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Research Article

A Note on Generalized $|A|_k$-Summability Factors for Infinite Series

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A general theorem concerning the $|A|_k$—summability factors of infinite series has been proved.

1. Introduction

A weighted mean matrix, denoted by $(\overline{N}, p_n)$, is a lower triangular matrix with entries p_k/P_n, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$, and $P_n := \sum_{k=0}^{n} p_k$.

Mishra and Srivastava [1] obtained sufficient conditions on a sequence $\{p_k\}$ and a sequence $\{\lambda_n\}$ for the series $\sum a_n P_n \lambda_n / np_n$ to be absolutely summable by the weighted mean matrix $(\overline{N}, p_n)$.

Recently Savaş and Rhoades [2] established the corresponding result for a nonnegative triangle, using the correct definition of absolute summability of order $k \geq 1$.

Let A be an infinite lower triangular matrix. We may associate with A two lower triangular matrices $\overline{A}$ and $\widehat{A}$, whose entries are defined by

$$\overline{a}_{nk} = \sum_{i=k}^{n} a_{ni}, \qquad \widehat{a}_{nk} = \overline{a}_{nk} - \overline{a}_{n-1,k}, \tag{1.1}$$

respectively. The motivation for these definitions will become clear as we proceed.

Let A be an infinite matrix. The series $\sum a_k$ is said to be absolutely summable by A, of order $k \geq 1$, written as $|A|_k$, if

$$\sum_{k=0}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty, \tag{1.2}$$

where Δ is the forward difference operator and t_n denotes the nth term of the matrix transform of the sequence $\{s_n\}$, where $s_n := \sum_{k=1}^{n} a_k$.

Thus

$$t_n = \sum_{k=1}^{n} a_{nk} s_k = \sum_{k=1}^{n} a_{nk} \sum_{\nu=1}^{k} a_\nu = \sum_{\nu=1}^{n} a_\nu \sum_{k=\nu}^{n} a_{nk} = \sum_{\nu=1}^{n} \overline{a}_{n\nu} a_\nu,$$
$$t_n - t_{n-1} = \sum_{\nu=1}^{n} \overline{a}_{n\nu} a_\nu - \sum_{\nu=1}^{n-1} \overline{a}_{n-1,\nu} a_\nu = \sum_{\nu=1}^{n} \widehat{a}_{n\nu} a_\nu, \tag{1.3}$$

since $\overline{a}_{n-1,n} = 0$.

A sequence $\{\lambda_n\}$ is said to be of bounded variation (bv) if $\sum_n |\Delta \lambda_n| < \infty$. Let $bv_0 = bv \cap c_0$, where c_0 denotes the set of all null sequences.

A positive sequence $\{b_n\}$ is said to be an almost increasing sequence if there exist an increasing sequence $\{c_n\}$ and positive constants A and B such that $Ac_n \le b_n \le Bc_n$, (see [3]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(-1)^n} n$.

A positive sequence $\gamma := \{\gamma_n\}$ is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \ge 1$ such that

$$K n^\beta \gamma_n \ge m^\beta \gamma_m \tag{1.4}$$

holds for all $n \ge m \ge 1$. It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking an example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$ (see [4]). If (1.4) stays with $\beta = 0$ then γ is simply called a quasi-increasing sequence. It is clear that if $\{\gamma_n\}$ is quasi β-power increasing then $\{n^\beta \gamma_n\}$ is quasi-increasing.

A positive sequence $\gamma = \{\gamma_n\}$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(\gamma, f) \ge 1$ such that $K f_n \gamma_n \ge f_m \gamma_m$ holds for all $n \ge m \ge 1$, where $f := \{f_n\} = \{n^\beta (\log n)^\mu\}$, $\mu > 0$, $0 < \beta < 1$ was considered instead of n^β(see [5, 6]).

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$.

Quite recently, Savaş and Rhoades [2] proved the following theorem for $|A|_k$-summability factors of infinite series.

