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On a Generalization of the Osgood Condition

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In this paper a generalization of the famous uniqueness Osgood condition is given. This new result is important for many applications.

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1. INTRODUCTION

We consider nonlinear Volterra equations of the following type:

$$u(x) = \int_0^x (x-s)^{\alpha-1} g(u(s))\,\mathrm{d}s \quad (x \geq 0,\ \alpha \geq 1), \tag{1.1}$$

where the kernel k and the nonlinearity g are nonnegative. Moreover $g(u) = 0$ for $u \leq 0$.

This type of equation appears in some applications such as nonlinear diffusion problems or shock wave propagation [1]. It is clear that $u(x) \equiv 0$ is the trivial solution of (1.1) but from the physical point of view only nonnegative solutions of the considered equation are interesting.

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This problem is a very special case of the problem of the uniqueness of the trivial solution of the equation

$$u(x) = \int_0^x k(x, s, u(s))\, \mathrm{d}s \quad (x \geq 0).$$

If the trivial solution is unique one says that k is a Kamke function and this question appears in many problems not directly connected with the uniqueness of the solution [2]. In this paper we will consider only $k(x, s, u) = (x - s)^{\alpha-1} g(u)$. If we put $\alpha = 1$ in (1.1), then the uniqueness of the trivial solution is equivalent to the uniqueness of the trivial solution to the problem: $u' = g(u)$, $u(0) = 0$. If g is a nondecreasing continuous function ($g(0) = 0$), then the uniqueness answer is given by

$$\int_0^\delta \frac{\mathrm{d}s}{g(s)} = \infty.$$

If the last integral is finite, the problem $u' = g(u)$, $u(0) = 0$ has a nontrivial solution.

Having in mind the physical applications of (1.1), different mathematicians since the eighties have tried to generalize the Osgood condition for (1.1). It has been shown [1,3–6] that for a nondecreasing continuous g ($g(0) = 0$) the trivial solution is unique for (1.1) if and only if

$$\int_0^\delta \frac{\mathrm{d}s}{\phi_0(s)} = \infty, \quad \text{where } \phi_0(s) = s\left[\frac{g(s)}{s}\right]^{1/\alpha}. \tag{1.2}$$

Let us note that for $\alpha = 1$ we obtain the classical Osgood condition. But in some applications [7,8] there appear nonlinearities g which behave like u^p ($p \in (-1, 0)$). In this case the generalized Osgood condition does not work. In recent papers [9,10] a new condition for the uniqueness of the trivial solution in the case of g not necessarily increasing has been presented. But this was done for an integer $\alpha \geq 2$. In this note we want to present the generalization of the condition (1.2) for all the $\alpha > 1$ and nonlinearities g general enough.

We assume

(i) $g(s)$ is continuous for $s > 0$ and $g(s)s^{1/(\alpha-1)} \to 0$ as $s \to 0+$;
(ii) there exists $m \geq 0$ such that $g(s)s^m$ is nondecreasing in the right-hand side vicinity of zero.

Now we can formulate

THEOREM *Let $\alpha > 1$ and let g satisfy (i) and (ii). Then the trivial solution $u(x) \equiv 0$ is unique if and only if*

$$\int_0^\delta \frac{\mathrm{d}s}{\phi(s)} = \infty, \quad \textit{where } \phi(s) = s^{(\alpha-2)/(\alpha-1)}[\psi(s)]^{1/\alpha} \tag{1.3}$$

and

$$\psi(s) = s^{2-\alpha} \int_0^s (s-t)^{\alpha-2} g(t) t^{-(\alpha-2)/(\alpha-1)}\,\mathrm{d}t. \tag{1.4}$$

Remark 1.1 We shall prove theorem in the following equivalent form:
Equation (1.1) has a nontrival solution, i.e. a continuous function u such that $u(x) > 0$ for $x > 0$, if and only if

$$\int_0^\delta \frac{\mathrm{d}s}{\phi(s)} < \infty.$$

Remark 1.2 If g is a nondecreasing continuous function, then an easy comparison of ϕ with $s(g(s)/s)^{1/\alpha}$ shows that the conditions (1.2) and (1.3) are equivalent.

