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Research Article

Upper Bounds for the Euclidean Operator Radius and Applications

S. S. Dragomir

Research Group in Mathematical Inequalities & Applications, School of Engineering & Science, Victoria University, P.O. Box 14428, Melbourne, VIC 8001, Australia

Correspondence should be addressed to S. S. Dragomir, sever.dragomir@vu.edu.au

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The main aim of the present paper is to establish various sharp upper bounds for the Euclidean operator radius of an n-tuple of bounded linear operators on a Hilbert space. The tools used are provided by several generalizations of Bessel inequality due to Boas-Bellman, Bombieri, and the author. Natural applications for the norm and the numerical radius of bounded linear operators on Hilbert spaces are also given.

1. Introduction

Following Popescu's work [1], we present here some basic properties of the *Euclidean operator radius* of an n-tuple of operators $(T_1, \dots, T_n)$ that are defined on a Hilbert space $(H; \langle \cdot, \cdot \rangle)$. This radius is defined by

$$w_e(T_1, \dots, T_n) := \sup_{\|h\|=1} \left(\sum_{i=1}^{n} \left| \langle T_i h, h \rangle \right|^2 \right)^{1/2}. \tag{1.1}$$

We can also consider the following norm and spectral radius on $B(H)^{(n)} := B(H) \times \cdots \times B(H)$, by setting [1]

$$\begin{aligned} \left\| (T_1, \dots, T_n) \right\|_e &:= \sup_{(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n} \left\| \lambda_1 T_1 + \cdots + \lambda_n T_n \right\|, \\ r_e(T_1, \dots, T_n) &= \sup_{(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n} r(\lambda_1 T_1 + \cdots + \lambda_n T_n), \end{aligned} \tag{1.2}$$

where $r(T)$ denotes the usual spectral radius of an operator $T \in B(H)$ and B_n is the closed unit ball in $\mathbb{C}^n$.

Notice that $\|\cdot\|_e$ is a norm on $B(H)^{(n)}$:

$$\|(T_1,\ldots,T_n)\|_e = \|(T_1^*,\ldots,T_n^*)\|_e, \qquad r_e(T_1,\ldots,T_n) = r_e(T_1^*,\ldots,T_n^*). \tag{1.3}$$

Now, if we denote by $\|[T_1,\ldots,T_n]\|$ the square root of the norm $\|\sum_{i=1}^n T_iT_i^*\|$, that is,

$$\|[T_1,\ldots,T_n]\| := \left\|\sum_{i=1}^n T_iT_i^*\right\|^{1/2}, \tag{1.4}$$

then we can present the following result due to Popescu [1] concerning some sharp inequalities between the norms $\|[T_1,\ldots,T_n]\|$ and $\|(T_1,\ldots,T_n)\|_e$.

Theorem 1.1 (see [1]). *If* $(T_1,\ldots,T_n) \in B(H)^{(n)}$, *then*

$$\frac{1}{\sqrt{n}}\|[T_1,\ldots,T_n]\| \le \|(T_1,\ldots,T_n)\|_e \le \|[T_1,\ldots,T_n]\|, \tag{1.5}$$

where the constants $1/\sqrt{n}$ *and* 1 *are best possible in* (1.5).

Following [1], we list here some of the basic properties of the Euclidean operator radius of an n-tuple of operators $(T_1,\ldots,T_n) \in B(H)^{(n)}$.

(i) $w_e(T_1,\ldots,T_n) = 0$ if and only if $T_1 = \cdots = T_n = 0$;

(ii) $w_e(\lambda T_1,\ldots,\lambda T_n) = |\lambda| w_e(T_1,\ldots,T_n)$ for any $\lambda \in \mathbb{C}$;

(iii) $w_e(T_1+T_1',\ldots,T_n+T_n') \le w_e(T_1,\ldots,T_n) + w_e(T_1',\ldots,T_n')$;

(iv) $w_e(U^*T_1U,\ldots,U^*T_nU) = w_e(T_1,\ldots,T_n)$ for any unitary operator $U : K \to H$;

(v) $w_e(X^*T_1X,\ldots,X^*T_nX) \le \|X\|^2 w_e(T_1,\ldots,T_n)$ for any operator $X : K \to H$;

(vi) $(1/2)\|(T_1,\ldots,T_n)\|_e \le w_e(T_1,\ldots,T_n) \le \|(T_1,\ldots,T_n)\|_e$;

(vii) $r_e(T_1,\ldots,T_n) \le w_e(T_1,\ldots,T_n)$;

(viii) $w_e(I_\varepsilon \otimes T_1,\ldots,I_\varepsilon \otimes T_n) = w_e(T_1,\ldots,T_n)$ for any separable Hilbert space ε;

(ix) w_e is a continuous map in the norm topology;

(x) $w_e(T_1,\ldots,T_n) = \sup_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{B}_n} w(\lambda_1T_1+\cdots+\lambda_nT_n)$;

(xi) $(1/2\sqrt{n})\|[T_1,\ldots,T_n]\| \le w_e(T_1,\ldots,T_n) \le \|[T_1,\ldots,T_n]\|$ and the inequalities are sharp.

Due to the fact that the particular cases $n = 2$ and $n = 1$ are related to some classical and new results of interest which naturally motivate the research, we recall here some facts of significance for our further considerations.

For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, and for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.6}$$

For other results concerning the numerical range and radius of bounded linear operators on a Hilbert space, see [2, 3].

In [4], Kittaneh has improved (1.6) in the following manner:

$$\frac{1}{4}\left\|A^*A + AA^*\right\| \leq w^2(A) \leq \frac{1}{2}\left\|A^*A + AA^*\right\|, \tag{1.7}$$

with the constants $1/4$ and $1/2$ as best possible.