Theorem 1.1. *Let A be a triangle with nonnegative entries satisfying*

(i) $\overline{a}_{n0} = 1$, $n = 0, 1, \ldots,$

(ii) $a_{n-1,\nu} \ge a_{n\nu}$ *for* $n \ge \nu + 1$,

(iii) $n a_{nn} \asymp O(1)$,

(iv) $\Delta(1/a_{nn}) = O(1)$, *and*

(v) $\sum_{\nu=0}^{n} a_{\nu\nu} |\widehat{a}_{n,\nu+1}| = O(a_{nn})$.

If $\{X_n\}$ is a positive nondecreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy

(vi) $|\Delta \lambda_n| \le \beta_n$,

(vii) $\lim \beta_n = 0$,

(viii) $|\lambda_n| X_n = O(1)$,

(ix) $\sum_{n=1}^{\infty} nX_n|\Delta\beta_n| < \infty$, *and*

(x) $T_n := \sum_{\nu=1}^{n} \dfrac{|s_\nu|^k}{\nu} = O(X_n)$,

then the series $\sum_{n=1}^{\infty} a_n\lambda_n/na_{nn}$ *is summable* $|A|_k, k \geq 1$.

It should be noted that if $\{X_n\}$ is an almost increasing sequence then (viii) implies that the sequence $\{\lambda_n\}$ is bounded. However, when $\{X_n\}$ is a quasi β-power increasing sequence or a quasi f-increasing sequence, (viii) does not imply $|\lambda_m| = O(1)$, $m \to \infty$. For example, since $X_m = m^{-\beta}$ is a quasi β-power increasing sequence for $0 < \beta < 1$, if we take $\lambda_m = m^{\delta}$, $0 < \delta < \beta < 1$ then $|\lambda_m|X_m = m^{\delta-\beta} = O(1)$, $m \to \infty$ holds but $|\lambda_m| = m^{\delta} \neq O(1)$ (see [7]).

The goal of this paper is to prove a theorem by using quasi f-increasing sequences. We show that the crucial condition of our proof, $\{\lambda_n\} \in bv_0$, can be deduced from another condition of the theorem.

2. The Main Results

We have the following theorem:

Theorem 2.1. *Let* A *be nonnegative triangular matrix satisfying conditions* (i)–(v) *and let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions (vi) and (vii) of Theorem 1.1 and*

$$\sum_{n=1}^{m}\lambda_n = o(m), \quad m \longrightarrow \infty. \tag{2.1}$$

If $\{X_n\}$ *is a quasi* f*-increasing sequence and condition (x) and*

$$\sum_{n=1}^{\infty} nX_n(\beta,\mu)|\Delta\beta_n| < \infty \tag{2.2}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n\lambda_n/na_{nn}$ *is summable* $|A|_k$, $k \geq 1$, *where* $\{f_n\} := \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0$, $0 \leq \beta < 1$, *and* $X_n(\beta,\mu) := (n^{\beta}(\log n)^{\mu}X_n)$.

Theorem 2.1 includes the following theorem with the special case $\mu = 0$.

Theorem 2.2. *Let* A *satisfying conditions* (i)–(v) *and let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions (vi), (vii), and* (2.1). *If* $\{X_n\}$ *is a quasi* β*-power increasing sequence for some* $0 \leq \beta < 1$ *and conditions (x) and*

$$\sum_{n=1}^{\infty} nX_n(\beta)|\Delta\beta_n| < \infty \tag{2.3}$$

are satisfied, where $X_n(\beta) := (n^{\beta}X_n)$, *then the series* $\sum_{\nu=1}^{\infty} a_n\lambda_n/na_{nn}$ *is summable* $|A|_k, k \geq 1$.

If we take that $\{X_n\}$ is an almost increasing sequence instead of a quasi β-power increasing sequence then our Theorem 2.2 reduces to [8, Theorem 1].

