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Research Article

Stability of Approximate Quadratic Mappings

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We investigate the general solution of the quadratic functional equation $f(2x+y)+3f(2x-y) = 4f(x-y)+12f(x)$, in the class of all functions between quasi-β-normed spaces, and then we prove the generalized Hyers-Ulam stability of the equation by using direct method and fixed point method.

1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let G_1 be a group and let G_2 be a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In 1941, the first result concerning the stability of functional equations was presented by Hyers [2]. And then Aoki [3] and Bourgin [4] have investigated the stability theorems of functional equations with unbounded Cauchy differences. In 1978, Th. M. Rassias [5] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. It was shown by Gajda [6] as well as by Th. M. Rassias and Šemrl [7] that one cannot prove the Rassias' type theorem when $p = 1$. Găvruta [8] obtained generalized result of Th. M. Rassias' Theorem which allow the Cauchy difference to be controlled by a general unbounded function. J. M. Rassias [9, 10] established a similar stability theorem linear and nonlinear mappings with the unbounded Cauchy difference.

Let E_1 and E_2 be real vector spaces. A function $f : E_1 \to E_2$ is called a quadratic function if and only if f is a solution function of the quadratic functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x, where the mapping B is given by $B(x,y) = (1/4)(f(x+y) - f(x-y))$. See [11, 12] for the details. The Hyers-Ulam stability of the quadratic functional (1.1) was first proved by Skof [13] for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [14] demonstrated that Skof's theorem is also valid if E_1 is replaced by an abelian group. Czerwik [15] proved the Hyers-Ulam stability of quadratic functional (1.1) by the similar way to Th. M. Rassias control function [5]. According to the theorem of Borelli and Forti [16], we obtain the following generalization of stability theorem for the quadratic functional (1.1): let G be an abelian group and E a Banach space; let $f : G \to E$ be a mapping with $f(0) = 0$ satisfying the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x,y) \tag{1.2}$$

for all $x, y \in G$. Assume that one of the following conditions

$$\Phi(x,y) := \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{2(k+1)}} \varphi(2^k x, 2^k y) < \infty, \\ \displaystyle\sum_{k=0}^{\infty} 2^{2k} \varphi\left(\frac{x}{2^{(k+1)}}, \frac{y}{2^{(k+1)}}\right) < \infty \end{cases} \tag{1.3}$$

holds for all $x, y \in G$, then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \leq \Phi(x,x) \tag{1.4}$$

for all $x \in G$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [17–23].

In this paper, we consider a new quadratic functional equation

$$f(2x+y) + 3f(2x-y) = 4f(x-y) + 12f(x), \tag{1.5}$$

for all vectors in quasi-β-normed spaces. First, we note that a function f is a solution of the functional (1.5) in the class of all functions between vector spaces if and only if the function f is quadratic. Further, we investigate the generalized Hyers-Ulam stability of (1.5) by using direct method and fixed point method. As a result of the paper, we have a much better possible estimation of approximate quadratic mappings by quadratic mappings than that of Czerwik [15] and Skof [13].

2. Stability of (1.5)

Now, we consider some basic concepts concerning quasi-β-normed spaces and some preliminary results. We fix a realnumber β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Let X be a linear space over $\mathbb{K}$. A *quasi- β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following.

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|^{\beta} \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-β-Banach space* is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a *(β, p)-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^{p} \leq \|x\|^{p} + \|y\|^{p} \tag{2.1}$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a *(β, p)-Banach space*. We can refer to [24, 25] for the concept of quasinormed spaces and p-Banach spaces. Given a p-norm, the formula $d(x, y) := \|x-y\|^{p}$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [25] (see also [24]), each quasinorm is equivalent to some p-norm. In [26], Tabor has investigated a version of the D. H. Hyers, Th. M. Rassias, and Z. Gajda theorem (see [5, 6]) in quasibanach spaces. Recently, J. M. Rassias and Kim [27] have obtained stability results of general additive equations in quasi-β-normed spaces.

From now on, let X be a quasi-α-normed space with norm $\|\cdot\|_{\alpha}$ and let Y be a (β, p)-Banach space with norm $\|\cdot\|_{\beta}$ unless we give any specific reference. Now, we are ready to investigate the generalized Hyers-Ulam stability problem for the functional (1.5) using direct method.

