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Nonlinear Singular Integral Inequalities for Functions in Two and n Independent Variables*

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In this paper nonlinear integral inequalities with weakly singular kernels for functions in two and n independent variables are solved. The obtained results are related to the well known Gronwall–Bihari and Henry inequalities for functions in one variable and the Wendroff inequality for functions in two variables. A modification of Ou–Iang–Pachpatte inequality and inequalities for functions in n independent variables are also treated here.

Keywords: Integral inequality; Weakly singular kernel; Henry inequality; Wendroff inequality; Gronwell–Bihari inequality

AMS Subject Classification (1991): 34D05, 35B35, 35K55

1. INTRODUCTION

D. Henry proposed in his book [7] a method to estimate solutions of linear integral inequality with weakly singular kernel. His inequality plays the same role in the geometric theory of parabolic partial differential equations (see [6,7,18]) as the well known Gronwall inequality in the theory of ordinary differential equations. In the paper [13] a new method to estimate solutions for nonlinear integral inequalities with

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singular kernels of Bihari type is proposed. The resulting estimation formulas are similar to those for classical integral inequalities (see [1,2,5,9–12,16]). For instance the estimate of solution of the Henry inequality is of exponential form in contrary to the Henry's estimate (see [7,18]) by an infinite series of a complicated form. The method has been applied in the paper [14] in the proof of global existence of solutions and a stability theorem for a class of parabolic PDEs.

In this paper we use the method proposed by the author in the paper [13] to obtain an analogue of the Wendroff inequality (see [1,5,9,10]) for functions in two variables. Thandapani and Agarwal [19] proved interesting results concerning inequalities for functions in n independent variables. Applying our method of desingularization of weakly singular inequalities we prove a singular version of one of them. We remark that the papers [3,4,15,19] contain many results on inequalities of Wendroff type and applying our desingularization method one can formulate and prove their singular versions in a similar way as we are doing this in Section 4. We also present an estimate of solutions of an analogue of Ou–Iang inequality whose generalization for the nonlinear case has been given by Pachpatte [16].

2. WENDROFF TYPE INEQUALITIES

First let us recall a definition of a class of functions from the paper [13].

DEFINITION 2.1 *Let $q > 0$ be a real number and $0 < T \le \infty$. We say that a function $\omega : R^+ \to R$ ($R^+ = \langle 0, \infty)$) satisfies a condition* (q) *if*

$$\mathrm{e}^{-qt}[\omega(u)]^q \le R(t)\omega(\mathrm{e}^{-qt}u^q) \quad \textit{for all } u \in R^+, \quad t \in \langle 0, T), \qquad \text{(q)}$$

where $R(t)$ is a continuous, nonnegative function.

Examples (see [13])

1. $\omega(u) = u^m$, $m > 0$ satisfies the condition (q) with $R(t) = \mathrm{e}^{(m-1)qt}$.
2. $\omega(u) = u + au^m$, where $0 \le a \le 1, m \ge 1$ satisfies the condition (q) with $R(t) = 2^{q-1}\mathrm{e}^{qmt}$.

We shall need the following well known consequence of the Jensen inequality:

$$(A_1 + A_2 + \cdots + A_n) \le n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r) \tag{1}$$

(see [8,17]).

We shall study an inequality of the type

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1} \times F(s,t)\omega(u(s,t))\,\mathrm{d}s\,\mathrm{d}t, \tag{2}$$

for $(x,y) \in \langle 0,T)^2 = \langle 0,T) \times \langle 0,T)$ $(0 < T \le \infty)$, where $\alpha > 0$, $\beta > 0$. Results on integral inequalities in two variables with regular kernels (i.e. with $\alpha = 1$, $\beta = 1$, F continuous) and $a(x,y)$ constant are contained in the books [1,5,9,10].

THEOREM 2.2 *Let $a(x,y)$ be a nonnegative, C^2-function,*

$$\frac{\partial^2 a(x,y)}{\partial x \partial y} \ge 0, \quad \frac{\partial a(x,y)}{\partial x} \ge 0 \quad \left(or \ \frac{\partial a(x,y)}{\partial y} \ge 0 \right) \tag{C}$$

on $\langle 0,T)^2 = \langle 0,T) \times \langle 0,T)$ $(0 < T \le \infty)$, $u(x,y)$, $F(x,y)$ be continuous, nonnegative functions on $\langle 0,T)^2$ satisfying the inequality (2), *where $\omega : R^+ \to R$ is a nonnegative C^1-function. Then the following assertions hold:*

(i) *Suppose $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ and ω satisfies the condition* (q) *with $q = 2$. Then*

$$u(x,y) \le \mathrm{e}^{x+y} \left\{ \Omega^{-1} \left[\Omega(2a(x,y)^2) + 2K \int_0^x \int_0^y F(s,t)^2 R(s+t)\,\mathrm{d}s\,\mathrm{d}t \right] \right\}^{1/2}, \tag{3}$$

where

$$K = \frac{\Gamma(2\beta - 1)\Gamma(2\alpha - 1)}{4^{\alpha+\beta-1}},$$

$$(x,y) \in \langle 0,T_1)^2 = \langle 0,T_1) \times \langle 0,T_1),$$

Γ *is the Gamma function,* $\Omega(v) = \int_{v_0}^{v} \mathrm{d}y/\omega(y)$, $v_0 > 0$, Ω^{-1} *is the inverse of* Ω *and* $T_1 > 0$ *is such that the argument of* Ω^{-1} *in* (3) *belongs to Dom*(Ω^{-1}) *for all* $(x, y) \in \langle 0, T_1)^2$.