Remark 2.3. The crucial condition, $\{\lambda_n\} \in bv_0$, and condition (viii) do not appear among the conditions of Theorems 2.1 and 2.2. By Lemma 3.3, under the conditions on $\{X_n\}, \{\beta_n\}$, and $\{\lambda_n\}$ as taken in the statement of the Theorem 2.1, also in the statement of Theorem 2.2 with the special case $\mu = 0$, conditions $\{\lambda_n\} \in bv_0$ and (viii) hold.

3. Lemmas

We shall need the following lemmas for the proof of our main Theorem 2.1.

Lemma 3.1 (see [9]). *Let $\{\varphi_n\}$ be a sequence of real numbers and denote*

$$\Phi_n := \sum_{k=1}^{n} \varphi_k, \qquad \Psi_n := \sum_{k=n}^{\infty} |\Delta \varphi_k|. \tag{3.1}$$

If $\Phi_n = o(n)$ then there exists a natural number $\mathbb{N}$ such that

$$|\varphi_n| \leq 2\Psi_n \tag{3.2}$$

for all $n \geq \mathbb{N}$.

Lemma 3.2 (see [7]). *If $\{X_n\}$ is a quasi f-increasing sequence, where $\{f_n\} = \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0$, $0 \leq \beta < 1$, then conditions* (2.1) *of Theorem 2.1,*

$$\sum_{n=1}^{m} |\Delta \lambda_n| = o(m), \quad m \longrightarrow \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} n X_n(\beta, \mu) |\Delta |\Delta \lambda_n|| < \infty, \tag{3.4}$$

where $X_n(\beta, \mu) = (n^{\beta}(\log n)^{\mu} X_n)$, imply conditions (viii) *and*

$$\lambda_n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.5}$$

Lemma 3.3 (see [7]). *If $\{X_n\}$ is a quasi f-increasing sequence, where $\{f_n\} = \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0$, $0 \leq \beta < 1$, then under conditions (vi), (vii),* (2.1) *and* (2.2)*, conditions* (viii) *and* (3.5) *are satisfied.*

Lemma 3.4 (see [7]). *Let $\{X_n\}$ be a quasi f-increasing sequence, where $\{f_n\} = \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0$, $0 \leq \beta < 1$. If conditions (vi), (vii), and* (2.2) *are satisfied, then*

$$n \beta_n X_n = O(1), \tag{3.6}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.7}$$

4. Proof of Theorem 2.1

Let T_n denote the nth term of the A-transform of the partial sums of the series $\sum_{n=1}^{\infty}(a_n\lambda_n)/(na_{nn})$. Then, we have

$$T_n = \sum_{\nu=1}^{n} a_{n\nu} \sum_{i=1}^{\nu} \frac{a_i\lambda_i}{a_{ii}i} = \sum_{i=1}^{m} \frac{a_i\lambda_i}{a_{ii}i} \sum_{\nu=i}^{n} a_{n\nu} = \sum_{i=1}^{n} \overline{a}_{ni} \frac{a_i\lambda_i}{a_{ii}i}. \tag{4.1}$$

Thus,

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=1}^{n} \overline{a}_{ni} \frac{a_i\lambda_i}{a_{ii}i} - \sum_{i=1}^{n-1} \overline{a}_{n-1,i} \frac{a_i\lambda_i}{a_{ii}i} \\
&= \sum_{i=1}^{n} (\overline{a}_{ni} - \overline{a}_{n-1,i}) \frac{a_i\lambda_i}{a_{ii}i} = \sum_{i=1}^{n} \widehat{a}_{ni} \frac{a_i\lambda_i}{a_{ii}i} \\
&= \sum_{i=1}^{n} \widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} (s_i - s_{i-1}) \\
&= \sum_{i=1}^{n-1} \widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} s_i + a_{nn} \frac{\lambda_n}{a_{nn}n} s_n - \sum_{i=1}^{n} \widehat{a}_{ni} \frac{\lambda_i s_{i-1}}{a_{ii}i} \\
&= \sum_{i=1}^{n-1} \widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} s_i + a_{nn} \frac{\lambda_n}{a_{nn}n} s_n - \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \frac{\lambda_{i+1} s_i}{(i+1)a_{i+1,i+1}} \\
&= \sum_{i=1}^{n-1} \left(\widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} - \widehat{a}_{n,i+1} \frac{\lambda_{i+1}}{(i+1)a_{i+1,i+1}} \right) s_i + a_{nn} \frac{\lambda_n}{na_{nn}}.
\end{aligned} \tag{4.2}$$