Theorem 2.1. *Assume that a function $f : X \to Y$ satisfies*

$$\left\|Df(x,y) := f(2x+y) + 3f(2x-y) - 4f(x-y) - 12f(x)\right\|_{\beta} \leq \varphi(x,y) \tag{2.2}$$

for all $x, y \in X$ and that φ satisfies the following control conditions

$$\Phi_1(x) := \sum_{i=0}^{\infty} \frac{\varphi(3^{i}x, 3^{i}x)^{p}}{9^{ip\beta}} < \infty, \qquad \lim_{n\to\infty} \frac{\varphi(3^{n}x, 3^{n}y)^{p}}{9^{np\beta}} = 0 \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\left\| f(x) + \frac{f(0)}{2} - Q(x) \right\|_{\beta} \leq \frac{1}{9^{\beta}} \sqrt[p]{\Phi_1(x)} \tag{2.4}$$

for all $x \in X$, where $\|f(0)\|_{\beta} \leq \varphi(0,0)/12^{\beta}$. The function Q is defined as

$$Q(x) = \lim_{k\to\infty} \frac{f(3^{k}x)}{3^{2k}} \tag{2.5}$$

for all $x \in X$.

Proof. Putting $x, y := 0$ in (2.2), we get $\|f(0)\|_\beta \leq \varphi(0,0)/12^\beta$. Replacing y by x in (2.2), we obtain

$$\left\|f(3x) - 9f(x) - 4f(0)\right\|_\beta \leq \varphi(x, x) \tag{2.6}$$

for all $x \in X$. Dividing (2.6) by 9^β, we get

$$\left\|\frac{1}{9}\overline{f}(3x) - \overline{f}(x)\right\|_\beta \leq \frac{1}{9^\beta}\varphi(x, x) \tag{2.7}$$

for all $x \in X$ where $\overline{f}(x) = f(x) + f(0)/2,\ x \in X$. Now letting $x := 3^i x$ and dividing $3^{2ip\beta}$ in (2.7), we have

$$\left\|\frac{1}{3^{2(i+1)}}\overline{f}(3^{i+1}x) - \frac{1}{3^{2i}}\overline{f}(3^i x)\right\|_\beta^p \leq \frac{1}{9^{(i+1)p\beta}}\varphi\left(3^i x, 3^i x\right)^p \tag{2.8}$$

for all $x \in X$. Therefore we prove from the inequality (2.8) that for any integers $m,\ n$ with $m > n \geq 0$

$$\begin{aligned} \left\|\frac{1}{3^{2m}}\overline{f}(3^m x) - \frac{1}{3^{2n}}\overline{f}(3^n x)\right\|_\beta^p &\leq \sum_{i=n}^{m-1}\left\|\frac{\overline{f}(3^{i+1}x)}{3^{2(i+1)}} - \frac{\overline{f}(3^i x)}{3^{2i}}\right\|_\beta^p \\ &\leq \sum_{i=n}^{m-1}\frac{1}{9^{(i+1)p\beta}}\varphi\left(3^i x, 3^i x\right)^p. \end{aligned} \tag{2.9}$$

Since the right-hand side of (2.9) tends to zero as $n \to \infty$, the sequence $\{(1/3^{2n})\overline{f}(3^n x)\}$ is Cauchy for all $x \in X$ and thus converges by the completeness of Y. Define $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{1}{3^{2n}}\left(f(3^n x) + \frac{f(0)}{2}\right) = \lim_{n\to\infty}\frac{f(3^n x)}{3^{2n}}, \quad x \in X. \tag{2.10}$$

Letting $x := 3^n x,\ y := 3^n y$ in (2.2), respectively, and dividing both sides by $3^{2np\beta}$ and after then taking the limit in the resulting inequality, we have

$$\begin{aligned} &\left\|Q(2x+y) + 3Q(2x-y) - 4Q(x-y) - 12Q(x)\right\|_\beta^p \\ &\quad= \lim_{n\to\infty}\frac{\left\|f(3^n(2x+y)) + 3f(3^n(2x-y)) - 4f(3^n(x-y)) - 12f(3^n x)\right\|_\beta^p}{9^{np\beta}} \\ &\quad\leq \lim_{n\to\infty}\frac{1}{9^{np\beta}}\varphi y(3^n x, 3^n)^p = 0, \end{aligned} \tag{2.11}$$

and so the function Q is quadratic.

