \le v$. Recursively, if $\tilde{\alpha}_{n-1} \le v$, then v is an upper solution of (19) and hence $\tilde{\alpha}_{n-1} \le \tilde{\alpha}_n \le v$ which concludes the proof of (i) and (ii).

To prove that every solution v of (16) such that $\alpha \le v \le \beta$ on Q_∞ satisfies $\tilde{\alpha} \le v$ on Q_∞, we proceed again recursively, observing first that v is an upper solution of (18) and as $\tilde{\alpha}_0$ is the infimal solution of (18) in $[\alpha, \beta]$, $\alpha \le \tilde{\alpha}_0 = \tilde{\alpha} \le v$ on $\bar{Q}_T$. Moreover $v(x, t+T)$ is also an upper solution of (18). Hence $v(x,t) \ge \tilde{\alpha}_0(x, t-T)$ on $\bar{\Omega} \times [T, 2T]$. Recursively, if $v(x,t) \ge \tilde{\alpha}(x,t)$ on $\bar{\Omega} \times [0, nT]$ and $v(x,t) \ge \tilde{\alpha}_{n-1}(x, t-nT)$ on $\bar{\Omega} \times [nT, (n+1)T]$, then $v(x, t+nT)$ is an upper solution of (19) and $v(x, t+nT) \ge \tilde{\alpha}_{n-1}(x,t)$. As $\tilde{\alpha}_n$ is the infimal solution of (19) in $[\tilde{\alpha}_{n-1}, \beta]$, we have $\tilde{\alpha}_{n-1} \le \tilde{\alpha}_n \le v(\cdot, \cdot + nT)$ on $\bar{Q}_T$ i.e. $\tilde{\alpha} \le v$ on $\bar{Q}_{(n+1)T}$ and again, as above, $v(x,t) \ge \tilde{\alpha}_n(x, t-(n+1)T)$ on $\bar{\Omega} \times [(n+1)T, (n+2)T]$. Now, an induction argument shows the assertion.

We note in passing that the results of this section extend those obtained in [7] for homogeneous Dirichlet boundary conditions.

4 DAMPING EFFECT OF THE DISSIPATIVE DYNAMICAL BOUNDARY CONDITION

The comparison techniques of Sections 2 and 3 enable the comparison of solutions under different boundary condition, especially for the Neumann boundary condition and (2). Let us discuss this in a model case given by a globally attractive equilibrium. Though the global attractor turns out to be independent of the condition on $\partial\Omega \times (0,\infty)$ with $\partial\Omega_2 = \partial\Omega$, the convergence rate decreases with respect to the coefficient σ. The reaction term is supposed to admit two equilibria $A < B$ and to be of the form

$$\begin{aligned} &f(x,t,A) = f(x,t,B) = 0 \quad \text{for all } (x,t) \in \Omega \times (0,\infty), \\ &f(x,t,u) > 0 \ \text{ if } u \in (A,B), \qquad f(x,t,u) < 0 \ \text{ if } u \in (B,\infty). \end{aligned} \tag{22}$$

If a and f do not depend on x and t, we can state the following:

THEOREM 4.1 *Suppose that* $a \in (0,\infty)$, $\sigma c^{-1}(\cdot, t) \in L^1(\Omega)$ *for all* $t \in [0,\infty)$ *and* $f \in C([A,\infty))$ *fulfills* (6) *and* (22). *Let* $u \in C(\bar{Q}_\infty) \cap C^{2,1}(Q_\infty)$ *be a solution of*

$$\begin{aligned} &\partial_t u = a\Delta u + f(u) && \text{in } Q_\infty, \\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = 0 && \text{on } \partial\Omega \times (0,\infty), \\ &u(x,0) \ge A, \not\equiv A && \text{in } \Omega. \end{aligned} \tag{23}$$

Then

$$\lim_{t\to\infty} \|u(\cdot,t) - B\|_{C^0(\bar{\Omega})} = 0.$$

Proof Observe first that $u \geq A$ for $t \geq 0$ by Theorem 2.2 and, therefore, by the strong minimum principle Theorem 2.8 applied locally to $\partial_t u - a\Delta u \geq 0, u > A$ for $t > 0$. As for the Neumann condition, the energy $E(u)$ can serve as a Lyapunov functional

$$E(u) = \int_\Omega \left(\frac{a}{2}\|\nabla u\|_2^2 - \int_A^u f(s)\,\mathrm{d}s\right)\mathrm{d}x.$$