It is easy to see that

$$\frac{\widehat{a}_{ni}\lambda_i}{ia_{ii}} - \frac{\widehat{a}_{n,i+1}\lambda_{i+1}}{(i+1)a_{i+1,i+1}} = \Delta_i\left(\frac{\widehat{a}_{ni}}{ia_{ii}}\right)\lambda_i + \frac{\widehat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \Delta(\lambda_i). \tag{4.3}$$

Also we may write

$$\Delta_i\left(\frac{\widehat{a}_{ni}}{ia_{ii}}\right)\lambda_i = \frac{\Delta_i(\widehat{a}_{ni})\lambda_i}{ia_{ii}} + a_{n,i+1}\lambda_i\left(\frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}}\right). \tag{4.4}$$

Therefore, for $n > 1$,

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=1}^{n-1} \frac{\Delta_i(\widehat{a}_{ni})}{ia_{ii}} \lambda_i s_i + \sum_{i=1}^{n-1} \widehat{a}_{n,i+1}\lambda_i \left(\frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}}\right) s_i \\
&\quad + \sum_{i=1}^{n-1} \frac{\widehat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \Delta_i(\lambda_i) s_i + \frac{\lambda_n}{n} s_n \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say}.
\end{aligned} \tag{4.5}$$

To complete the proof of the theorem, it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{nr}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{4.6}$$

Using Hölder's inequality and condition (iii),

$$\begin{aligned}
I_1 = \sum_{n=1}^{m+1} n^{k-1}|T_{n1}|^k &\le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=1}^{n-1}\left|\frac{\Delta_i(\widehat{a}_{ni})}{i a_{ii}}\lambda_i s_i\right|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=1}^{n-1}|\Delta_i(\widehat{a}_{ni})\lambda_i s_i|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=1}^{n-1}|\Delta_i(\widehat{a}_{ni})||\lambda_i|^k|s_i|^k\right) \times \left(\sum_{i=1}^{n-1}|\Delta_i(\widehat{a}_{ni})|\right)^{k-1}.
\end{aligned} \tag{4.7}$$

Since (λ_n) is bounded by Lemma 3.3, using (ii), (iii), (vi), (x), and property (3.7) of Lemma 3.4,

$$\begin{aligned}
I_1 &= O(1)\sum_{n=1}^{m+1}(n a_{nn})^{k-1}\sum_{i=1}^{n-1}|\lambda_i|^k|s_i|^k|\Delta_i(\widehat{a}_{ni})| \\
&= O(1)\sum_{n=1}^{m+1}(n a_{nn})^{k-1}\left(\sum_{i=1}^{n-1}|\lambda_i|^{k-1}|\lambda_i||\Delta_i(\widehat{a}_{ni})||s_i|^k\right) \\
&= O(1)\sum_{i=1}^{m}|\lambda_i||s_i|^k\sum_{n=i+1}^{m+1}(n a_{nn})^{k-1}|\Delta_i(\widehat{a}_{ni})| \\
&= O(1)\sum_{i=1}^{m}|\lambda_i||s_i|^k a_{ii} = O(1)\sum_{i=1}^{m}\frac{|\lambda_i||s_i|^k}{i} \\
&= O(1)\left[\sum_{i=1}^{m}|\lambda_i|\sum_{r=1}^{i}\frac{|s_r|^k}{r} - \sum_{i=0}^{m-1}|\lambda_{i+1}|\sum_{r=1}^{i}\frac{|s_r|^k}{r}\right] \\
&= O(1)\sum_{i=1}^{m-1}\Delta(|\lambda_i|)\sum_{r=1}^{i}\frac{1}{r}|s_r|^k + O(1)|\lambda_m|\sum_{i=1}^{m}\frac{|s_i|^k}{i} \\
&= O(1)\sum_{i=1}^{m-1}\Delta(|\lambda_i|)X_i + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{i=1}^{m}\beta_i X_i + O(1)|\lambda_m|X_m = O(1).
\end{aligned} \tag{4.8}$$