Taking the limit in (2.9) with $n = 0$ as $m \to \infty$, we obtain that

$$\left\| f(x) + \frac{f(0)}{2} - Q(x) \right\|_\beta^p \le \frac{1}{9^{p\beta}} \sum_{i=0}^{\infty} \frac{\varphi(3^i x, 3^i x)^p}{9^{ip\beta}}, \tag{2.12}$$

which yields the estimation (2.4).

To prove the uniqueness of the quadratic function Q subject to (2.4), let us assume that there exists a quadratic function $Q' : X \to Y$ which satisfies (1.5) and the inequality (2.4). Obviously, we obtain that

$$Q(x) = 3^{-2n} Q(3^n x), \qquad Q'(x) = 3^{-2n} Q'(3^n x) \tag{2.13}$$

for all $x \in X$. Hence it follows from (2.4) that

$$\begin{aligned} \|Q(x) - Q'(x)\|_\beta^p &\le \frac{1}{3^{2np\beta}} \left(\left\| Q(3^n x) - f(3^n x) - \frac{f(0)}{2} \right\|_\beta^p + \left\| f(3^n x) + \frac{f(0)}{2} - Q'(3^n x) \right\|_\beta^p \right) \\ &\le \frac{2}{9^{p\beta}} \sum_{i=0}^{\infty} \frac{1}{3^{2(n+i)p\beta}} \varphi\left(3^{n+i} x, 3^{n+i} x\right)^p = \frac{2}{9^{p\beta}} \sum_{j=n}^{\infty} \frac{1}{3^{2jp\beta}} \varphi\left(3^j x, 3^j x\right)^p \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$. Therefore letting $n \to \infty$, one has $Q(x) - Q'(x) = 0$ for all $x \in X$, completing the proof of uniqueness. □

Theorem 2.2. *Assume that a function $f : X \to Y$ satisfies*

$$\|Df(x,y)\|_\beta \le \varphi(x,y) \tag{2.15}$$

for all $x, y \in X$ and that φ satisfies conditions

$$\Phi_2(x) := \sum_{i=1}^{\infty} 9^{ip\beta} \varphi\left(\frac{x}{3^i}, \frac{x}{3^i}\right)^p < \infty, \qquad \lim_{n\to\infty} 9^{np\beta} \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}\right)^p = 0 \tag{2.16}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\|_\beta^p \le \frac{1}{9^\beta} \sqrt[p]{\Phi_2(x)} \tag{2.17}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{n\to\infty} 3^{2n} f\left(\frac{x}{3^n}\right) \tag{2.18}$$

for all $x \in X$.

Proof. In this case, $f(0) = 0$ since $\sum_{i=1}^{\infty}(1/9^i)\varphi(0,0) < \infty$ and so $\varphi(0,0) = 0$ by assumption.

Replacing x by $x/3$ in (2.6), we obtain

$$\left\| f(x) - 9f\left(\frac{x}{3}\right)\right\|_{\beta} \leq \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \tag{2.19}$$

for $x \in X$. Therefore we prove from inequality (2.19) that for any integers m, n with $m > n \geq 0$

$$\begin{aligned}\left\|9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right)\right\|_{\beta}^{p} &\leq \sum_{i=n}^{m-1}\left\|9^i f\left(\frac{x}{3^i}\right) - 9^{i+1} f\left(\frac{x}{3^{i+1}}\right)\right\|_{\beta}^{p} \\ &\leq \sum_{i=n}^{m-1} 9^{ip\beta}\varphi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right)^{p} \\ &= \frac{1}{9^{p\beta}}\sum_{i=n}^{m-1} 9^{(i+1)p\beta}\varphi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right)^{p}\end{aligned} \tag{2.20}$$

for all $x \in X$. Since the right-hand side of (2.20) tends to zero as $n \to \infty$, the sequence $\{3^{2n} f(x/3^n)\}$ is Cauchy for all $x \in X$ and thus converges by the completeness of Y. Define $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 3^{2n} f\left(\frac{x}{3^n}\right) \tag{2.21}$$

for all $x \in X$.

Thereafter, applying the same argument as in the proof of Theorem 2.1, we obtain the desired result. □

We now introduce a fundamental result of fixed point theory. We refer to [28] for the proof of it, and the reader is referred to papers [29–31].