Then $E(u) \geq E(B) = -|\Omega| \int_A^B f(s)\,\mathrm{d}s$ and $E(u) = E(B)$ iff $u = B$. Green's formula yields,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(u) &= \int_\Omega \left\{a\left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla u\right)\cdot\nabla u - \partial_t u f(u)\right\}\mathrm{d}x \\ &= -\int_\Omega (\partial_t u)^2\,\mathrm{d}x + \int_{\partial\Omega} \partial_t u\, a\, \partial_\nu u\,\mathrm{d}s \\ &= -\int_\Omega (\partial_t u)^2\,\mathrm{d}x - \int_{\partial\Omega} \sigma c^{-1} a(\partial_t u)^2\,\mathrm{d}s\end{aligned}$$

Thus, $E(u)$ is a Lyapunov functional in the considered function class and the La Salle invariance principle (see e.g. [1]) yields the assertion.

Note that we did not use any consistency condition between the diffusion coefficient a and the boundary conductivity $c(x,t)$ as it was necessary in the systems considered in [6, Section 5]. For different dynamical boundary terms let us compare the solutions in the following simple case.

THEOREM 4.2 *Suppose that f, a, c and $0 \leq \sigma_1 \leq \sigma_2$ fulfill the hypotheses of Theorem* 4.1. *Let $u_1, u_2 \in C(\bar{Q}_\infty; [A,B]) \cap C^{2,1}(Q_\infty)$ be solutions of the IBVP* (23) *with $\sigma = \sigma_1$ and $\sigma = \sigma_2$, respectively and $u_2(\cdot,0) \leq u_1(\cdot,0)$ in $\bar{\Omega}$. Then $u_2 \leq u_1$ in $\Omega \times (0,\infty)$.*

Proof For $\mathcal{B}(u;\sigma) := \sigma\partial_t u + c\partial_\nu u$ the different time lateral boundary conditions yield

$$\mathcal{B}(u_1;\sigma_2) = (\sigma_2 - \sigma_1)\partial_t u_1.$$

Thus, if $\partial_t u_1 \geq 0$, the comparison principle Theorem 2.2 permits to conclude.

With the same argument as in the proof of Theorem 4.1, we see that $u_1 > A$ for $t > 0$. If $\partial_t u_1(\tilde{x}, \tilde{t}) < 0$ at some point $(\tilde{x}, \tilde{t}) \in Q_\infty$, then u attains a local minimum $m < B$ somewhere in $\bar{\Omega} \times [\tilde{t}, \infty)$ due to Theorem 4.1. The hypotheses and the differential equation imply that $m = A$, which is impossible. Thus, in fact, $\partial_t u_1 \geq 0$ in Q_∞.

Clearly, the corresponding result $u_2 \geq u_1$ holds for solutions taking values in $[B, \infty)$.

Let us illustrate the damping effect of the dissipative dynamical condition by means of the following simple example that has been computed with the aid of the convergent finite difference method similar to the one in [6, (78–79)]. Consider the equation

$$\partial_t u = \Delta u + u(1 - u) \quad \text{in } \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$$

under the condition

$$\sigma \partial_t u + \partial_\nu u = 0 \quad \text{on } \Gamma = \{0\} \times (0, 1)$$

and the Neumann condition on the remaining parts of $\partial\Omega$. Figure 1 displays the solutions on the cross-section $y = 0.5$ for values of σ varying in $[0, 2]$. Note that σ is not continuous on $\partial\Omega$. Though, for all $\sigma \geq 0$, the solutions with $u(\cdot, 0) \geq 0, \not\equiv 0$ tend to the equilibrium 1, the damping of the convergence rate increases with increasing σ.

Under higher regularity assumptions, the attractivity result holds also in the following nonautonomous case.