Now

$$I_2 = \sum_{n=1}^{m+1} n^{k-1} |T_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \lambda_i \Delta\left(\frac{1}{ia_{ii}}\right) s_i \right|^k$$
$$= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\widehat{a}_{n,i+1}| |\lambda_i| \left| \Delta\left(\frac{1}{ia_{ii}}\right) \right| |s_i| \right\}^k. \tag{4.9}$$

From [2],

$$\Delta\left(\frac{1}{ia_{ii}}\right) = \frac{1}{(i+1)} \left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}} \right]. \tag{4.10}$$

Thus, using (iv) and (ii),

$$\left| \Delta\left(\frac{1}{ia_{ii}}\right) \right| = \left| \frac{1}{i+1} \left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}} \right] \right|$$
$$= \frac{1}{i+1} [O(1) + O(1)]. \tag{4.11}$$

Hence, using Hölder's inequality, (v), (iii), and the fact that the λ_n's are bounded,

$$I_2 = O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\widehat{a}_{n,i+1}| |\lambda_i| \frac{1}{i+1} |s_i| \right\}^k$$
$$= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\widehat{a}_{n,i+1}| a_{ii} |\lambda_i| |s_i| \right\}^k$$
$$= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=1}^{n-1} |\widehat{a}_{n,i+1}| a_{ii} |\lambda_i|^k |s_i|^k \right) \left(\sum_{i=1}^{n-1} a_{ii} |\widehat{a}_{n,i+1}| \right)^{k-1}$$
$$= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=1}^{n-1} |\widehat{a}_{n,i+1}| a_{ii} |\lambda_i|^k |s_i|^k$$
$$= O(1) \sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=i+1}^{m+1} (na_{nn})^{k-1} |\widehat{a}_{n,i+1}|$$
$$= O(1) \sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=i+1}^{m+1} |\widehat{a}_{n,i+1}|$$
$$= O(1) \sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii}$$

$$\begin{aligned}
&= O(1)\sum_{i=1}^{m}|\lambda_i||\lambda_i|^{k-1}|s_i|^k\frac{1}{i}\\
&= \sum_{i=1}^{m}|\lambda_i|\frac{|s_i|^k}{i} = O(1),
\end{aligned} \tag{4.12}$$

as in the proof of I_1.

It follows from (3.6) that $\beta_n = O(1/n)$ and hence that $|\Delta\lambda_n| = O(1/n)$ by condition (vi).