Theorem 2.3. *Let (Ω, d) be a generalized complete metric space (i.e., d may assume infinite values). Assume that $\Lambda : \Omega \to \Omega$ is a strictly contractive operator with the Lipschitz constant $0 < L < 1$. Then for a given element $x \in \Omega$ one of the following assertions is true:*

(A_1) *$d(\Lambda^{k+1}x, \Lambda^k x) = \infty$ for all $k \geq 0$;*

(A_2) *there exists a nonnegative integer n_0 such that*

- $(A_{2.1})$ *$d(\Lambda^{n+1}x, \Lambda^n x) < \infty$ for all $n \geq n_0$;*
- $(A_{2.2})$ *the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*
- $(A_{2.3})$ *x^* is the unique fixed point of Λ in the set $\Delta = \{y \in \Omega : d(\Lambda^{n_0}x, y) < \infty\}$;*
- $(A_{2.4})$ *$d(y, x^*) \leq (1/1 - L)d(y, \Lambda y)$ for all $y \in \Delta$.*

For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [32]. In 1996, Isac and Th. M. Rassias [33] applied the stability theory of functional equations to prove fixed point theorems and study some new applications in nonlinear analysis. Cădariu and Radu [29, 31] and Radu [34] applied the fixed point theorem of alternative to the investigation of Cauchy and Jensen functional equations. Recently, Jung et al. [35–40] and Jung and Rassias [41] have obtained the generalized Hyers-Ulam stability of functional equations via the fixed point method.

Now we are ready to investigate the generalized Hyers-Ulam stability problem for the functional (1.5) using the fixed point method.

Theorem 2.4. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists a constant L, $0 < L < 1$, satisfying the inequalities*

$$\|Df(x,y)\|_\beta \le \varphi(x,y), \tag{2.22}$$

$$\varphi(3x,3y) \le 9^\beta L\varphi(x,y) \tag{2.23}$$

for all $x,y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ defined by $\lim_{k\to\infty}(f(3^k x)/3^{2k}) = Q(x)$ such that

$$\|f(x) - Q(x)\|_\beta \le \frac{1}{9^\beta(1-L)}\varphi(x,x) \tag{2.24}$$

for all $x \in X$.

Proof. Let us define Ω to be the set of all functions $g : X \to Y$ and introduce a generalized metric d on Ω as follows:

$$d(g,h) = \inf\left\{C \in [0,\infty] : \|g(x) - h(x)\|_\beta \le C\varphi(x,x),\ \forall x \in X\right\}. \tag{2.25}$$

Then it is easy to show that (Ω, d) is complete (see [37, Proof of Theorem 3.1]). Now we define an operator $\Lambda : \Omega \to \Omega$ by

$$\Lambda g(x) = \frac{g(3x)}{9}, \quad g \in \Omega \tag{2.26}$$

for all $x \in X$. First, we assert that Λ is strictly contractive with constant L on Ω. Given $g, h \in \Omega$, let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C$, that is, $\|g(x) - h(x)\|_\beta \le C\varphi(x,x)$. Then it follows from (2.23) that

$$\begin{aligned}\|\Lambda g(x) - \Lambda h(x)\|_\beta &= \frac{1}{9^\beta}\|g(3x) - h(3x)\|_\beta \le \frac{1}{9^\beta}C\varphi(3x,3x)\\ &\le LC\varphi(x,x)\end{aligned} \tag{2.27}$$

for all $x \in X$, that is, $d(\Lambda g, \Lambda h) \le LC$ for any $C \in [0,\infty]$ with $d(g,h) \le C$. Thus we see that $d(\Lambda g, \Lambda h) \le Ld(g,h)$ for any $g, h \in \Omega$ and so Λ is strictly contractive with constant L on Ω.

Next, if we put $(x,y) := (x,x)$ in (2.22) and we divide both sides by 9, then we get

$$\begin{aligned}\left\|\frac{f(3x)}{9}-f(x)\right\|_\beta &= \frac{1}{9^\beta}\|f(3x)-9f(x)\|_\beta \\ &\le \frac{1}{9^\beta}\varphi(x,x)\end{aligned} \tag{2.28}$$

for all $x \in X$, which implies $d(\Lambda f, f) \le 1/9^\beta < \infty$.

Thus applying Theorem 2.3 to the complete generalized metric space (Ω, d) with contractive constant L, we see from $(A_{2.2})$ of Theorem 2.3 that there exists a function $Q : X \to Y$ which is a fixed point of Λ, that is, $Q(x) = \Lambda Q(x) = Q(3x)/9$, such that $d(\Lambda^k f, Q) \to 0$ as $k \to \infty$. By mathematical induction we know that

$$\Lambda^k Q(x) = \frac{Q(3^k x)}{3^{2k}} = Q(x) \tag{2.29}$$

for all $k \in \mathbb{N}$.

