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Research Article

Characterization of P-Core and Absolute Equivalence of Double Sequences

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The P-core of a double sequence has been defined and it is studied by many authors. In this paper, we have determined two permutations π_1 and π_2 on the set of natural numbers for which $P\text{-core}A_{\pi_1}(x) = P\text{-core}A_{\pi_2}(x)$ for all $x \in \ell_\infty^2$.

1. Introduction and Preliminaries

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ is said to be convergent in the Pringsheim sense or P-convergent if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > N$ [1]. In this case, we write $P - \lim x = l$. By c_2, we mean the space of all P-convergent sequences.

A double sequence x is said to be bounded if there exists a positive number M such that $|x_{jk}| < M$ for all j, k, that is, if

$$\|x\| = \sup_{j,k} \left| x_{jk} \right| < \infty. \tag{1.1}$$

By ℓ_∞^2 we will denote the set of all bounded double sequences. We note that in contrast to the case for single sequences, a convergent double sequence needs not to be bounded. So, by c_2^∞ we will denote the space of all real bounded and convergent double sequences.

Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a four-dimensional infinite matrix of real numbers for all $m, n = 0, 1, \ldots$. The sums

$$y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk} \tag{1.2}$$

are called the A-transforms of the double sequence x and we will denote it by $[Ax]$. We say that a sequence x is A-summable to the limit l if the A-transform of x exists for all $m, n = 0, 1, \ldots$ and convergent to l in the Pringsheim sense, that is,

$$\lim_{p,q\to\infty}\sum_{j=0}^{p}\sum_{k=0}^{q}a_{jk}^{mn}x_{jk} = y_{mn}, \qquad \lim_{m,n\to\infty}y_{mn} = l. \tag{1.3}$$

Móricz and Rhoades [2] have defined almost convergence of a double sequence as follows.

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ of real numbers is said to be almost convergent to a limit l if

$$\lim_{p,q\to\infty}\left|\frac{1}{pq}\sum_{j=0}^{p}\sum_{k=0}^{q}x_{j+s,k+t} - l\right| = 0 \tag{1.4}$$

uniformly in s, t. By f_2 we denote the set of all almost convergent double sequences.

Recall that Knopp's Core of a single bounded sequence x is the closed interval $[\liminf x, \limsup x]$ in [3, page 138]. In the sense of Knopp's Core, P-core of a double sequence was introduced by Patterson as the closed interval $[P-\liminf x, P-\limsup x]$ in [4], where the definitions of $P-\liminf x$ (Pringsheim limit inferior) and $P-\limsup x$ (Pringsheim limit superior) are given as follows. Let $\alpha_n = \sup_n\{x_{jk} : j, k \geq n\}$ and $\beta_n = \inf_n\{x_{jk} : j, k \geq n\}$. Then

$$P-\limsup x = \begin{cases} +\infty, & \alpha_n = +\infty \text{ for each } n, \\ \inf\limits_n \alpha_n, & \alpha_n < \infty \text{ for some } n, \end{cases}$$
$$P-\liminf x = \begin{cases} -\infty, & \beta_n = +\infty \text{ for each } n, \\ \sup\limits_n \beta_n, & \beta_n < \infty \text{ for some } n. \end{cases} \tag{1.5}$$

After this definition, this concept has been studied by many authors. For example see in [5–11] and the others.

Let $\mathbb{N}$ denote the set of all natural numbers. A bijective function $\pi : \mathbb{N} \to \mathbb{N}$ is said to be a permutation. In this paper we have determined two permutations π_1 and π_2 for which P-core $(A\pi_1(x)) = P$-core $(A\pi_2(x))$ for all $x \in \ell_{\infty}^2$.

A two-dimensional matrix transformation is said to be regular (see [3, page 64]) if it maps every convergent sequence into a convergent sequence with the same limit. In 1926, Robison presented a four-dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness. A four-dimensional matrix $A = [a_{jk}^{mn}]$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Lemma 1.1 (see [12, 13]). *The four-dimensional matrix A is bounded-regular or RH-regular if and only if*

$$
\begin{gathered}
P-\lim_{m,n} a_{jk}^{mn}=0 \quad (j,k=0,1,\ldots), \\
P-\lim_{m,n} \sum_{j,k} a_{jk}^{mn}=1, \\
P-\lim_{m,n} \sum_{j} \left| a_{jk}^{mn} \right| = 0 \quad (k=0,1,\ldots), \\
P-\lim_{m,n} \sum_{k} \left| a_{jk}^{mn} \right| = 0 \quad (j=0,1,\ldots), \\
\sum_{j,k} \left| a_{jk}^{mn} \right| \le C < \infty \quad (m,n=0,1,\ldots).
\end{gathered}
\tag{1.6}
$$