Using (iii), Hölder's inequality, and (v),

$$\begin{aligned}
I_3 &= \sum_{n=1}^{m+1}n^{k-1}|T_{n3}|^k = \sum_{n=1}^{m+1}n^{k-1}\left|\sum_{i=1}^{n-1}\frac{\widehat{a}_{n,i+1}(\Delta\lambda_i)s_i}{(i+1)a_{i+1,i+1}}\right|^k\\
&= O(1)\sum_{n=1}^{m+1}n^{k-1}\left(\sum_{i=1}^{n-1}|\widehat{a}_{n,i+1}||\Delta\lambda_i||s_i|\right)^k\\
&= O(1)\sum_{n=1}^{m+1}n^{k-1}\left\{\sum_{i=1}^{n-1}\frac{a_{ii}}{a_{ii}}|\widehat{a}_{n,i+1}||\Delta\lambda_i||s_i|\right\}^k\\
&= O(1)\sum_{n=1}^{m+1}n^{k-1}\left\{\sum_{i=1}^{n-1}a_{ii}\frac{|\widehat{a}_{n,i+1}|}{a_{ii}^k}|\Delta\lambda_i|^k|s_i|^k\right\}\left\{\sum_{i=1}^{n-1}a_{ii}|\widehat{a}_{n,i+1}|\right\}^{k-1}\\
&= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{i=1}^{n-1}a_{ii}\frac{|\widehat{a}_{n,i+1}|}{a_{ii}^k}|\Delta\lambda_i|^k|s_i|^k\\
&= O(1)\sum_{n=1}^{m+1}\sum_{i=1}^{n-1}|\widehat{a}_{n,i+1}||\Delta\lambda_i|^k|s_i|^k\frac{1}{a_{ii}^k}a_{ii}\\
&= O(1)\sum_{i=1}^{m}\frac{a_{ii}}{a_{ii}^k}|\Delta\lambda_i|^k|s_i|^k\sum_{n=i+1}^{m+1}|\widehat{a}_{n,i+1}|\\
&= O(1)\sum_{i=0}^{m}\left(\frac{|\Delta\lambda_i|}{a_{ii}}\right)^{k-1}|\Delta\lambda_i||s_i|^k\\
&= O(1)\sum_{i=1}^{m}|\Delta\lambda_i|\ |s_i|^k = O(1)\sum_{i=0}^{m}|s_i|^k\beta_i.
\end{aligned} \tag{4.13}$$

Since $|s_i|^k = i(T_i - T_{i-1})$ by (x), we have

$$I_3 = O(1)\sum_{i=1}^{m}i(T_i - T_{i-1})\beta_i. \tag{4.14}$$

Using Abel's transformation, (vi), (2.2), and properties (3.7) and (3.6) of Lemma 3.4,

$$\begin{aligned} I_3 &= O(1)\sum_{i=1}^{m-1} T_i \Delta\left(i\beta_i\right) + O(1) m T_n \beta_n \\ &= O(1)\sum_{i=1}^{m-1} i\left|\Delta\beta_i\right| X_i + O(1)\sum_{i=1}^{m-1} X_i\beta_i + O(1) m X_n \beta_n = O(1). \end{aligned} \tag{4.15}$$

Using the boundedness of λ_n and (x),

$$\begin{aligned} I_4 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n4}|^k = \sum_{n=1}^{m+1} n^{k-1} \left|\frac{s_n \lambda_n}{n}\right|^k \\ &= \sum_{n=1}^{m+1} |s_n|^k |\lambda_n|^k \frac{1}{n} = \sum_{n=1}^{m+1} \frac{|s_n|^k}{n} |\lambda_n| |\lambda_n|^{k-1} = O(1), \end{aligned} \tag{4.16}$$

as in the proof of I_1.

A weighted mean matrix, written $(\overline{N}, p_n)$, is a lower triangular matrix with entries $a_{nv} = p_v / P_n$, where $\{p_n\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{i=0}^{n} p_i \to \infty$, as $n \to \infty$.

Corollary 4.1. *Let* $\{p_n\}$*be a positive sequence satisfying*

(i) $np_n \asymp O(P_n)$ *and*

(ii) $\Delta(P_n / p_n) = O(1)$.

and let $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions* (vi), (vii), *and* (2.1). *If* $\{X_n\}$ *is a quasi* f*-increasing sequence, where* $\{f_n\} := \{n^\beta (\log n)^\mu\}$, $\mu \geq 0, 0 \leq \beta < 1$, *and conditions (x) and* (2.2) *are satisfied, then the series* $\sum_{n=1}^{\infty} (a_n P_n \lambda_n)/(np_n)$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

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