(ii) *Suppose* $\alpha = \beta = 1/(z+1)$ *for some real number* $z \geq 1$ *and* ω *satisfies the condition* (q) *with* $q = z+2$. *Then*

$$u(x,y) \leq \mathrm{e}^{x+y}\left\{\Omega^{-1}\left[\Omega(2a(x,y)^2) + M_z \int_0^x \int_0^y F(s,t)^q R(s+t)\,\mathrm{d}s\,\mathrm{d}t\right]\right\}^{1/q},$$

$(x, y) \in \langle 0, T_2)$, *where*

$$p = \frac{z+2}{z+1}, \quad M_z = \left(\frac{\Gamma(1-p\delta)}{p^{(1-p\delta)}}\right)^{2/p}, \quad \delta = 1 - \beta = \frac{z}{z+1},$$

$T_2 > 0$ *is such that the argument of* Ω^{-1} *belongs to Dom*(Ω^{-1}) *for all* $(x, y) \in \langle 0, T_2)$.

Proof First let us prove the assertion (i). Using the Cauchy–Schwarz inequality we obtain from (2)

$$\begin{aligned} u(x,y) &\leq a(x,y) + \int_0^x \int_0^y (x-s)^{\alpha-1}\mathrm{e}^s (y-t)^{\beta-1} \\ &\quad \times \mathrm{e}^t[\mathrm{e}^{-(s+t)}F(s,t)\omega(u(s,t))]\,\mathrm{d}s\,\mathrm{d}t \\ &\leq a(x,y) + \left[\int_0^x \int_0^y (x-s)^{2\alpha-2}\mathrm{e}^{2s}(y-t)^{2\beta-2}\mathrm{e}^{2t}\,\mathrm{d}s\,\mathrm{d}t\right]^{1/2} \\ &\quad \times \left[\int_0^x \int_0^y \mathrm{e}^{-2(s+t)}F(s,t)^2\omega(u(s,t))^2\,\mathrm{d}s\,\mathrm{d}t\right]^{1/2}. \end{aligned} \tag{4}$$

For the first integral in (4) we have the estimate

$$\begin{aligned} &\int_0^x \int_0^y (x-s)^{2\alpha-2}\mathrm{e}^{2s}(y-t)^{2\beta-2}\mathrm{e}^{2t}\,\mathrm{d}s\,\mathrm{d}t \\ &\quad = \mathrm{e}^{2(x+y)} \int_0^x \sigma^{2\alpha-2}\mathrm{e}^{-2\sigma} \int_0^y \eta^{2\beta-2}\mathrm{e}^{-2\eta}\,\mathrm{d}\sigma\,\mathrm{d}\eta \end{aligned}$$

$$= \frac{e^{2(x+y)}}{2^{2(\alpha+\beta)-2}} \int_0^x \sigma^{2\alpha-2} e^{-\sigma} \int_0^y \eta^{2\beta-2} e^{-\xi} \, d\sigma \, d\xi$$
$$< \frac{e^{2(x+y)}}{2^{2(\alpha+\beta)-2}} \Gamma(2\beta - 1)\Gamma(2\alpha - 1).$$

Therefore we obtain from (4)

$$u(x,y) \leq a(x,y) + e^{x+y} K^{1/2} \left[\int_0^x \int_0^y F(s,t)^2 e^{-2(s+t)} \omega(u(s,t))^2 \, ds \, dt \right]^{1/2},$$

where K is as in Theorem 2.2. Using the inequality (2) with $n=2$, $r=2$ and applying the condition (q) with $q=2$ we obtain

$$v(x,y) \leq \alpha(x,y) + 2K \int_0^x \int_0^y F(s,t)^2 R(s+t) \omega(v(s,t)) \, ds \, dt, \qquad (5)$$

where

$$v(x,y) = (e^{-(x+y)} u(x,y))^2, \quad \alpha(x,y) = 2a(x,y)^2. \qquad (6)$$

We need the following lemma.

LEMMA 2.3 *Let $\omega: R^+ \to R$ be a nonnegative, nondecreasing C^1-function, $a(x,y)$ be a nonnegative C^2-function on $\langle 0,T)^2$ $(0 < T \leq \infty)$ such that*

$$\frac{\partial^2 a(x,y)}{\partial x \partial y} \geq 0, \quad \frac{\partial a(x,y)}{\partial y} \geq 0 \quad \left(or \ \frac{\partial a(x,y)}{\partial x} \geq 0 \right)$$

on $\langle 0,T)^2$ $(0 < T \leq \infty)$. Let $k(x,y)$ be a continuous, nonnegative C^2-function and $z(x,y)$ be a continuous, nonnegative function on $\langle 0,T)^2$ with

$$z(x,y) \leq a(x,y) + \int_0^x \int_0^y k(s,t) \omega(z(s,t)) \, ds \, dt, \qquad (7)$$

$(x,y) \in \langle 0,T)^2$. Then

$$z(x,y) \leq \Omega^{-1} \left[\Omega(a(x,y)) + \int_0^x \int_0^y k(s,t) \, ds \, dt \right], \quad (x,y) \in \langle 0,T_1)^2,$$

where $T_1 > 0$ is such that the argument of Ω^{-1} in the above inequality belongs to Dom(Ω^{-1}) for all $(x,y) \in \langle 0,T_1)^2$.

Remark If $a(x,y)$ is constant then the lemma is a consequence of [9, Theorem 7.8]. In this case it suffices to assume that ω is continuous only.

Proof Let $V(x,y)$ be the right-hand side of (7). Then

$$\frac{\partial^2 V(x,y)}{\partial x \partial y} = \frac{\partial^2 a(x,y)}{\partial x \partial y} + k(x,y)\omega(z(x,y)), \tag{8}$$

$$\frac{\partial^2 \Omega(V(x,y))}{\partial x \partial y} = \Omega'(V(x,y))\frac{\partial^2 V(x,y)}{\partial x \partial y} + \Omega''(V(x,y))\frac{\partial V(x,y)}{\partial x}\frac{\partial V(x,y)}{\partial y}. \tag{9}$$

Since $\Omega'(V) = 1/\omega(V)$ and $\Omega''(V) \leq 0$ we obtain from (8) and (9)

$$\begin{aligned}\frac{\partial^2 \Omega(V(x,y))}{\partial x \partial y} &\leq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(V)} + k(x,y)\\ &\leq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))} + k(x,y).\end{aligned} \tag{10}$$