Since $d(\Lambda^k f, Q) \to 0$ as $k \to \infty$ by $(A_{2.3})$ of Theorem 2.3, there exists a sequence $\{C_k\}$ such that $C_k \to 0$ as $k \to \infty$, and $d(\Lambda^k f, Q) \le C_k$ for every $k \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\left\|\Lambda^k f(x) - Q(x)\right\|_\beta \le C_k \varphi(x,x) \tag{2.30}$$

for all $x \in X$. This implies

$$\lim_{k\to\infty}\left\|\Lambda^k f(x) - Q(x)\right\|_\beta = 0, \quad \text{i.e.,} \quad \lim_{k\to\infty}\frac{f(3^k x)}{3^{2k}} = Q(x) \tag{2.31}$$

for all $x \in X$.

In turn, it follows from (2.22) and (2.23) that

$$\begin{aligned}\|DQ(x,y)\|_\beta &= \lim_{k\to\infty}\frac{1}{3^{2k\beta}}\left\|Df(3^k x, 3^k y)\right\|_\beta \\ &\le \lim_{k\to\infty}\frac{1}{3^{2k\beta}}\varphi\left(3^k x, 3^k y\right) \le \lim_{k\to\infty} L^k \varphi(x,y) \\ &= 0\end{aligned} \tag{2.32}$$

for all $x, y \in X$, which implies that Q is a solution of (1.5) and so the mapping Q is quadratic.

By $(A_{2.4})$ of Theorem 2.3, we obtain

$$d(f,Q) \le \frac{1}{1-L}d(\Lambda f, f) \le \frac{1}{9^\beta(1-L)}, \tag{2.33}$$

which yields the inequality (2.24).

To prove the uniqueness of Q, assume now that $Q_1 : X \to Y$ is another quadratic mapping satisfying the inequality (2.24). Then Q_1 is a fixed point of Λ with $d(f, Q_1) < \infty$ in view of the inequality (2.24). This implies that $Q_1 \in \Delta = \{g \in \Omega : d(f, g) < \infty\}$ and so $Q = Q_1$ by $(A_{2.3})$ of Theorem 2.3. The proof is complete. □

By a similar way, one can prove the following theorem using the fixed point method.

Theorem 2.5. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that there exists a constant L, $0 < L < 1$, satisfying the inequalities*

$$\|Df(x,y)\|_\beta \le \varphi(x,y), \tag{2.34}$$

$$\varphi(x,y) \le \frac{L}{9^\beta}\varphi(3x,3y) \tag{2.35}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ defined by $\lim_{k\to\infty} 3^{2k} f(x/3^k) = Q(x)$ such that

$$\|f(x) - Q(x)\|_\beta \le \frac{L}{9^\beta(1-L)}\varphi(x,x) \tag{2.36}$$

for all $x \in X$.

Proof. We use the same notations for Ω and d as in the proof of Theorem 2.4. Thus (Ω, d) is a complete generalized metric space. Let us define an operator $\Lambda : \Omega \to \Omega$ by

$$\Lambda g(x) = 9g\left(\frac{x}{3}\right), \quad g \in \Omega \tag{2.37}$$

for all $x \in X$. Then it follows from (2.35) that

$$\|\Lambda g(x) - \Lambda h(x)\|_\beta = 9^\beta \left\| g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right)\right\|_\beta \le 9^\beta C\varphi\left(\frac{x}{3}, \frac{x}{3}\right) \le LC\varphi(x,x) \tag{2.38}$$

for all $x \in X$, that is, $d(\Lambda g, \Lambda h) \le LC$. Thus we see that $d(\Lambda g, \Lambda h) \le Ld(g,h)$ for any $g, h \in \Omega$ and so Λ is strictly contractive with constant L on Ω.

Next, if we put $(x,y) := (x/3, x/3)$ in (2.34) and we divide both sides by $1/9$, then we get by virtue of (2.35)

$$\left\| f(x) - 9f\left(\frac{x}{3}\right)\right\|_\beta = \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \le \frac{L}{9^\beta}\varphi(x,x) \tag{2.39}$$

for all $x \in X$, which implies $d(f, \Lambda f) \le L/9^\beta < \infty$. Thereafter, applying the same argument as in the proof of Theorem 2.4, we obtain the desired results. □

3. Applications of Main Results

In the following corollary, we have a stability result of (1.5) in the sense of Th. M. Rassias.