Let (C, D) be a pair of bounded linear operators on H, the Euclidean operator radius is

$$w_e(C, D) := \sup_{\|x\|=1}\left(\left|\langle Cx, x\rangle\right|^2 + \left|\langle Dx, x\rangle\right|^2\right)^{1/2} \tag{1.8}$$

and, as pointed out in [1], $w_e : B^2(H) \to [0, \infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4}\left\|C^*C + D^*D\right\|^{1/2} \leq w_e(C, D) \leq \left\|C^*C + D^*D\right\|^{1/2}, \tag{1.9}$$

where the constants $\sqrt{2}/4$ and 1 are best possible in (1.9).

We observe that, if C and D are self-adjoint operators, then (1.9) becomes

$$\frac{\sqrt{2}}{4}\left\|C^2 + D^2\right\|^{1/2} \leq w_e(C, D) \leq \left\|C^2 + D^2\right\|^{1/2}. \tag{1.10}$$

We observe also that if $A \in B(H)$ and $A = B + iC$ is the *Cartesian decomposition* of A, then

$$\begin{aligned} w_e^2(B, C) &= \sup_{\|x\|=1}\left[\left|\langle Bx, x\rangle\right|^2 + \left|\langle Cx, x\rangle\right|^2\right] \\ &= \sup_{\|x\|=1}\left|\langle Ax, x\rangle\right|^2 \\ &= w^2(A). \end{aligned} \tag{1.11}$$

By the inequality (1.10) and since (see [4])

$$A^*A + AA^* = 2\left(B^2 + C^2\right), \tag{1.12}$$

then we have

$$\frac{1}{16}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|. \tag{1.13}$$

We remark that the lower bound for $w^2(A)$ in (1.13) provided by Popescu's inequality (1.9) is not as good as the first inequality of Kittaneh from (1.7). However, the upper bounds for $w^2(A)$ are the same and have been proved using different arguments.

In order to get a natural generalization of Kittaneh's result for the Euclidean operator radius of two operators, we have obtained in [5] the following result.

Theorem 1.2. *Let $B, C : H \to H$ be two bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Then*

$$\frac{\sqrt{2}}{2}\left[w\left(B^2 + C^2\right)\right]^{1/2} \le w_e(B, C)\left(\le \|B^*B + C^*C\|^{1/2}\right). \tag{1.14}$$

The constant $\sqrt{2}/2$ is best possible in the sense that it cannot be replaced by a larger constant.

Corollary 1.3. *For any two self-adjoint bounded linear operators B, C on H,one has*

$$\frac{\sqrt{2}}{2}\left\|B^2 + C^2\right\|^{1/2} \le w_e(B, C)\left(\le \left\|B^2 + C^2\right\|^{1/2}\right). \tag{1.15}$$

The constant $\sqrt{2}/2$ is sharp in (1.15).

Remark 1.4. The inequality (1.15) is better than the first inequality in (1.10) which follows from Popescu's first inequality in (1.9). It also provides, for the case that B, C are the self-adjoint operators in the Cartesian decomposition of A, exactly the lower bound obtained by Kittaneh in (1.7) for the numerical radius $w(A)$.

For other inequalities involving the Euclidean operator radius of two operators and their applications for one operator, see the recent paper [5], where further references are given.

Motivated by the useful applications of the Euclidean operator radius concept in multivariable operator theory outlined in [1], we establish in this paper various new sharp upper bounds for the general case $n \ge 2$. The tools used are provided by several generalizations of Bessel inequality due to Boas-Bellman, Bombieri, and the author. Also several reverses of the Cauchy-Bunyakovsky-Schwarz inequalities are employed. The case $n = 2$, which is of special interest since it generates for the Cartesian decomposition of a bounded linear operator various interesting results for the norm and the usual numerical radius, is carefully analyzed.

2. Upper bounds via the Boas-Bellman-type inequalities

The following inequality that naturally generalizes Bessel's inequality for the case of nonorthonormal vectors $y_1,\dots,y_n$ in an inner product space is known in the literature as the *Boas-Bellman inequality* (see [6, 7], or [8, chapter 4]):

$$\sum_{i=1}^{n}\left|\langle x,y_i\rangle\right|^2 \le \|x\|^2\left[\max_{1\le i\le n}\left\|y_i\right\|^2+\left(\sum_{1\le i\ne j\le n}\left|\langle y_i,y_j\rangle\right|^2\right)^{1/2}\right], \tag{2.1}$$

for any $x\in H$.

Obviously, if $\{y_1,\dots,y_n\}$ is an orthonormal family, then (2.1) becomes the classical *Bessel's inequality*

$$\sum_{i=1}^{n}\left|\langle x,y_i\rangle\right|^2 \le \|x\|^2,\quad x\in H. \tag{2.2}$$

The following result provides a natural upper bound for the Euclidean operator radius of n bounded linear operators.

Theorem 2.1. *If $(T_1,\dots,T_n)\in B(H)^{(n)}$, then*

$$w_e(T_1,\dots,T_n)\le\left[\max_{1\le i\le n}\left\|T_i\right\|^2+\left\{\sum_{1\le i\ne j\le n}w^2\left(T_j^*T_i\right)\right\}^{1/2}\right]^{1/2}. \tag{2.3}$$

Proof. Utilizing the Boas-Bellman inequality for $x=h,\ \|h\|=1$ and $y_i=T_ih,\ i=1,\dots,n,$ we have

$$\sum_{i=1}^{n}\left|\langle T_ih,h\rangle\right|^2\le\max_{1\le i\le n}\left\|T_ih\right\|^2+\left(\sum_{1\le i\ne j\le n}\left|\langle T_j^*T_ih,h\rangle\right|^2\right)^{1/2}. \tag{2.4}$$