However

$$\begin{aligned}\frac{\partial}{\partial x \partial y}\Omega(a(x,y)) &= \frac{\partial}{\partial x \partial y}\int_0^{a(x,y)} \frac{\mathrm{d}\sigma}{\omega(\sigma)}\\ &= \frac{\partial}{\partial x}\left[\frac{\partial a(x,y)}{\partial y}\frac{1}{\omega(a(x,y))}\right]\\ &= \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))} - \omega'(a(x,y))\frac{\partial a(x,y)}{\partial x}\frac{1}{\omega(a(x,y))^2}\\ &\geq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))},\end{aligned}$$

i.e.

$$\frac{\partial}{\partial x \partial y}\Omega(a(x,y)) \geq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))}. \tag{11}$$

(If $\partial a/\partial y \geq 0$ then one can obtain (11) by estimating $(\partial/\partial x\partial y)\Omega(a(x, y))$.) Thus we obtain from (10) and (11)

$$\frac{\partial^2\Omega(V(x, y))}{\partial x\partial y} \leq \frac{\partial^2\Omega(a(x, y))}{\partial x\partial y} + k(x, y)$$

and this yields

$$\Omega(v(x, y)) \leq \Omega(a(x, y)) + \int_0^x \int_0^y k(s, t)\, \mathrm{d}s\, \mathrm{d}t.$$

From this inequality we obtain

$$z(x, y) \leq V(x, y) \leq \Omega^{-1}\left[\Omega(a(x, y)) + \int_0^x \int_0^y k(s, t)\, \mathrm{d}s\, \mathrm{d}t\right].$$

Now let us continue the proof of Theorem 2.2. Applying Lemma 2.3 to the inequality (5) we obtain

$$v(x, y) \leq \Omega^{-1}\left[\Omega(\alpha(x, y)) + 2K\int_0^x \int_0^y F(s, t)^2 R(s + t)\, \mathrm{d}t\, \mathrm{d}s\right].$$

Using (6) we obtain

$$u(x, y) \leq \mathrm{e}^{x+y}\left\{\Omega^{-1}\left[\Omega(2a(x, y)^2 + 2K\int_0^x \int_0^y F(s, t)^2 R(t + s)\, \mathrm{d}t\, \mathrm{d}s\right]\right\}^{1/2}.$$

Now we shall prove the assertion (ii). Let $p = (z + 2)/(z + 1)$, $q = z + 2$. Then

$$u(x, y) \leq a(x, y) + \left[\int_0^x \int_0^y (x - s)^{-p\delta}\mathrm{e}^{ps}(y - t)^{-p\delta}\mathrm{e}^{pt}\, \mathrm{d}s\, \mathrm{d}t\right]^{1/p}$$
$$\times \left[\int_0^x \int_0^y \mathrm{e}^{-q(s+t)}F(s, t)^q\omega(u(s, t))^q\, \mathrm{d}t\, \mathrm{d}s\right]^{1/q}.$$

We have

$$\begin{aligned}
&\int_0^x \int_0^y (x-s)^{-p\delta} \mathrm{e}^{ps} (y-t)^{-p\delta} \mathrm{e}^{pt} \, \mathrm{d}s \, \mathrm{d}t \\
&\quad = \int_0^x (x-s)^{-p\delta} \mathrm{e}^{ps} \int_0^y (y-t)^{-p\delta} \mathrm{e}^{-pt} \, \mathrm{d}t \, \mathrm{d}s \\
&\quad \le \frac{\mathrm{e}^y}{p^{1-p\delta}} \Gamma(1-p\delta) \int_0^x (x-s)^{-p\delta} \mathrm{e}^{ps} \, \mathrm{d}s \\
&\quad \le \frac{\mathrm{e}^{x+y}}{p^{2(1-p\delta)}} \Gamma(1-p\delta)^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&u(x,y) \\
&\quad \le a(x,y) + K\mathrm{e}^{x+y} \left[\int_0^x \int_0^y F(s,t)^q R(t+s) \omega(\mathrm{e}^{-q(s+t)} u(s,t)^q) \, \mathrm{d}s \, \mathrm{d}t \right]^{1/q}
\end{aligned}$$

and this yields

$$v(x,y) \le \alpha(x,y) + 2K^2 \int_0^x \int_0^y F(s,t)^q R(t+s) \omega(v(s,t)) \, \mathrm{d}s \, \mathrm{d}t,$$

where

$$\alpha(x,y) = 2a(x,y)^2, \quad v(x,y) = (\mathrm{e}^{-(x+y)} u(x,y))^q,$$
$$M_z = \left(\frac{\Gamma(1-p\delta)}{p^{1-p\delta}} \right)^{2/p}$$

and this yields the inequality for $u(x,y)$ from the assertion (ii).

If $\alpha \neq \beta$, $\alpha, \beta < \frac{1}{2}$, then there are some technical problems and we omit this case.

THEOREM 2.4 *Let functions a, F be as in Theorem 2.2 and $u(x,y)$ be a continuous, nonnegative function on $\langle 0, T)^2$ satisfying the inequality*

$$\begin{aligned}
u(x,y) \le a(x,y) + \int_0^x \int_0^y & (x-s)^{\beta-1} \\
& \times (y-t)^{\beta-1} s^{\gamma-1} t^{\gamma-1} F(s,t) u(s,t) \, \mathrm{d}s \, \mathrm{d}t, \quad (12)
\end{aligned}$$

where $\beta > 0, \gamma > 0$. *Then the following assertions hold:*

(i) *If* $\beta > \frac{1}{2}, \gamma > 1 - (1/2p)$ *then*

$$u(x,y) \le \mathrm{e}^{x+y}\Phi(x,y), \tag{13}$$

for $(x,y) \in \langle 0, T)$, *where*

$$\Phi(x,y) = 2^{1-(1/2q)} \exp\left[\frac{4^{q-1}}{q} K^q L^q \int_0^x \int_0^y F(s,t)^{2q} \mathrm{e}^{q(s+t)} \,\mathrm{d}s\,\mathrm{d}t\right], \tag{14}$$