Lemma 1.2 (see [4]). *If A is a four-dimensional matrix, then for all real-valued double sequences x,*

$$P-\limsup(Ax) \le P-\limsup(x), \tag{1.7}$$

if and only if A is RH-regular and

$$P-\lim_{m,n} \sum_{j,k} \left| a_{jk}^{mn} \right| = 1. \tag{1.8}$$

Now let us state the definition given in [14] for absolutely equivalent RH-regular matrices.

Definition 1.3. Two RH-regular matrices A and B are said to be absolutely equivalent for a given class of sequences $x=[x_{jk}]$ whenever

$$P-\lim(Ax-Bx)=0. \tag{1.9}$$

This means that Ax and Bx have the same limit or neither Ax nor Bx have a limit but their difference goes to zero.

The following proposition, lemma, and theorem characterizing the relationship between absolutely equivalent matrices and the P-core are given in [14].

Proposition 1.4. *A necessary and sufficient condition for the RH-regular matrices $A=[a_{jk}^{mn}]$ and $B=[b_{jk}^{mn}]$ to be absolutely equivalent for all bounded sequences is that*

$$P-\lim_{m,n} \sum_{j,k} \left| a_{jk}^{mn} - b_{jk}^{mn} \right| = 0. \tag{1.10}$$

Lemma 1.5. *If two double sequences $x=[x_{jk}]$ and $y=[y_{jk}]$ are such that $P-\lim_{j,k}|x_{jk}-y_{jk}|=0$, then P-core $(x)=P$-core (y).*

Theorem 1.6. *P-core $(Ax) \subseteq$ P-core (x) for all bounded sequences $x = [x_{jk}]$ if and only if A is RH-regular and is absolutely equivalent to a nonnegative matrix $B = [b_{jk}^{mn}]$ for all bounded sequences.*

If $A \in (f_2, c_2^{\infty})$, then the four-dimensional matrix $A = [a_{jk}^{mn}]$ is said to be strongly RH-regular. In [2] the characterization of strongly RH-regular has been given as follows.

Theorem 1.7. *A matrix $A = [a_{jk}^{mn}]$ is strongly RH-regular if and only if A is RH-regular and*

$$P-\lim_{m,n}\sum_{j,k}\left|a_{jk}^{mn}-a_{j+1,k}^{mn}\right|=0,$$
$$P-\lim_{m,n}\sum_{j,k}\left|a_{jk}^{mn}-a_{j,k+1}^{mn}\right|=0. \tag{1.11}$$

2. The Main Results

Theorem 2.1. *If $A = [a_{jk}^{mn}]$ is a strongly Rx-regular matrix and $\pi_1,\ \pi_2$ are two permutations such that*

$$1 \le |\pi_1(k)-\pi_2(k)| \le M \quad \forall k \in \mathbb{N}, \tag{2.1}$$

then P-core $(A\pi_1(x))$ = P-core $(A\pi_2(x))$ for all $x \in \ell_{\infty}^2$, where M is a positive integer and

$$A\pi_1(x)=\sum_{j,k}a_{\pi_1(j),\pi_1(k)}^{mn}x_{jk}, \qquad A\pi_2(x)=\sum_{j,k}a_{\pi_2(j),\pi_2(k)}^{mn}x_{jk}. \tag{2.2}$$

Proof. In the light of Lemma 1.5, it is enough to show that $P-\lim|A\pi_1(x)-A\pi_2(x)|=0$.