Taking the supremum over $\|h\|=1$ and observing that

$$\begin{aligned}\sup_{\|h\|=1}\left[\max_{1\le i\le n}\left\|T_ih\right\|^2\right]&=\max_{1\le i\le n}\left\|T_i\right\|^2,\\ \sup_{\|h\|=1}\left(\sum_{1\le i\ne j\le n}\left|\langle T_j^*T_ih,h\rangle\right|^2\right)^{1/2}&\le\left[\sum_{1\le i\ne j\le n}\sup_{\|h\|=1}\left|\langle T_j^*T_ih,h\rangle\right|^2\right]^{1/2}\\ &=\left[\sum_{1\le i\ne j\le n}w^2\left(T_j^*T_i\right)\right]^{1/2},\end{aligned} \tag{2.5}$$

then by (2.4) we deduce the desired inequality (2.3). □

Remark 2.2. If $(T_1, \ldots, T_n) \in B(H)^{(n)}$ is such that $T_j^* T_i = 0$ for $i, j \in \{1, \ldots, n\}$, then from (2.3), we have the inequality:

$$w_e(T_1, \ldots, T_n) \leq \max_{1 \leq i \leq n} \|T_i\|. \tag{2.6}$$

We observe that a sufficient condition for $T_j^* T_i = 0$, with $i \neq j$, $i, j \in \{1, \ldots, n\}$ to hold, is that $\mathrm{Range}(T_i) \perp \mathrm{Range}(T_j)$ for $i, j \in \{1, \ldots, n\}$, with $i \neq j$.

Remark 2.3. If we apply the above result for two bounded linear operators on $H, B, C : H \to H$, then we get the simple inequality

$$w_e^2(B, C) \leq \max\left\{\|B\|^2, \|C\|^2\right\} + \sqrt{2} w(B^* C). \tag{2.7}$$

Remark 2.4. If $A : H \to H$ is a bounded linear operator on the Hilbert space H and if we denote by

$$B := \frac{A + A^*}{2}, \qquad C := \frac{A - A^*}{2i} \tag{2.8}$$

its Cartesian decomposition, then

$$\begin{gathered} w_e^2(B, C) = w^2(A), \\ w(B^* C) = w(C^* B) = \frac{1}{4} w\left[(A^* - A)(A + A^*)\right], \end{gathered} \tag{2.9}$$

and from (2.7), we get the inequality

$$w^2(A) \leq \frac{1}{4}\left\{ \max\left\{ \|A + A^*\|^2, \|A - A^*\|^2 \right\} + \sqrt{2} w\left[(A^* - A)(A + A^*)\right] \right\}. \tag{2.10}$$

In [9], the author has established the following Boas-Bellman type inequality for the vectors $x, y_1, \ldots, y_n$ in the real or complex inner product space $(H, \langle \cdot, \cdot \rangle)$:

$$\sum_{i=1}^{n} \left|\langle x, y_i \rangle\right|^2 \leq \|x\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} \left|\langle y_i, y_j \rangle\right| \right\}. \tag{2.11}$$

For orthonormal vectors, (2.11) reduces to Bessel's inequality as well. It has also been shown in [9] that the Boas-Bellman inequality (2.1) and the inequality (2.11) cannot be compared in general, meaning that in some instances the right-hand side of (2.1) is smaller than that of (2.11) and vice versa.

Now, utilizing the inequality (2.11) and making use of the same argument from the proof of Theorem 2.1, we can state the following result as well.

Theorem 2.5. *If* $(T_1,\ldots,T_n)\in B(H)^{(n)}$*, then*

$$w_e(T_1,\ldots,T_n)\le\left[\max_{1\le i\le n}\|T_i\|^2+(n-1)\max_{1\le i\ne j\le n}w(T_j^*T_i)\right]^{1/2}. \tag{2.12}$$

If in (2.12) one assumes that $T_j^*T_i=0$ for each $i,j\in\{1,\ldots,n\}$ with $i\ne j$, then one gets the result from (2.6).

Remark 2.6. We observe that, for $n=2$, we get from (2.12) a better result than (2.7), namely,

$$w_e^2(B,C)\le\max\{\|B\|^2,\|C\|^2\}+w(B^*C), \tag{2.13}$$

where $B,\ C$ are arbitrary linear bounded operators on H. The inequality (2.13) is sharp. This follows from the fact that for $B=C=A\in B(H),\ A$ a normal operator, we have

$$\begin{gathered}w_e^2(A,A)=2w^2(A)=2\|A\|^2,\\ w(A^*A)=\|A\|^2,\end{gathered} \tag{2.14}$$

and we obtain in (2.13) the same quantity in both sides. The inequality (2.13) has been obtained in [5, (12.23)] on utilizing a different argument.

Also, for the operator $A:H\to H$, we can obtain from (2.13) the following inequality:

$$w^2(A)\le\frac{1}{4}\{\max\{\|A+A^*\|^2,\|A-A^*\|^2\}+w[(A^*-A)(A+A^*)]\}, \tag{2.15}$$

which is better than (2.10). The constant $1/4$ in (2.15) is sharp. The case of equality in (2.15) follows, for instance, if A is assumed to be self-adjoint.

Remark 2.7. If in (2.13) we choose $C=A,\ B=A^*,\ A\in B(H)$, and take into account that

$$w_e^2(A^*,A)=2w^2(A), \tag{2.16}$$

then we get the inequality

$$w^2(A)\le\frac{1}{2}[\|A\|^2+w(A^2)]\,(\le\|A\|^2), \tag{2.17}$$

for any $A\in B(H)$. The constant $1/2$ is sharp.

Note that this inequality has been obtained in [10] by the use of a different argument based on the Buzano inequality [11].

A different approach is incorporated in the following result.