K is as in Theorem 2.2,

$$L = \left(\frac{\Gamma((2\gamma-2)p+1)}{p^{(2\gamma-2)p+1}}\right)^{2/q}, \quad p \ge 1, \quad q \ge 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) *Let* $\beta = 1/(z+1)$ *for some real number* $z \ge 1$, $p = (z+2)/(z+1)$, $q = z+1, \gamma > 1 - 1/\kappa q$, *where* $\kappa > 1$. *Then*

$$u(x,y) \le \mathrm{e}^{x+y}\Psi(x,y), \tag{15}$$

where

$$\Psi(x,y) = 2^{1-1/rq} a(x,y) \exp\left[\frac{Q^{rq}}{rq} \int_0^x \int_0^y \mathrm{e}^{r(s+t)} F(s,t)^{rq} \,\mathrm{d}s\,\mathrm{d}t\right],$$

$r > 1$ *is such that* $1/\kappa + 1/r = 1$, $Q = M_z P$, M_z *is as in Theorem* 2.2, $P = [\Gamma(sq(\gamma-1)+1)]^{2/\kappa}$ *and* $\alpha = -z/(z+1) = \beta - 1$.

Proof We shall prove the assertion (i). From

$$\begin{aligned}
u(x,y) &\le a(x,y) + \left[\int_0^x \int_0^y (x-s)^{2\gamma-2} \mathrm{e}^{2s} (y-t)^{2\beta-2} \mathrm{e}^{2t} \,\mathrm{d}s\,\mathrm{d}t\right]^{1/2} \\
&\quad \times \left[\int_0^x \int_0^y s^{2\gamma-2} t^{2\gamma-2} F(s,t)^2 (\mathrm{e}^{-(s+t)} u(s,t))^2 \,\mathrm{d}s\,\mathrm{d}t\right]^{1/2} \\
&\le a(x,y) + \mathrm{e}^{x+y} K^{1/2} \\
&\quad \times \left[\int_0^x \int_0^y s^{2\gamma-2} t^{2\gamma-2} F(s,t)^2 (\mathrm{e}^{-(s+t)} u(s,t))^2 \,\mathrm{d}s\,\mathrm{d}t\right]^{1/2},
\end{aligned}$$

where K is as in Theorem 2.2. This yields

$$v(x,y) \leq c(x,y) + 2K \int_0^x \int_0^y s^{2\gamma-2} t^{2\gamma-2} F(s,t)^2 v(s,t)\, \mathrm{d}s\, \mathrm{d}t, \tag{16}$$

where

$$v(x,y) = (\mathrm{e}^{-(x+y)} u(x,y))^2, \qquad c(x,y) = 2a(x,y)^2. \tag{17}$$

From (16) we have

$$\begin{aligned} v(x,y) \leq c(x,y) + 2K \int_0^x \int_0^y s^{(2\gamma-2)p} t^{(2\gamma-2)p} \mathrm{e}^{-p(s+t)}\, \mathrm{d}s\, \mathrm{d}t \Bigg]^{1/p} \\ \times \left[\int_0^x \int_0^y F(s,t)^{2q} \mathrm{e}^{q(s+t)} v(s,t)^q\, \mathrm{d}s\, \mathrm{d}t \right]^{1/q}, \end{aligned} \tag{18}$$

where p, q are as in theorem. For the first integral in (18) we have

$$\begin{aligned} &\int_0^x \int_0^y s^{(2\gamma-2)p} t^{(2\gamma-2)p} \mathrm{e}^{-p(s+t)}\, \mathrm{d}s\, \mathrm{d}t \\ &= \frac{1}{(p^{(2\gamma-2)p+1})^2} \int_0^{px} \sigma^{(2\gamma-2)p} \mathrm{e}^{-\sigma} \int_0^{py} \tau^{(2\gamma-2)p} \mathrm{e}^{-\tau}\, \mathrm{d}\tau\, \mathrm{d}\sigma \\ &< \left(\frac{\Gamma((2\gamma-2)p+1)}{p^{(2\gamma-2)p+1}} \right)^2 \end{aligned}$$

and thus we obtain from (18) that

$$v(x,y) \leq c(x,y) + 2KL \int_0^x \int_0^y F(s,t)^{2q} \mathrm{e}^{q(s+t)} v(s,t)^q\, \mathrm{d}s\, \mathrm{d}t, \tag{19}$$

where L is defined in theorem. This yields

$$v(x,y)^q \leq 2^{q-1} \left[c(x,y)^q + 2^q K^q L^q \int_0^x \int_0^y F(s,t)^{2q} \mathrm{e}^{q(s+t)} v(s,t)^q\, \mathrm{d}s\, \mathrm{d}t \right]. \tag{20}$$

One can check that from the assumptions of theorem it follows that

$$\frac{\partial c(x,y)}{\partial x \partial y} \geq 0, \quad \frac{\partial c(x,y)}{\partial x} \geq 0 \ \left(\text{or } \frac{\partial c(x,y)}{\partial y} \geq 0\right).$$

Thus from Lemma 2.3 and (20) we obtain

$$v(x,y)^q \leq 2^{q-1} c(x,y)^q \exp\left[\frac{4^q}{2} K^q L^q \int_0^x \int_0^y F(s,t)^{2q} \mathrm{e}^{q(s+t)} \,\mathrm{d}s\,\mathrm{d}t\right]$$

and the equalities (17) yield (20).