Remark 1.3 One can check easily that in the case $g(u) = u^{-\beta}$, $\beta \geq 1/(\alpha-1)$ Eq. (1.1) only has the trivial solution. Because of this we assume in (i) that $\lim_{s\to 0+} g(s)s^{1/(\alpha-1)} = 0$. If (1.1) has a nontrivial solution, then the condition $\lim_{s\to 0+} g(s)s^{1/(\alpha-1)} = 0$ is equivalent to the following one $\int_0^\delta g(s)s^{-(\alpha-2)/(\alpha-1)}\mathrm{d}s < \infty$. It is also known [10] that the last condition is necessary for the existence of nontrivial solutions of (1.1) in the case $\alpha \geq 2$. The case $\alpha \in (1, 2)$ is still open.

Remark 1.4 Slight modifications of assumptions (i) and (ii) allow us also to consider g which behave at the origin like $|\sin(1/x)|$ [10].

2. MAIN STEPS OF THE PROOF OF THE THEOREM

The proof of the theorem is based mainly on some *a priori* estimates of nontrivial solutions and properties of auxiliary functions. Since similar arguments to those used in [10] apply to the case $\alpha \geq 2$, we concentrate on $\alpha \in (1,2)$. As in [11] we can show

LEMMA 2.1 *Let μ be a Borel measure on $[0,a]$ $(a>0)$. Then the function*

$$u(x) = \int_0^x (x-s)^\beta \, \mathrm{d}\mu(s) \quad (\beta > 0)$$

is absolutely continuous and there exists constants $c_1, c_2 > 0$ such that

$$c_1 u'(x)^\beta \leq \int_0^x (u(x)-u(s))^{\beta-1} \, \mathrm{d}\mu(s) \leq c_2 u'(x)^\beta$$

for $x \in [0,a]$.

Remark 2.1 The function $x^{-\beta}u(x)$ is nondecreasing.

LEMMA 2.2 *Let $\alpha > 1$. Then the nontrivial solution of (1.1) is increasing and there exist constants $c_1, c_2 > 0$ such that*

$$c_1 v(x)^{\alpha-1} \leq \int_0^x (x-s)^{\alpha-2} g(s)[v(s)]^{-1} \mathrm{d}s \leq c_2 v(x)^{\alpha-1}, \tag{2.1}$$

where $v(x) = u'(u^{-1}(x))$.

To prove Lemma 2.2 we apply the results of Lemma 2.1 to (1.1) with $\beta = \alpha - 1$ and $\mathrm{d}\mu(s) = g(u(s))\,\mathrm{d}s$.

Throughout, a function $f\colon [0,a] \to [0,\infty)$ for which there exists a constant $c > 0$ such that

$$f(x) \leq cf(y) \quad \text{for } 0 < x < y \leq a$$

will be called an almost monotonous function.

LEMMA 2.3 *Let $\alpha \in (1,2)$. Then the function ψ defined by (1.4) is almost monotonous.*

Proof of Lemma 2.3 First we note that

$$\psi(s) = \int_0^s (s-t)^{\alpha-2}[(s-t)+t]^{2-\alpha}\psi_1(t)\,\mathrm{d}t,$$

where $\psi_1(s) = g(s)s^{-(\alpha-2)/(\alpha-1)}$.