Theorem 2.8. *If* $(T_1, \ldots, T_n) \in B(H)^{(n)}$, *then*

$$w_e^2(T_1, \ldots, T_n) \le \max_{1\le i\le n} w(T_i)\cdot \left\{ \left\| \sum_{i=1}^{n} T_i^* T_i \right\| + \sum_{1\le i\ne j\le n} w(T_j^* T_i) \right\}^{1/2}. \tag{2.18}$$

Proof. We use the following Boas-Bellman-type inequality obtained in [9] (see also [8, page 132]):

$$\sum_{i=1}^{n} \left|\langle x, y_i\rangle\right|^2 \le \|x\| \max_{1\le i\le n} \left|\langle x, y_i\rangle\right| \left\{ \sum_{i=1}^{n} \|y_i\|^2 + \sum_{1\le i\ne j\le n} \left|\langle y_i, y_j\rangle\right| \right\}^{1/2}, \tag{2.19}$$

where $x, y_1, \ldots, y_n$ are arbitrary vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

Now, for $x = h$, $\|h\| = 1$, $y_i = T_i h$, $i = 1, \ldots, n$, we get from (2.19) that

$$\sum_{i=1}^{n} \left|\langle T_i h, h\rangle\right|^2 \le \max_{1\le i\le n} \left|\langle T_i h, h\rangle\right| \left\{ \sum_{i=1}^{n} \|T_i h\|^2 + \sum_{1\le i\ne j\le n} \left|\langle T_i h, T_j h\rangle\right| \right\}^{1/2}. \tag{2.20}$$

Observe that

$$\sum_{i=1}^{n} \|T_i h\|^2 = \sum_{i=1}^{n} \langle T_i h, T_i h\rangle = \sum_{i=1}^{n} \left\langle (T_i^* T_i) h, h\right\rangle = \left\langle \sum_{i=1}^{n} (T_i^* T_i) h, h \right\rangle, \tag{2.21}$$

for $h \in H, \|h\| = 1$.

Therefore, on taking the supremum in (2.20) and noticing that $w(\sum_{i=1}^{n} T_i^* T_i) = \|\sum_{i=1}^{n} T_i^* T_i\|$, we get the desired result (2.18). □

Remark 2.9. If $(T_1, \ldots, T_n) \in B(H)^{(n)}$ satisfies the condition that $T_i^* T_j = 0$ for each $i, j \in \{1, \ldots, n\}$ with $i \ne j$, then from (2.18) we get

$$w_e^2(T_1, \ldots, T_n) \le \max_{1\le i\le n} w(T_i)\cdot \left\| \sum_{i=1}^{n} T_i^* T_i \right\|^{1/2}. \tag{2.22}$$

Remark 2.10. If we apply Theorem 2.8 to $n = 2$, then we can state the following simple inequality:

$$w_e^2(B, C) \le \max\{\|B\|, \|C\|\} \left[\|B^* B + C^* C\| + 2w(B^* C) \right]^{1/2}, \tag{2.23}$$

for any bounded linear operators $B, C \in B(H)$.

Moreover, if B and C are chosen as the Cartesian decomposition of the bounded linear operator $A \in B(H)$, then we can state that

$$w^2(A) \le \frac{1}{2}\max\{\|A + A^*\|, \|A - A^*\|\}\left\{\left\|\frac{A^*A + AA^*}{2}\right\| + \frac{1}{2}w[(A + A^*)(A - A^*)]\right\}^{1/2}. \tag{2.24}$$

The constant $1/2$ is best possible in (2.24). The equality case is obtained if A is a self-adjoint operator on H.

If we choose in (2.23), $C = A$, $B = A^*$, $A \in B(H)$, then we get

$$w^2(A) \le \frac{1}{2}\|A\|\left[\|AA^* + A^*A\| + 2w(A^2)\right]^{1/2}(\le \|A\|^2). \tag{2.25}$$

The constant $1/2$ is best possible in (2.25).

3. Upper bounds via the Bombieri-type inequalities

A different generalization of Bessel's inequality for nonorthogonal vectors than the one mentioned above and due to Boas and Bellman is the *Bombieri inequality* (see [12], [13, page 394], or [8, page 134])

$$\sum_{i=1}^{n}\left|\langle x, y_i\rangle\right|^2 \le \|x\|^2 \max_{1\le i\le n}\left\{\sum_{j=1}^{n}\left|\langle y_i, y_j\rangle\right|\right\}, \tag{3.1}$$

where $x, y_1, \ldots, y_n$ are vectors in the real or complex inner product space $(H; \langle\cdot,\cdot\rangle)$.

Note that the Bombieri inequality was not stated in the general case of inner product spaces in [12]. However, the inequality presented there easily leads to (3.1) which, apparently, was firstly mentioned as is in [13, page 394].

The following upper bound for the Euclidean operator radius may be obtained as follows.

Theorem 3.1. *If* $(T_1, \ldots, T_n) \in B(H)^{(n)}$, *then*

$$w_e^2(T_1, \ldots, T_n) \le \max_{1\le i\le n}\left\{\sum_{j=1}^{n} w(T_j^* T_i)\right\}. \tag{3.2}$$

Proof. Follows by Bombieri's inequality applied for $x = h$, $\|h\| = 1$ and $y_i = T_i h$, $i = 1, \ldots, n$. Then taking the supremum over $\|h\| = 1$ and utilizing its properties, we easily deduce the desired inequality (3.2). □

Remark 3.2. If we apply the above theorem for two operators B and C, then we get

$$\begin{aligned} w_e^2(B,C) &\le \max\{w(B^*B)+w(C^*B), w(B^*C)+w(C^*C)\} \\ &= \max\{\|B\|^2+w(B^*C), w(B^*C)+\|C\|^2\} \\ &= \max\{\|B\|^2,\|C\|^2\}+w(B^*C), \end{aligned} \tag{3.3}$$

which is exactly the inequality (2.13) that has been obtained in a different manner above.

In order to get other bounds for the Euclidean operator radius, we may state the following result as well.