Now let us prove the assertion (ii). From the inequality (12) we obtain

$$\begin{aligned}
u(x,y) &\leq a(x,y) + \left[\int_0^x \int_0^y (x-s)^{-p\alpha} (y-t)^{-p\alpha} \mathrm{e}^{p(s+t)} \,\mathrm{d}s\,\mathrm{d}t\right]^{1/p} \\
&\quad \times \left[\int_0^x \int_0^y s^{q(\gamma-1)} t^{q(\gamma-1)} \mathrm{e}^{-q(s+t)} F(s,t)^q u(s,t)^q \,\mathrm{d}s\,\mathrm{d}t\right]^{1/q} \\
&\leq a(x,y) + \mathrm{e}^{x+y} \left(\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right)^{2/p} \\
&\quad \times \left[\int_0^x \int_0^y s^{\kappa q(\gamma-1)} t^{\kappa q(\gamma-1)} \mathrm{e}^{-(s+t)} \,\mathrm{d}s\,\mathrm{d}t\right]^{1/\kappa} \\
&\quad \times \left[\int_0^x \int_0^y \mathrm{e}^{r(s+t)} F(s,t)^{rq} (\mathrm{e}^{-(s+t)} u(s,t)^{rq} \,\mathrm{d}s\,\mathrm{d}t\right]^{1/rq} \\
&\leq a(x,y) + \mathrm{e}^{x+y} Q \left[\int_0^x \int_0^y \mathrm{e}^{r(s+t)} F(s,t)^{rq} (\mathrm{e}^{-(s+t)} u(s,t))^{rq} \,\mathrm{d}s\,\mathrm{d}t\right]^{1/rq},
\end{aligned}$$

where $Q = M_z P$, M_z is as in Theorem 2.2, P is as in theorem and r, κ are as in the assertion (ii). The above inequality yields

$$v(x,y) \leq 2^{qr-1} \left[a(x,y)^{rq} + Q^{rq} \int_0^x \int_0^y \mathrm{e}^{r(s+t)} F(s,t)^{rq} v(s,t) \,\mathrm{d}s\,\mathrm{d}t\right],$$

where

$$v(x,y) = (\mathrm{e}^{-(x+y)} u(x,y))^{rq}. \tag{21}$$

Therefore we have

$$v(x,y) \le 2^{qr-1}a(x,y)^{rq}\exp\left[Q^{rq}\int_0^x\int_0^y \mathrm{e}^{r(s+t)}F(s,t)^{rq}\,\mathrm{d}s\,\mathrm{d}t\right]$$

and using (21) we obtain (15).

3. OU–IANG–PACHPATTE TYPE INEQUALITY

We shall prove a theorem corresponding to an analog of Ou–Iang–Pachpatte inequality (see [13,16]).

THEOREM 3.1 *Let $T>0$, F and ω be as in Theorem 2.2 and a be a positive constant. Let $u(x,y)$ be a continuous, nonnegative function on $\langle 0,T)^2$ satisfying the inequality*

$$u(x,y)^2 \le a + \int_0^x\int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1}F(s,t)\omega(u(s,t))\,\mathrm{d}s\,\mathrm{d}t, \quad (22)$$

$(x,y)\in\langle 0,T)^2$. Then the following assertions hold:

(i) *Suppose $\alpha>\frac{1}{2}$, $\beta>\frac{1}{2}$ and ω satisfies the condition* (q) *with $q=2$. Then*

$$u(x,y) \le \mathrm{e}^{x+y}\Phi(x,y), \quad (x,y)\in\langle 0,T_1)^2, \quad (23)$$

where

$$\Phi(x,y) = \left[\Lambda^{-1}\left(\Lambda(2a^2) + 2K\int_0^x\int_0^y F(s,t)^2R(s+t)\,\mathrm{d}s\,\mathrm{d}t\right)\right]^{1/4},$$
$$(x,y)\in\langle 0,T_1)^2,$$

K is the number from Theorem 2.2 and $\Lambda(v)=\int_{v_0}^v \mathrm{d}\sigma/\omega(\sqrt{\sigma})$, $v_0>0$, $T_1>0$ is such that the argument of Λ^{-1} belongs to $Dom(\Lambda^{-1})$ for all $\langle 0,T_1)^2$.

(ii) *Suppose $\alpha=\beta=1/(z+1)$ for some real numbers $z\ge 1$ and let $p=(z+2)/(z+1)$, $q=z+2$. Assume that ω satisfies the condition*

(q) *with* $q = z + 2$. *Then*

$$u(x,y) \leq \mathrm{e}^{x+y}\Psi(x,y), \quad (x,y) \in \langle 0, T_2)^2, \tag{24}$$

where

$$\Psi(x,y) = \left[\Lambda^{-1}(\Lambda(2^{q-1}a^q)) + 2^{q-1}M_z^q \int_0^x \int_0^y F(s,t)^q R(s+t)\,\mathrm{d}s\,\mathrm{d}t\right]^{1/2q},$$

$(x,y) \in \langle 0, T_2)$, $T_2 > 0$ *is such that the argument of* Λ^{-1} *in the above inequality belongs to* $Dom(\Lambda^{-1})$ *for all* $(x,y) \in \langle 0, T_2)$, M_z *is as in Theorem* 2.2.

Proof Let us prove (ii). Using the Cauchy–Schwarz inequality and inequality (1) we obtain

$$\begin{aligned}
u(x,y)^2 &\leq a + \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1}\mathrm{e}^{s+t}F(s,t)\mathrm{e}^{-(s+t)}\omega(u(s,t))\,\mathrm{d}s\,\mathrm{d}t \\
&\leq a + \left(\int_0^x \int_0^y (x-s)^{2\alpha-2}(y-t)^{2\beta-2}\mathrm{e}^{2(s+t)}\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2} \\
&\quad \times \left(\int_0^x \int_0^y F(s,t)^2 R(s+t)\omega(\mathrm{e}^{-2(s+t)}u(s,t)^2)\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2} \\
&\leq a + K\mathrm{e}^{-(x+y)}\left(\int_0^x \int_0^y F(s,t)^2 R(s+t)\omega(\mathrm{e}^{-2(s+t)}u(s,t)^2)\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2},
\end{aligned}$$

where K is as in Theorem 2.2. Applying the inequality (1) similarly as in the proof of Theorem 2.2 we obtain the inequality

$$\mathrm{e}^{-(x+y)}u(x,y)^2 \leq 2a^2 + 2K\int_0^x \int_0^y F(s,t)^2 R(s+t)\omega(\mathrm{e}^{-(s+t)}u(s,t))\,\mathrm{d}s\,\mathrm{d}t,$$

where K is an in Theorem 2.2. This yields

$$v(x,y)^2 \leq c + 2K\int_0^x \int_0^y F(s,t)^2 R(s+t)\omega(v(s,t))\,\mathrm{d}s\,\mathrm{d}t, \tag{25}$$

where

$$v(x,y) = (\mathrm{e}^{-(x+y)}u(x,y))^2, \qquad c = 2a^2. \tag{26}$$

Let $V(x,y)$ be the right-hand side of (25). Then

$$v(x,y) \leq \sqrt{V(x,y)}, \qquad \omega(v(x,t)) \leq \omega(\sqrt{V(x,y)}). \tag{27}$$