We introduce the following auxiliary functions:

$$\psi_2(s) = \int_0^s \psi_1(t)\,\mathrm{d}t + \int_0^s (s-t)^{\alpha-2} t^{2-\alpha} \psi_1(t)\,\mathrm{d}t,$$

$$\psi_3(s) = \psi_1(s)s + m\int_0^s \psi_1(t)\,\mathrm{d}t,$$

where m is given by (ii) and

$$\psi_4(s) = \int_0^s \psi_1(t)\,\mathrm{d}t + \int_0^s (s-t)^{\alpha-2} t^{1-\alpha} \psi_3(t)\,\mathrm{d}t.$$

Making the following observations

$$\psi_3(x) = \lim_{\delta\to 0+} \int_\delta^s t^{-m}\mathrm{d}(t^{m+1}\psi_1(t))$$

and

$$\int_0^s (s-t)^{\alpha-2} t^{1-\alpha} \psi_3(t)\,\mathrm{d}t = \int_0^1 (1-t)^{\alpha-2} t^{1-\alpha} \psi_3(st)\,\mathrm{d}t,$$

we infer that the functions ψ_3 and ψ_4 are nondecreasing. Furthermore, we note that

$$\psi_2(s) \le \psi_4(s) \le \max(\gamma, 1+\gamma m)\psi_2(s) \quad (s \in (0,a]),$$

where $\gamma = \int_0^a (s-t)^{\alpha-2} t^{1-\alpha}\,\mathrm{d}t$. Thus ψ_2 is almost monotonous.

Finally, we easily see that

$$c_1\psi_2(s) \le \psi(s) \le c_2\psi_2(s) \quad (s \in (0,a])$$

for some constants $c_1, c_2 > 0$, which gives our assertion.

Now we can prove the lemma:

LEMMA 2.4 *Let ϕ be given by* (1.3) *and u be a nontrivial solution to* (1.1). *Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\phi(x) \le v(x) \le c_2\phi(x) \quad (x \in (0,a]), \tag{2.2}$$

where $v(x) = u'(u^{-1}(x))$.

Proof of Lemma 2.4 Let $\alpha \in (1,2)$. We shall denote

$$h(x) = \int_0^x (x-s)^{\alpha-2} g(s)[v(s)]^{-1} \mathrm{d}s \quad \text{and} \quad h_1(x) = \int_0^x g(s)[v(s)]^{-1} \mathrm{d}s.$$

We have the following relations

$$h_1(x) = \text{const} \int_0^x (x-s)^{1-\alpha} h(s)\, \mathrm{d}s \quad \text{and} \quad h(x) = \int_0^x (x-s)^{\alpha-2} h_1'(s)\, \mathrm{d}s.$$

By (2.1) we can write

$$\begin{aligned} \psi_1(s) &= h_1'(s)(s^{2-\alpha} v(s)^{\alpha-1})^{1/(\alpha-1)} \\ &\geq \text{const}\, h_1'(s)(s^{2-\alpha} h(s))^{1/(\alpha-1)}. \end{aligned} \tag{2.3}$$

Since

$$\omega(s;x) = \int_0^s (x-t)^{\alpha-2} t^{2-\alpha} h_1'(t)\, \mathrm{d}t \leq s^{2-\alpha} h(s) \quad (0 < s < x),$$

by (2.3) we get

$$h_1'(s)\omega(s;x)^{1/(\alpha-1)} \leq \text{const}\, \psi_1(s) \tag{2.4}$$

for $s \in (0, x]$. We also have the inequality

$$\begin{aligned} \psi(x) &= \int_0^x ((x-s)+s)^{2-\alpha}(x-s)^{\alpha-2} \psi_1(s)\, \mathrm{d}s \\ &\geq \text{const} \int_0^x \psi_1(s)\, \mathrm{d}s + \text{const} \int_0^x (x-s)^{\alpha-2} s^{2-\alpha} \psi_1(s)\, \mathrm{d}s \end{aligned}$$

(the constants are positive). By (2.4) we can write

$$\begin{aligned} \psi(x) \geq{} & \text{const} \int_0^x h_1'(s) h_1(s)^{1/(\alpha-1)}\, \mathrm{d}s \\ & + \text{const} \int_0^x (x-s)^{\alpha-2} s^{2-\alpha} h_1'(s) \omega(s;x)^{1/(\alpha-1)}\, \mathrm{d}s. \end{aligned} \tag{2.5}$$