Theorem 3.3. *If $(T_1,\ldots,T_n) \in B(H)^{(n)}$, then*

$$w_e^2(T_1,\ldots,T_n) \le \begin{cases} D_w, \\ E_w, \\ F_w, \end{cases} \tag{3.4}$$

meaning that the left side is less than each of the quantities in the right side, where

$$D_w := \begin{cases} \max\limits_{1\le k\le n}\{w(T_k)\}\left[\sum\limits_{i,j=1}^{n} w(T_j^*T_i)\right]^{1/2}, \\ \max\limits_{1\le k\le n}\{w(T_k)\}^{1/2}\left(\sum\limits_{i=1}^{n}[w(T_i)]^r\right)^{1/2r}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} w(T_j^*T_i)\right)^s\right]^{1/2s}, \\ \qquad \text{where } r,s>1,\ \dfrac{1}{r}+\dfrac{1}{s}=1, \\ \max\limits_{1\le k\le n}\{w(T_k)\}^{1/2}\left(\sum\limits_{i=1}^{n} w(T_i)\right)^{1/2}\max\limits_{1\le i\le n}\left[\sum\limits_{j=1}^{n} w(T_j^*T_i)\right]^{1/2}, \end{cases}$$

$$E_w := \begin{cases} \left(\sum\limits_{k=1}^{n}[w(T_k)]^p\right)^{1/2p}\max\limits_{1\le k\le n}\{w(T_k)\}^{1/2}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} w(T_j^*T_i)\right)^q\right]^{1/2q}, \\ \qquad \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \left(\sum\limits_{k=1}^{n}[w(T_k)]^p\right)^{1/2p}\left(\sum\limits_{i=1}^{n}[w(T_i)]^t\right)^{1/2t}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}[w(T_j^*T_i)]^q\right)^{u/q}\right]^{1/2u}, \\ \qquad \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1,\ t>1,\ \dfrac{1}{t}+\dfrac{1}{u}=1, \\ \left(\sum\limits_{k=1}^{n}[w(T_k)]^p\right)^{1/2p}\left(\sum\limits_{i=1}^{n} w(T_i)\right)^{1/2}\max\limits_{1\le i\le n}\left\{\left(\sum\limits_{j=1}^{n}[w(T_j^*T_i)]^q\right)^{1/2p}\right\}, \\ \qquad \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \end{cases}$$

$$F_w := \begin{cases} \left(\sum_{k=1}^{n} w(T_k)\right)^{1/2} \max_{1\le i\le n}\{w(T_i)\}^{1/2} \sum_{i=1}^{n}\left[\max_{1\le j\le n}\{w(T_j^*T_i)\}^{1/2}\right], \\ \left(\sum_{k=1}^{n} w(T_k)\right)^{1/2} \left(\sum_{i=1}^{n}[w(T_i)]^m\right)^{1/2m} \sum_{i=1}^{n}\left[\max_{1\le j\le n}[w(T_j^*T_i)]^l\right]^{1/2l}, \\ \qquad\qquad \text{where } m>1,\ \dfrac{1}{m}+\dfrac{1}{l}=1, \\ \sum_{k=1}^{n} w(T_k)\cdot \max_{1\le i,j\le n}\{w(T_j^*T_i)\}^{1/2}. \end{cases} \tag{3.5}$$

Proof. In our paper [14] (see also [8, page 141-142]), we have established the following sequence of inequalities for the vectors $x, y_1, \ldots, y_n$ in the inner product space $(H, \langle\cdot,\cdot\rangle)$ and the scalars $c_1, \ldots, c_n \in \mathbb{K}$:

$$\left|\sum_{i=1}^{n} c_i\langle x, y_i\rangle\right|^2 \le \|x\|^2 \times \begin{cases} D, \\ E, \\ F, \end{cases} \tag{3.6}$$

where

$$D := \begin{cases} \max_{1\le k\le n}|c_k|^2\left(\sum_{i,j=1}^{n}|\langle y_i, y_j\rangle|\right), \\ \max_{1\le k\le n}|c_k|\left(\sum_{i=1}^{n}|c_i|^r\right)^{1/r}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|\right)^s\right]^{1/s}, \quad \text{where } r,s>1,\ \dfrac{1}{r}+\dfrac{1}{s}=1, \\ \max_{1\le k\le n}|c_k|\left(\sum_{i=1}^{n}|c_i|\right)\max_{1\le i\le n}\left[\sum_{j=1}^{n}|\langle y_i, y_j\rangle|\right], \end{cases}$$

$$E := \begin{cases} \left(\sum_{k=1}^{n}|c_k|^p\right)^{1/p}\max_{1\le i\le n}|c_i|\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|\right)^q\right]^{1/q}, & \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \left(\sum_{k=1}^{n}|c_k|^p\right)^{1/p}\left(\sum_{i=1}^{n}|c_i|^t\right)^{1/t}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|^q\right)^{u/q}\right]^{1/u}, & \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ & t>1,\ \dfrac{1}{t}+\dfrac{1}{u}=1, \\ \left(\sum_{k=1}^{n}|c_k|^p\right)^{1/p}\left(\sum_{i=1}^{n}|c_i|\right)\max_{1\le i\le n}\left\{\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|^q\right)^{1/q}\right\}, & \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \end{cases}$$

$$F := \begin{cases} \left(\sum_{k=1}^{n}|c_k|\right)\max_{1\le i\le n}|c_i|\sum_{i=1}^{n}\left[\max_{1\le j\le n}|\langle y_i, y_j\rangle|\right], \\ \sum_{k=1}^{n}|c_k|\left(\sum_{i=1}^{n}|c_i|^m\right)^{1/m}\sum_{i=1}^{n}\left[\max_{1\le j\le n}|\langle y_i, y_j\rangle|^l\right]^{1/l}, \quad \text{where } m>1,\ \frac{1}{m}+\frac{1}{l}=1, \\ \left(\sum_{k=1}^{n}|c_k|\right)^2\max_{1\le i,j\le n}|\langle y_i, y_j\rangle|. \end{cases} \tag{3.7}$$