We have

$$\frac{\partial^2 V(x,y)}{\partial x \partial y} = 2KF(x,y)^2 R(x+y)\omega(v(x,y)) \tag{28}$$

and

$$\begin{aligned}\frac{\partial}{\partial x \partial y}\int_0^{V(x,y)} \frac{\mathrm{d}t}{\omega(\sqrt{t})} &= \frac{\partial}{\partial x}\frac{\partial V(x,y)/\partial y}{\omega(\sqrt{V(x,y)})} \\ &= \frac{\partial^2 V(x,y)}{\partial x \partial y}\cdot\frac{1}{\omega(\sqrt{V(x,y)})} \\ &\quad - \frac{\partial V(x,y)}{\partial y}\cdot\frac{\partial V(x,y)}{\partial x}\cdot\frac{\omega'(\sqrt{V(x,y)})}{2\sqrt{V(x,y)}\omega(\sqrt{V(x,y)})^2} \\ &\leq \frac{\partial^2 V(x,y)}{\partial x \partial y}\cdot\frac{1}{\omega(\sqrt{V(x,y)})}\end{aligned}$$

i.e.

$$\frac{\partial^2}{\partial x \partial y}\Lambda(V(x,y)) \leq \frac{\partial^2 V(x,y)}{\partial x \partial y}\cdot\frac{1}{\omega(\sqrt{V(x,y)})}. \tag{29}$$

From this inequality and (28) we have

$$\frac{\partial}{\partial x \partial y}\Lambda(V(x,y)) \leq 2K\int_0^x \int_0^y F(s,t)^2 R(s+t)\,\mathrm{d}s\,\mathrm{d}t$$

and using (26), (27) we obtain the inequality (22).

Now let us prove (ii). Following the proof of the assertion (ii) of Theorem 2.2 one can show that

$$w(x,y)^2 \leq \alpha + 2K^2\int_0^x \int_0^y F(s,t)^q R(s+t)\omega(w(s,t))\,\mathrm{d}s\,\mathrm{d}t, \tag{30}$$

where

$$\alpha = 2a^2, \qquad w(x,y) = (\mathrm{e}^{-(x+y)}u(x,y))^q.$$

Applying the same procedure to (30) as we have used in the proof of the assertion (ii) as well as that one from the proof of (ii) of Theorem 2.2 one can prove the inequality (24).

4. ON A LINEAR INTEGRAL INEQUALITY IN n INDEPENDENT VARIABLES

In this section we state and prove a result on a singular integral inequality in n variable. In the proof of this result we apply our method of desingularization of weakly singular inequalities and the well known result by Thandapani and Agarwal [19, Theorem 2.3]. First let us formulate this result.

Let $\Omega \subset R^n$ be an open bounded set and let a point $(x_1^i, \ldots, x_n^i) \in \Omega$ be denoted by x^i. Let $y = (y_1, \ldots, y_n), x = (x_1, \ldots, x_n) \in \Omega$ ($y < x$, i.e. $y_i < x_i$, $i = 1, 2, \ldots, n$) and denote by D parallelepiped defined by $y < s < x$. The $\int_y^x \cdot \,\mathrm{d}s$ indicates the n-fold integral $\int_{y_1}^{x_1} \cdots \int_{y_n}^{x_n} \cdot \,\mathrm{d}s_1 \cdots \mathrm{d}s_n$ and $u_x(x)$ denotes $\partial^n u(x)/(\partial x_1 \cdots \partial x_n)$.

THEOREM 4.1 [19, Theorem 2.3] *Let $V(s,x)$ be the solution of characteristic initial value problem*

$$(-1)^n V_s(s,x) - \sum_{r=1}^{m} E_s^r(s,b)\,V(s,x) = 0 \quad in\ \Omega, \tag{31}$$

$$V(s,x) = 1 \quad on\ s_i = x_i,\ 1 \le i \le n \tag{32}$$

and let D^+ be a connected subdomain of Ω containing x such that $V(s,x) \ge 0$ for $s \in D^+$. Let $D \subset D^+$ be a parallelepiped and $u : D^+ \to R$ be a continuous function satisfying the inequality

$$u(x) \le a(x) + b(x) \sum_{r=1}^{m} E^r(x,u), \quad x \in D, \tag{33}$$

where

$$E^r(x,u) = \int_y^x f_{r1}(x^1) \int_y^{x^1} f_{r2}(x^2) \cdots \int_y^{x^{r-1}} f_{rr}(x^r) u(x^r)\,\mathrm{d}x^r \cdots \mathrm{d}x^1, \tag{34}$$

$a, b, f_{rj}: D^+ \to R$, $j = 1, 2, \ldots, r$ are continuous, nonnegative functions. Then

$$u(x) \le a(x) + b(x) \int_y^x \sum_{r=1}^m E_s^r(s, a) V(s, x)\, \mathrm{d}s, \quad x \in D. \tag{35}$$

In the sequel we use the notations: $\mathrm{e}^x := \mathrm{e}^{|x|}$, $x^\gamma := x_1^{\gamma 1} \cdot x_2^{\gamma 2} \cdots x_n^{\gamma n}$ for $x = (x_1, x_2, \ldots, x_n) \in R^n$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in R_+^n = \{(k_1, k_2, \ldots, k_n) \mid k_i \in R,\ k_i \ge 0,\ i = 1, 2, \ldots, n\}$, where $|x| = x_1 + x_2 + \cdots + x_n$. We also denote by $[\beta]$ the vector $(\beta, \beta, \ldots, \beta) \in R^n$, by $\mathbf{1}, \mathbf{2}, \ldots$ the vectors $(1, 1, \ldots, 1) \in R^n$, $(2, 2, \ldots, 2) \in R^n$, $\ldots$ and by $\mathbf{p/q}$ we mean the vector $(p/q, \ldots, p/q)$.