Let $m(k)=\min\{\pi_1(k),\pi_2(k)\}$ and $M(k)=\max\{\pi_1(k),\pi_2(k)\}$. Then, it is clear that $|\pi_1(k)-\pi_2(k)|=M(k)-m(k)$ for each $k\in\mathbb{N}$. Now, for $j,k,m,n\in\mathbb{N}$, we can write

$$\begin{aligned}
\left|a_{\pi_1(j),\pi_1(k)}^{mn}-a_{\pi_2(j),\pi_2(k)}^{mn}\right| &= \left|a_{M(j),M(k)}^{mn}-a_{m(j),m(j)k}^{mn}\right| \\
&\le \left|a_{M(j),M(k)}^{mn}-a_{M(j)-1,M(k)-1}^{mn}\right| \\
&\quad+\left|a_{M(j)-1,M(k)-1}^{mn}-a_{M(j)-2,M(k)-2}^{mn}\right| \\
&\quad+\cdots+\left|a_{m(j)+1,m(k)+1}^{mn}-a_{m(j),m(k)}^{mn}\right| \\
&= \sum_{r=1}^{M(j)-m(j)}\ \sum_{s=1}^{M(k)-m(k)}\left|a_{m(j)+r,m(k)+s}^{mn}-a_{m(j)+r-1,m(k)+s-1}^{mn}\right|.
\end{aligned} \tag{2.3}$$

On the other hand, since A is strongly RH-regular, by an easy calculation it can be seen that

$$
\begin{aligned}
|A\pi_1(x) - A\pi_2(x)| &\leq \|x\| \sum_{j,k} \left| a^{mn}_{\pi_1(j),\pi_1(k)} - a^{mn}_{\pi_2(j),\pi_2(k)} \right| \\
&\leq \|x\| \sum_{j,k} \sum_{r=1}^{M(j)-m(j)} \sum_{s=1}^{M(k)-m(k)} \left| a^{mn}_{m(j)+r,m(k)+s} - a^{mn}_{m(j)+r-1,m(k)+s-1} \right| \\
&\leq \|x\| \sum_{j,k} \sum_{r=1}^{M} \sum_{s=1}^{M} \left| a^{mn}_{m(j)+r,m(k)+s} - a^{mn}_{m(j)+r-1,m(k)+s-1} \right| \\
&= \|x\| \sum_{r=1}^{M(j)-m(j)} \sum_{s=1}^{M(k)-m(k)} \sum_{j,k} \left| a^{mn}_{m(j)+r,m(k)+s} - a^{mn}_{m(j)+r-1,m(k)+s-1} \right| \\
&\leq 4\|x\| \sum_{r=1}^{M} \sum_{s=1}^{M} \sum_{j,k} \left| a^{mn}_{j,k} - a^{mn}_{j+1,k} \right|.
\end{aligned}
\tag{2.4}
$$

By the same way, one can also see that

$$
|A\pi_1(x) - A\pi_2(x)| \leq 4\|x\| \sum_{r=1}^{M} \sum_{s=1}^{M} \sum_{j,k} \left| a^{mn}_{j,k} - a^{mn}_{j,k+1} \right|. \tag{2.5}
$$

Now, conditions (1.11) imply that $P - \lim |A\pi_1(x) - A\pi_2(x)| = 0$. This completes the proof. □

Here, let us specialize the permutations π_1 and π_2. Let $I_r = [2^{r-1}, 2^r - 1] = \{k \in \mathbb{N} : 2^{r-1} \leq k \leq 2^r - 1\}$, $r \in \mathbb{N}$, and π_1 be a permutation on I_1 such that $\pi_1(1) = 1$ and

$$
\pi_1\left(2^{r-1} + z\right) = \begin{cases} 2^{r-1} + z + 1 & \text{if } z \text{ is even,} \\ 2^{r-1} + z - 1 & \text{if } z \text{ is odd,} \end{cases} \tag{2.6}
$$

for $z = 0, 1, 2, \ldots, 2^{r-1} - 1$ (see in [15, 16]). Also, let choose the permutation π_2 such that $\pi_2(k) = k$ for all $k \in \mathbb{N}$. Then, we have the following theorem.

Theorem 2.2. *If $A = [a^{mn}_{jk}]$ is a strongly RH-regular nonnegative matrix and π_1, π_2 are two permutations as above, then P-core $(A\pi_1(x)) =$ P-core (x) for all $x \in \ell^2_\infty$.*

Proof. By the same technique used in Theorem 2.1, one can see that $P - \lim |A\pi_1(x) - A\pi_2(x)| = 0$. So, Lemma 1.5 implies that P-core $(A\pi_1(x)) = P$-core $(A\pi_2(x))$. But, since A is an RH-regular nonnegative matrix, P-core $(A(x)) = P$-core (x) for all $x \in \ell^2_\infty$. This step completes the proof. □

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