THEOREM 4.3 *Suppose that $a \in C(\Omega, (0, \infty))$ and that f, c and σ satisfy* (6), (14), (15) *and* (22). *Let $u \in C(\bar{Q}_\infty) \cap C^{2,1}(Q_\infty)$ be a solution of*

$$\begin{aligned} &\partial_t u = a(x)\Delta u + f(x, t, u) && \text{in } Q_\infty, \\ &\sigma(x, t)\partial_t u + c(x, t)\partial_\nu u = 0 && \text{on } \partial\Omega \times (0, \infty), \\ &u(x, 0) \geq A, \not\equiv A && \text{in } \Omega. \end{aligned} \tag{24}$$

Then

$$\lim_{t \to \infty} \|u(\cdot, t) - B\|_{C^0(\bar{\Omega})} = 0.$$

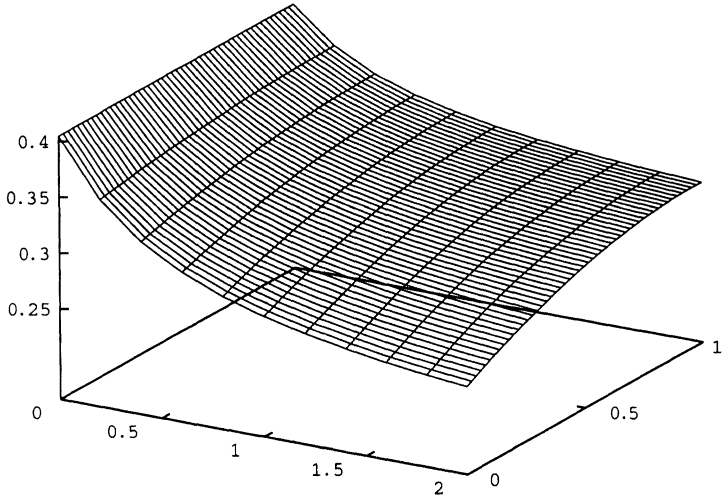

FIGURE 1 The damping effect of the dynamical condition.

Proof Again, by Theorem 2.2, $u \geq A$ for $t \geq 0$. Then by the strong maximum and minimum principle Theorem 2.8, there exist $d \geq B > c > A$ such that $d \geq u(x,t) \geq c$ on $\bar{\Omega} \times [T, \infty)$. Moreover c and d are lower and upper solutions of

$$\begin{aligned} &\partial_t u = a(x)\Delta u + f(x,t,u) && \text{in } \Omega \times (0,\infty),\\ &\sigma(x,t)\partial_t u + c(x,t)\partial_\nu u = 0 && \text{on } \partial\Omega \times (0,\infty),\\ &u(x,0) = u(x,T) && \text{on } \Omega. \end{aligned} \tag{25}$$

If we prove that B is the only solution of (25) in $[c,d]$, we can conclude by Theorem 3.7.

If B is not the only solution we have an infimal solution u_{inf} of (25) with $c \leq u_{\text{inf}} \leq B$ as B is a solution. Hence, on $\bar{\Omega} \times (0, T]$,

$$\partial_t u_{\text{inf}} \geq a(x)\Delta u_{\text{inf}}.$$

By the minimum principle and by periodicity

$$\min_{\bar{Q}_T} u_{\text{inf}} = \min_{\bar{\Omega}\times\{0\}} u_{\text{inf}} = \min_{\bar{\Omega}\times\{T\}} u_{\text{inf}}.$$

Again, by application of the minimum principle, we conclude that $u_{\inf}$ is constant in $\bar{\Omega} \times [0, T]$, hence $u_{\inf} = B$. In the same way, we prove that $u_{\sup} = B$.

With the same argument as for Theorem 4.2 we obtain another monotonicity result concerning different dynamical conditions for the problem (24).

COROLLARY 4.4 *Under the hypotheses of Theorem* 4.3, *two solutions* u_1, u_2 *belonging to* $C(\bar{Q}_\infty; [A, B]) \cap C^{2,1}(Q_\infty)$ *of the IBVP* (24) *with* $\sigma = \sigma_1$ *and* $\sigma = \sigma_2$, *respectively and satisfying* $u_2(\cdot, 0) \leq u_1(\cdot, 0)$ *on* $\bar{\Omega}$ *fulfill the latter inequality of* Q_∞.

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