THEOREM 4.2 *Let $\Omega, D, D^+, V(s, x), a(x), b(x), f_{r1}(x), \ldots, f_{rr}(x)$ be as in Theorem 4.1 and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in R_+^n$, $0 < \alpha < 1$ (i.e. $0 < \alpha_i < 1$, $i = 1, 2, \ldots, n$). Let $u: D^+ \to R$ be a continuous, nonnegative function satisfying the inequality*

$$u(x) \le a(x) + b(x) \sum_{r=1}^m F^r(x, u), \quad x \ge 0, \tag{36}$$

where

$$F^r(x, u) = \int_0^x f_{r1}(x^1) \int_0^{x^1} f^{r2}(x^2) \cdots \int_0^{x^{r-1}} (x^{r-1} - x^r)^{\alpha - 1} f_{rr}(x^r) u(x^r)\, \mathrm{d}x^r \cdots \mathrm{d}x^1. \tag{37}$$

Then the following assertions hold:

(i) *Suppose $\alpha = (\alpha_1, \ldots, \alpha_n) > \frac{1}{2}$. Then*

$$u(x) \le \mathrm{e}^x \Bigg[2a(x)^2 + 4b(x)^2 S^2 \sum_{r=1}^m \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)^2\, \mathrm{d}\sigma \right) \times \int_0^x \left(\int_0^{x^1} \cdots \int_0^{x^{r-2}} f_{rr}(x^{r-1})^2 a(x^{r-1})^2\, \mathrm{d}x^{r-1}\, \mathrm{d}x^{r-1} \cdots \mathrm{d}x^2 \right) \times W(x^1, x)\, \mathrm{d}x^1 \Bigg]^{1/2}, \tag{38}$$

where

$$S = \frac{1}{2^{2|\alpha|-2}} \prod_{i=1}^{n} \Gamma(2\alpha_i - 1)$$

and $W(s,x)$ is the solution of characteristic initial value problem

$$(-1)^n W_s(s,x) - \sum_{r=1}^{m} K_s^r(s,B) W(s,x) = 0 \quad in\ \Omega \tag{39}$$

$$W(s,x) = 1 \quad on\ s_i = x_i,\ i = 1,2,\ldots,n, \tag{40}$$

$$B(x) = 2b(x)^2,$$

$$K^r(s,B) = S^2\left(\prod_{j=1}^{r-1}\int_0^x f_{rj}(\sigma)^2\,\mathrm{d}\sigma\right) \times \int_0^x\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}(x^r)^2 B(x^r)^2\,\mathrm{d}x^r\cdots\mathrm{d}x^1.$$

(ii) *Suppose $\alpha = [1/(z+1)]$ for a real number $z > 1$ and let $q = z+2$, $p = (z+2)/(z+1)$, i.e. $1/p + 1/q = 1$. Then*

$$u(x) \le 2^{1-1/q}\mathrm{e}^x\Bigg[a(x)^q + b(x)^q T_p^q \sum_{r=1}^{m}\left(\prod_{j=1}^{r-1}\int_0^x f_{rj}(\sigma)^p\,\mathrm{d}\sigma\right)^{q/p} \times \int_0^x\left(\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}(x^r)^q 2^{q-1} a(x)^q\,\mathrm{d}x^r\cdots\mathrm{d}x^2\right) \times Z(x^1,x)\,\mathrm{d}x^1\Bigg]^{1/q}, \tag{41}$$

where

$$T_p := \left(\frac{\Gamma(1-p\delta)^n}{p^{n(1-p\delta)}}\right)^{1/p}, \quad \delta = 1-\alpha, \tag{42}$$

$Z(s,x)$ is the solution of characteristic initial value problem

$$(-1)^n Z_s(s,x) - \sum_{r=1}^{r} R_s^r(s,C)Z(s,x) = 0 \quad \text{in } \Omega \tag{43}$$

$$Z(s,x) = 0 \quad \text{on } s_i = x_i,\ i = 1,2,\ldots,n, \tag{44}$$

where $C(x) = 2^{q-1}a(x)^q$,

$$R^r(s,x) = T_p^q\left(\prod_{j=1}^{r-1}\int_0^x f_{rj}(\sigma)^p \,\mathrm{d}\sigma\right)^{q/p} \times \int_0^x\left(\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}(x^r)^q 2^{q-1} b(x^r)^q \,\mathrm{d}x^r \cdots \mathrm{d}x^2\right)\mathrm{d}x^1.$$

Proof We shall prove (i). Let us estimate the function $F^r(x,u)$ using the Cauchy–Schwarz inequality and the inequality

$$\int_0^x (x^1-\sigma)^{2\alpha-2}\mathrm{e}^{2\sigma}\,\mathrm{d}\sigma < \mathrm{e}^{2x}S,$$

where S is as in theorem.