If in this inequality we choose $c_i = \langle x, y_i\rangle,\ i = 1, \ldots, n$ and take the square root, then we get the inequalities

$$\sum_{i=1}^{n}|\langle x, y_i\rangle|^2 \le \|x\| \times \begin{cases} \widetilde{D}, \\ \widetilde{E}, \\ \widetilde{F}, \end{cases} \tag{3.8}$$

where

$$\widetilde{D} := \begin{cases} \max_{1\le k\le n}|\langle x, y_k\rangle|\left(\sum_{i,j=1}^{n}|\langle y_i, y_j\rangle|\right)^{1/2}, \\ \max_{1\le k\le n}|\langle x, y_k\rangle|^{1/2}\left(\sum_{i=1}^{n}|\langle x, y_i\rangle|^r\right)^{1/2r}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|\right)^s\right]^{1/2s}, \\ \qquad \text{where } r,s>1,\ \frac{1}{r}+\frac{1}{s}=1, \\ \max_{1\le k\le n}|\langle x, y_k\rangle|^{1/2}\left(\sum_{i=1}^{n}|\langle x, y_i\rangle|\right)^{1/2}\max_{1\le i\le n}\left[\sum_{j=1}^{n}|\langle y_i, y_j\rangle|\right]^{1/2}, \end{cases}$$

$$\widetilde{E} := \begin{cases} \left(\sum_{k=1}^{n}|\langle x, y_k\rangle|^p\right)^{1/2p}\max_{1\le i\le n}|\langle x, y_i\rangle|^{1/2}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|\right)^q\right]^{1/2q}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left(\sum_{k=1}^{n}|\langle x, y_k\rangle|^p\right)^{1/2p}\left(\sum_{i=1}^{n}|\langle x, y_i\rangle|^t\right)^{1/2t}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|^q\right)^{u/q}\right]^{1/2u}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ t>1,\ \frac{1}{t}+\frac{1}{u}=1, \\ \left(\sum_{k=1}^{n}|\langle x, y_k\rangle|^p\right)^{1/2p}\left(\sum_{i=1}^{n}|\langle x, y_i\rangle|\right)^{1/2}\max_{1\le i\le n}\left\{\left(\sum_{j=1}^{n}|\langle y_i, y_j\rangle|^q\right)^{1/q}\right\}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases}$$

$$\widetilde{F} := \begin{cases} \left(\sum_{k=1}^{n}|\langle x, y_k\rangle|\right)^{1/2} \max_{1\le i\le n}|\langle x, y_i\rangle|^{1/2} \sum_{i=1}^{n}\left[\max_{1\le j\le n}|\langle y_i, y_j\rangle|\right]^{1/2}, \\ \left(\sum_{k=1}^{n}|\langle x, y_k\rangle|\right)^{1/2} \left(\sum_{i=1}^{n}|\langle x, y_i\rangle|^{m}\right)^{1/2m} \left[\sum_{i=1}^{n}\left[\max_{1\le j\le n}|\langle y_i, y_j\rangle|^{l}\right]\right]^{1/2l}, \\ \qquad \text{where } m > 1,\ \frac{1}{m} + \frac{1}{l} = 1, \\ \sum_{k=1}^{n}|\langle x, y_k\rangle| \max_{1\le i,j\le n}|\langle y_i, y_j\rangle|^{1/2}. \end{cases} \tag{3.9}$$

By making use of the inequality (3.8) for the choices $x = h$, $\|h\| = 1$, $y_i = T_i h$, $i = 1, \dots, n$ and taking the supremum, we get the following result (3.4). □

Remark 3.4. For $n = 2$, the above inequalities (3.4) provide various upper bounds for the Euclidean operator radius $w_e(B, C)$, for any $B, C \in B(H)$. Out of these results and for the sake of brevity, we only mention the following ones:

$$w_e^2(B, C) \le \max\{\|B\|, \|C\|\}^{1/2}[w(B) + w(C)]^{1/2}\left[\max\{\|B\|^2, \|C\|^2\} + w(B^*C)\right]^{1/2}, \tag{3.10}$$

$$w_e^2(B, C) \le [w(B) + w(C)] \max\{\|B\|, \|C\|, [w(B^*C)]^{1/2}\}, \tag{3.11}$$

for any $B, C \in B(H)$.

Both inequalities are sharp. This follows by the fact that for $B = C = A \in B(H)$, A a normal operator, we get in both sides of (3.10) and (3.11) the same quantity $2\|A\|^2$.

Remark 3.5. If we choose in (3.10), the Cartesian decomposition of the operator A, then we get

$$\begin{aligned} w^2(A) \le \frac{1}{4} \max\{\|A + A^*\|, \|A - A^*\|\}^{1/2} [\|A + A^*\| + \|A - A^*\|]^{1/2} \\ \times \left[\max\{\|A + A^*\|^2, \|A - A^*\|^2\} + w[(A^* - A)(A^* + A)]\right]^{1/2}. \end{aligned} \tag{3.12}$$

The constant $1/4$ is sharp. The equality case holds if $A = A^*$. The same choice in (3.11) will give

$$\begin{aligned} w^2(A) \le \frac{1}{4} [\|A + A^*\| + \|A - A^*\|] \\ \times \max\{\|A + A^*\|, \|A - A^*\|, w^{1/2}[(A^* - A)(A^* + A)]\}. \end{aligned} \tag{3.13}$$

The constant $1/4$ is also sharp.