Since the last integral is equal to const $[\omega(x;x)]^{\alpha/(\alpha-1)}$, by (2.5) we get

$$\psi(x) \geq \text{const}(h_1(x) + \omega(x;x))^{\alpha/(\alpha-1)}. \tag{2.6}$$

Noting that

$$h_1(x) = \int_0^x (x-t)^{\alpha-2}(x-t)^{2-\alpha} h_1'(t)\,\mathrm{d}t,$$

from (2.6) and the left-hand side of (2.1) we get

$$\psi(x) \geq \text{const}[x^{2-\alpha}h(x)]^{\alpha/(\alpha-1)} \geq \text{const}\, x^{(2-\alpha)/(\alpha-1)\alpha} v(x)^{\alpha}.$$

Hence we obtain the right-hand side of (2.2) for $\alpha \in (1,2)$. By the right-hand side of (2.2) and the monotonous properties of ψ we have

$$h(x) \geq \text{const} \int_0^x (x-s)^{\alpha-2} g(s) s^{-(\alpha-2)/(\alpha-1)} \mathrm{d}s\ \psi(x)^{-1/\alpha},$$

which gives

$$h(x) \geq \text{const}\, x^{\alpha-2}[\psi(x)]^{(\alpha-1)/\alpha}. \tag{2.7}$$

From (2.7) and the right-hand side of (2.1) we get the left-hand side of (2.2) for $\alpha \in (1,2)$. The lemma is proved.

Remark 2.2 If we consider the equation

$$u_\epsilon(x) = \epsilon x^{\alpha-1} + \int_0^x (x-s)^{\alpha-1} g(u_\epsilon(s))\,\mathrm{d}s \quad (\alpha > 1) \tag{2.8}$$

then putting $\mu(s) = \epsilon\delta_0 + g(u_\epsilon(s))\,\mathrm{d}s$ and repeating our considerations we have

$$c_1\left(\epsilon x^{\alpha-1} + \phi(x)^{\alpha-1}\right)^{1/(\alpha-1)} \leq v_\epsilon(x) \leq c_2\left(\epsilon x^{\alpha-1} + \phi(x)^{\alpha-1}\right)^{1/(\alpha-1)}, \tag{2.9}$$

where $c_1, c_2 > 0$ and $v_\epsilon(x) = u_\epsilon'(u_\epsilon^{-1}(x))$.

Sketch of the Proof of Theorem If (1.1) has a nontrivial solution u, then

$$u^{-1}(x) = \int_0^x (u^{-1})'(s)\,\mathrm{d}s = \int_0^x [v(s)]^{-1}\mathrm{d}s.$$

By (2.2) we get

$$\infty > u^{-1}(x) \geq \int_0^x [\phi(s)]^{-1} \mathrm{d}s$$

and the necessary condition for the existence of nontrivial solutions is proved.

By Schauder-type arguments it can be shown that for every $\epsilon \in (0, \epsilon_0)$ Eq. (2.8) has a nontrivial solution u_ϵ. Since all solutions satisfy (2.9), by the Arzela–Ascoli theorem [12] there exists a sequence $\epsilon_n \to 0$, as $n \to \infty$ and the corresponding solutions u_n of (2.8) such that $u_n(x)$ converges uniformly to a solution $u(x)$ of (1.1) on the interval $[0, a]$ $(a > 0)$ as $n \to \infty$.

Since by (2.9)

$$u_n^{-1}(x) \leq \text{const} \int_0^x \frac{\mathrm{d}s}{\phi(s)} = F^{-1}(x),$$

or equivalently $u_n(x) \geq F(x)$ on $[0, a]$ for all n. This implies $u(x) \geq F(x)$ on $[0, a]$ and u is a nontrivial solution to (1.1). Thus the sufficient condition for the existence of nontrivial solutions is proved.

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