We have

$$\begin{aligned}
F(x,u) \le{}& \int_0^x f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-2}} f_{rr-1}(x^{r-1}) \\
&\times\left[\int_0^{x^{r-1}}(x^{r-1}-x^r)^{(2\alpha-2)}\mathrm{e}^{2x^r}\,\mathrm{d}x^r\right]^{1/2} \\
&\times\left[\int_0^{x^{r-1}} f_{rr}(x^r)^2\mathrm{e}^{-2x^r}u(x^r)^2\,\mathrm{d}x^r\right]^{1/2}\mathrm{d}x^{r-1}\cdots\mathrm{d}x^1 \\
\le{}& S^{1/2}\mathrm{e}^x\int_0^x f_{r1}(x^1)\int_0^{x^1} f_{r_2}(x^2)\cdots\int_0^{x^{r-2}} f_{rr-1}(x^{r-1}) \\
&\times\left[\int_0^{x^{r-1}} f_{rr}(x^r)^2\mathrm{e}^{-2x^r}u(x^r)^2\,\mathrm{d}x^r\right]^{1/2}\mathrm{d}x^{r-1}\cdots\mathrm{d}x^1
\end{aligned}$$

$$
\begin{aligned}
&\leq S^{1/2}\mathrm{e}^{x}\int_0^{x} f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-3}} f_{rr-2}(x^{r-2}) \\
&\quad\times\left[\int_0^{x^{r-2}} f_{rr-1}(x^{r-1})^2\,\mathrm{d}x^{r-1}\right]^{1/2} \\
&\quad\times\left[\int_0^{x^{r-2}}\int_0^{x^{r-1}} f_{rr}(x^r)^2\mathrm{e}^{-2x^r}u(x^r)^2\,\mathrm{d}x^{r-1}\right]^{1/2}\mathrm{d}x^{r-2}\cdots\mathrm{d}x^1 \\
&\leq S^{1/2}\mathrm{e}^{x}\left(\int_0^{x} f_{rr-1}(\sigma)^2\,\mathrm{d}\sigma\right)^{1/2}\cdots \\
&\quad\times\int_0^{x} f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-3}} f_{rr-2}(x^{r-2}) \\
&\quad\times\left(\int_0^{x^{r-2}}\int_0^{x^{r-1}} f_{rr}(x^r)^2\mathrm{e}^{-2x^r}u(x^r)^2\,\mathrm{d}x^{r-1}\right)^{1/2}\mathrm{d}x^{r-2}\cdots\mathrm{d}x^1.
\end{aligned}
$$

Proceeding in this way using the Cauchy–Schwarz inequality one can prove that

$$
\begin{aligned}
F(x,u) &\leq S^{1/2}\mathrm{e}^{x}\left(\prod_{j=1}^{r-1}\int_0^{x} f_{rj}(\sigma)^2\,\mathrm{d}\sigma\right)^{1/2} \\
&\quad\times\left[\int_0^{x}\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}(x^r)^2\mathrm{e}^{-2x^r}u(x^r)^2\,\mathrm{d}x^r\,\mathrm{d}x^{r-1}\cdots\mathrm{d}x^1\right]^{1/2}.
\end{aligned}
$$

From this inequality and (36) we have

$$
\begin{aligned}
v(x) &\leq a(x) + S^{1/2}\mathrm{e}^{x}b(x)\sum_{r=1}^{m}\left(\prod_{j=1}^{r-1}\int_0^{x} f_{rj}(\sigma)^2\,\mathrm{d}\sigma\right)^{1/2} \\
&\quad\times\left[\int_0^{x}\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}(x^r)^2 v(x^r)^2\,\mathrm{d}x^r\cdots\mathrm{d}x^1\right]^{1/2},
\end{aligned}
$$

where $v(x) = e^{-x}u(x)$. Then using the Jensen inequality (1) we obtain

$$v(x)^2 \le 2a(x)^2 + 2Sb(x)^2 \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)^2 \,d\sigma \right) \times \int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}(x^{r-1})^2 v(x^r)^2 \,dx^r \cdots dx^1.$$

From Theorem 4.1 it follows that

$$v(x)^2 \le 2a(x)^2 + 2Sb(x)^2 \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)^2 \,d\sigma \right) \times \int_0^x \left(\int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}(x^r)^2 (2a(x^{r-1}))^2 \,dx^{r-1} \cdots dx^2 \right) \times W(x^1, x) \,dx^1,$$

where $W(s, x)$ is as in theorem and from definition of $v(x)$ we obtain the inequality (38).

Now let us prove the assertion (ii). We shall estimate the function $F^r(x, u)$ using the Hölder inequality:

$$F^r(x,u) \le \int_0^x f_{r1}(x^1) \int_0^{x^1} f_{r2}(x^2) \cdots \times \int_0^{x^{r-1}} f_{rr-1}(x^{r-1}) \left[\int_0^{x^{r-1}} (x^{r-1} - x^r)^{[p\alpha - p]} e^{px^r} \,dx^r \right]^{1/p} \times \left[\int_0^{x^{r-1}} f(x^r)^q e^{-qx^r} u(x^r)^q \,dx^r \right]^{1/q} dx^{r-1} \cdots dx_1$$

$$\le T_p^{1/p} e^x \int_0^x f_{r1}(x^1) \int_0^{x^1} f_{r_2}(x^2) \cdots \int_0^{x^{r-2}} f_{rr-1}(x^{r-1}) \times \left[\int_0^{x^{r-1}} f(x^r)^q e^{-qx^r} u(x^r)^q \,dx^r \right]^{1/q} dx^{r-1} \cdots dx^1,$$

where T_p is as in theorem. Similarly as in the case (i) using the Hölder inequality one can prove that

$$F^r(x,u) \leq T_p e^x \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(s)^p \, ds \right)^{1/p} \times \left[\int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}(x^r)^q v(x^r)^q \, dx^r \cdots dx^1 \right]^{1/q}.$$

From this inequality, (36) and the Jensen inequality (1) it follows that

$$v(x)^q \leq 2^{q-1} \left[a(x)^q + b(x)^q T_p^q \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)^p \, d\sigma \right)^{q/p} \times \int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}(x^r)^q v(x^r)^q \, dx^r \cdots dx^1 \right]$$

and from Theorem 4.1 we have

$$v(x)^q \leq 2^{q-1} \left[a(x)^q + b(x)^q T_p^q \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)^p \, d\sigma \right)^{q/p} \times \int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}(x^r)^q (2^{q-1} a(x^r)^q) \, dz^r \cdots dx^2) Z(x^1, x) \, dx^1 \right],$$

where $Z(s,x)$ is as in theorem and from definition of $v(x)$ we obtain the inequality (41).

Remark The case $\alpha < \frac{1}{2}$, α not equal to some $[\beta]$, is much more complicated than the case (ii) from the above theorem and we do not solve it.

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