In [15] (see also [8, page 233]), we obtained the following inequality of Bombieri type:

$$\sum_{i=1}^{n}\left|\langle x, y_i\rangle\right|^2 \le \|x\|\left(\sum_{k=1}^{n}\left|\langle x, y_k\rangle\right|^p\right)^{1/p}\left[\sum_{i,j=1}^{n}\left|\langle y_i, y_j\rangle\right|^q\right]^{1/2q}, \tag{3.14}$$

for any $x, y_1, \dots, y_n$ vectors in the inner product space $(H, \langle\cdot,\cdot\rangle,)$ where $p > 1$ and $1/p + 1/q = 1$. Out of this inequality, one can get for $p = q = 2$ the following inequality of Bessel-type firstly obtained in [16]:

$$\sum_{i=1}^{n}\left|\langle x, y_i\rangle\right|^2 \le \|x\|^2\left(\sum_{i,j=1}^{n}\left|\langle y_i, y_j\rangle\right|^2\right)^{1/2}. \tag{3.15}$$

The following upper bound for the Euclidean operator radius may be stated.

Theorem 3.6. *If* $(T_1, \dots, T_n) \in B(H)^{(n)}$, *then*

$$\begin{aligned} w_e^2(T_1, \dots, T_n) &\le \left\{w_e^2(T_1^*T_1, \dots, T_n^*T_n) + \sum_{1\le i\ne j\le n} w^2(T_j^*T_i)\right\}^{1/2} \\ &\le \left\{\left\|\sum_{i=1}^{n}(T_i^*T_i)^2\right\| + \sum_{1\le i\ne j\le n} w^2(T_j^*T_i)\right\}^{1/2} \\ &\left(\le \left\{\sum_{i=1}^{n}\|T_i\|^4 + \sum_{1\le i\ne j\le n} w^2(T_j^*T_i)\right\}^{1/2}\right). \end{aligned} \tag{3.16}$$

Proof. Utilizing (3.15), for $x = h$, $\|h\| = 1$, $y_i = T_i h$, $i = 1, \dots, n$, we get

$$\begin{aligned} \sum_{i=1}^{n}\left|\langle T_i h, h\rangle\right|^2 &\le \left\{\sum_{i=1}^{n}\|T_i h\|^4 + \sum_{1\le i\ne j\le n}\left|\langle T_i h, T_j h\rangle\right|^2\right\}^{1/2} \\ &= \left\{\sum_{i=1}^{n}\left|\langle T_i^*T_i h, h\rangle\right|^2 + \sum_{1\le i\ne j\le n}\left|\langle T_j^*T_i h, h\rangle\right|^2\right\}^{1/2}. \end{aligned} \tag{3.17}$$

Now, taking the supremum over $\|h\| = 1$, we deduce the first inequality in (3.16).

The second inequality follows by the property (xi) from Introduction applied for the self-adjoint operators $V_1 = T_1^*T_1, \dots, V_n = T_n^*T_n$.

The last inequality is obvious. □

Remark 3.7. If in (3.16) we assume that the operators $(T_1, \ldots, T_n)$ satisfy the condition $T_j^* T_i = 0$ for $i, j \in \{1, \ldots, n\}$, with $i \neq j$, then we get the inequality

$$w_e^2(T_1, \ldots, T_n) \le w_e^2(T_1^* T_1, \ldots, T_n^* T_n) \le \left\| \sum_{i=1}^n (T_i^* T_i)^2 \right\|^{1/2} \left(\le \left\{ \sum_{i=1}^n \|T_i\|^4 \right\}^{1/2} \right). \tag{3.18}$$

Remark 3.8. For $n = 2$, the above inequality (3.16) provides

$$\begin{aligned} w_e^2(B, C) &\le \left[w_e^2(B^* B, C^* C) + 2w^2(B^* C) \right]^{1/2} \\ &\le \left[\|(B^* B)^2 + (C^* C)^2\| + 2w^2(B^* C) \right]^{1/2} \\ &\le \left[\|B\|^4 + \|C\|^4 + 2w^2(B^* C) \right]^{1/2} (\le \|B\|^2 + \|C\|^2), \end{aligned} \tag{3.19}$$

for any $B, C \in B(H)$.

If in (3.19) we choose B and C to be the Cartesian decompositions of the operator $A \in B(H)$, then we get

$$\begin{aligned} w^2(A) &\le \frac{1}{4} \left\{ \|(A + A^*)^4 + (A - A^*)^4\| + 2w^2[(A^* - A)(A^* + A)] \right\}^{1/2} \\ &\left(\le \frac{1}{4} \left\{ \|A + A^*\|^4 + \|A - A^*\|^4 + 2w^2[(A^* - A)(A^* + A)] \right\}^{1/2} \right). \end{aligned} \tag{3.20}$$

Here, the constant $1/4$ is best possible.

Remark 3.9. If in (3.19) we choose $B = A^*$ and $C = A$, where $A \in B(H)$, then we get

$$w^4(A) \le \frac{1}{2} \left[\left\| \frac{(AA^*)^2 + (A^* A)^2}{2} \right\| + w^2(A^2) \right]. \tag{3.21}$$

The constant $1/2$ infront of the square bracket is best possible in (3.21).

4. Other upper bounds

For an n-tuple of operators $(T_1, \ldots, T_n) \in B(H)^{(n)}$, we use the notation $\Delta T_k := T_{k+1} - T_k$, $k = 1, \ldots, n-1$.

The following result may be stated as follows.