Corollary 3.1. *Let r_i and ε_i be real numbers such that $\alpha(\max\{r_i : i = 1,2\}) < 2\beta$ and $\varepsilon_i \geq 0$ for $i = 1,2$. Assume that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\|_\beta \leq \varepsilon_1\|x\|_\alpha^{r_1} + \varepsilon_2\|y\|_\alpha^{r_2} \tag{3.1}$$

for all $x, y \in X$, and for all $x, y \in X \setminus \{0\}$ if $r_1, r_2 < 0$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\left\|f(x) + \frac{f(0)}{2} - Q(x)\right\|_\beta \leq \left[\frac{\varepsilon_1^p\|x\|_\alpha^{pr_1}}{3^{p2\beta} - 3^{p\alpha r_1}} + \frac{\varepsilon_2^p\|x\|_\alpha^{pr_2}}{3^{p2\beta} - 3^{p\alpha r_1}}\right]^{1/p} \tag{3.2}$$

for all $x \in X$, and for all $x \in X \setminus \{0\}$ if $r_1, r_2 < 0$. The function Q is given by

$$Q(x) = \lim_{n\to\infty}\frac{f(3^n x)}{3^{2n}}, \tag{3.3}$$

for all $x \in X$, where $f(0) = 0$ if $r_1, r_2 > 0$.

Proof. If $r_1, r_2 > 0$, then we get $f(0) = 0$ by putting $x, y := 0$ in (3.1). Letting $\varphi(x,y) := \varepsilon_1\|x\|_\alpha^{r_1} + \varepsilon_2\|y\|_\alpha^{r_2}$ for all $x, y \in X$ and then applying Theorem 2.1 we obtain easily the desired results. □

Corollary 3.2. *Let r_i and ε_i be real numbers such that $\alpha(\min\{r_i : i = 1,2\}) > 2\beta$ and $\varepsilon_i \geq 0$ for $i = 1,2$. Assume that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\|_\beta \leq \varepsilon_1\|x\|_\alpha^{r_1} + \varepsilon_2\|y\|_\alpha^{r_2} \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\|f(x) - Q(x)\|_\beta \leq \left[\frac{\varepsilon_1^p\|x\|_\alpha^{pr_1}}{3^{p\alpha r_1} - 3^{p2\beta}} + \frac{\varepsilon_2^p\|x\|_\alpha^{pr_2}}{3^{p\alpha r_1} - 3^{p2\beta}}\right]^{1/p} \tag{3.5}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{n\to\infty} 3^{2n} f\left(\frac{x}{3^n}\right) \tag{3.6}$$

for all $x \in X$.

In the following corollary, we have a stability result of (1.5) in the sense of Hyers.

Corollary 3.3. *Let δ be a nonnegative real number. Assume that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\|_\beta \le \delta \tag{3.7}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$, defined by $Q(x) = \lim_{n\to\infty}(f(3^n x)/3^{2n})$, which satisfies the inequality

$$\left\| f(x) + \frac{f(0)}{2} - Q(x) \right\|_\beta \le \frac{\delta}{9^{p\beta} - 1} \tag{3.8}$$

for all $x \in X$.

In the next corollary, we get a stability result of (1.5) in the sense of J. M. Rassias.

Corollary 3.4. *Let ε, r_1, r_2 be real numbers such that $\varepsilon \ge 0$ and $\alpha r \ne 2\beta$, where $r := r_1 + r_2$. Suppose that a function $f : X \to Y$ satisfies*

$$\|Df(x,y)\|_\beta \le \varepsilon \|x\|_\alpha^{r_1} \|y\|_\alpha^{r_2} \tag{3.9}$$

for all $x, y \in X$, and for all $x, y \in X \setminus \{0\}$ if $r_1, r_2 < 0$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\left\| f(x) + \frac{f(0)}{2} - Q(x) \right\|_\beta \le \frac{\varepsilon \|x\|_\alpha^r}{\sqrt[p]{\left|3^{p\alpha r} - 3^{p2\beta}\right|}} \tag{3.10}$$

for all $x \in X$ and all $x, y \in X \setminus \{0\}$ if $r_1, r_2 < 0$, where $f(0) = 0$ if $r_1, r_2 > 0$.

Proof. Letting $\varphi(x,y) := \varepsilon \|x\|_\alpha^{r_1} \|y\|_\alpha^{r_2}$ and applying Theorems 2.1 and 2.2, we get the results. □

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