Theorem 4.1. *If* $(T_1, \ldots, T_n) \in B(H)^{(n)}, \; n \geq 2$, *then*

$$(0 \leq) \frac{1}{n} w_e^2(T_1, \ldots, T_n) - w^2\left(\frac{1}{n}\sum_{j=1}^{n} T_j\right)$$

$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{1\leq k\leq n-1} w^2(\Delta T_k) \\ \dfrac{n^2-1}{6n}\left[\sum\limits_{k=1}^{n-1} w^p(\Delta T_k)\right]^{1/p}\left[\sum\limits_{k=1}^{n-1} w^q(\Delta T_k)\right]^{1/q} & \text{if } p>1, \; \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{n-1}{2n}\left[\sum\limits_{k=1}^{n-1} w(\Delta T_k)\right]^2. \end{cases} \tag{4.1}$$

Proof. We use the following scalar inequality that provides reverses of the Cauchy-Bunyakovsky-Schwarz result for n complex numbers:

$$\frac{1}{n}\sum_{j=1}^{n}|z_j|^2 - \left|\frac{1}{n}\sum_{j=1}^{n} z_j\right|^2 \leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{1\leq k\leq n-1} |\Delta z_k|^2, \\ \dfrac{n^2-1}{6n}\left[\sum\limits_{k=1}^{n-1}|\Delta z_k|^p\right]^{1/p}\left[\sum\limits_{k=1}^{n-1}|\Delta z_k|^q\right]^{1/q} & \text{if } p>1, \; \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \dfrac{n-1}{2n}\left[\sum\limits_{k=1}^{n-1}|\Delta z_k|\right]^2, \end{cases} \tag{4.2}$$

and the constants 1/12, 1/6, and 1/2 above cannot be replaced by smaller quantities for general n. For complete proofs in the general setting of real or complex inner product spaces, see [8, page 196–200].

Now, writing the inequality (4.2) for $z_j = \langle T_j h, h\rangle$, where $h \in H$, $\|h\| = 1$, $j \in \{1, \ldots, n\}$, yields

$$(0 \leq) \frac{1}{n}\sum_{j=1}^{n}|\langle T_j h, h\rangle|^2 - \left|\frac{1}{n}\left\langle\left(\sum_{j=1}^{n} T_j\right)h, h\right\rangle\right|^2$$

$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{1\leq k\leq n-1} |\langle(\Delta T_k)h, h\rangle|^2, \\ \dfrac{n^2-1}{6n}\left[\sum\limits_{k=1}^{n-1}|\langle(\Delta T_k)h, h\rangle|^p\right]^{1/p}\left[\sum\limits_{k=1}^{n-1}|\langle(\Delta T_k)h, h\rangle|^q\right]^{1/q} & \text{if } p>1, \; \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \dfrac{n-1}{2n}\left[\sum\limits_{k=1}^{n-1}|\langle(\Delta T_k)h, h\rangle|\right]^2, \end{cases} \tag{4.3}$$

for any $h \in H, \; \|h\| = 1$.

Taking the supremum over $h, \; \|h\| = 1$ in (4.3), a simple calculation reveals that (4.1) holds true and the theorem is proved. □

Remark 4.2. We observe that if $p = q = 2$ in (4.3), then we have the inequality

$$(0 \leq) \frac{1}{n} \sum_{j=1}^{n} |\langle T_j h, h\rangle|^2 - \left| \frac{1}{n} \left\langle \left(\sum_{j=1}^{n} T_j \right) h, h \right\rangle \right|^2 \leq \frac{n^2-1}{6n} \sum_{k=1}^{n-1} |\langle (\Delta T_k) h, h\rangle|^2, \tag{4.4}$$

which implies, by taking the supremum over $h \in H$, $\|h\| = 1$, that

$$(0 \leq) \frac{1}{n} w_e^2(T_1, \ldots, T_n) - w^2\left(\frac{1}{n} \sum_{j=1}^{n} T_j \right) \leq \frac{n^2-1}{6n} w_e^2(T_2 - T_1, \ldots, T_n - T_{n-1}). \tag{4.5}$$

Remark 4.3. The case $n = 2$ in all the inequalities (4.1) and (4.5) produces the simple inequality

$$w_e^2(B, C) \leq \frac{1}{2} \left[w^2(B + C) + w^2(B - C) \right], \tag{4.6}$$

for any $B, C \in B(H)$, that has been obtained in a different manner in [5] (see (2.11)).

The following result providing other upper bounds for the Euclidean operator radius holds.

Theorem 4.4. *For any n-tuple of operator $(T_1, \ldots, T_n) \in B(H)^{(n)}$, one has*

$$\begin{aligned} &(0 \leq) \frac{1}{n} w_e^2(T_1, \ldots, T_n) - w^2\left(\frac{1}{n} \sum_{j=1}^{n} T_j \right) \\ &\leq \begin{cases} \inf\limits_{T \in B(H)} \left[\max\limits_{1 \leq j \leq n} \{ w(T_j - T) \} \right] \sum\limits_{j=1}^{n} w\left(T_j - \frac{1}{n} \sum\limits_{k=1}^{n} T_k \right), \\ \inf\limits_{T \in B(H)} \left[\left(\sum\limits_{j=1}^{n} w^q(T_j - T) \right)^{1/q} \right] \left(\sum\limits_{j=1}^{n} w^p\left(T_j - \frac{1}{n} \sum\limits_{k=1}^{n} T_k \right) \right)^{1/p} \quad \text{with } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \inf\limits_{T \in B(H)} \left[\left(\sum\limits_{j=1}^{n} w(T_j - T) \right) \right] \max\limits_{1 \leq j \leq n} \left\{ w\left(T_j - \frac{1}{n} \sum\limits_{k=1}^{n} T_k \right) \right\}. \end{cases} \end{aligned} \tag{4.7}$$

Proof. Utilizing the elementary identity for complex numbers,

$$\frac{1}{n} \sum_{j=1}^{n} |z_j|^2 - \left| \frac{1}{n} \sum_{j=1}^{n} z_j \right|^2 = \frac{1}{n} \sum_{j=1}^{n} \left(z_j - \frac{1}{n} \sum_{k=1}^{n} z_k \right) (\overline{z_j} - \overline{z}), \tag{4.8}$$

which holds for any $z_1, \ldots, z_n$ and z, we can write that

$$\frac{1}{n}\sum_{j=1}^{n}\left|\langle T_j h, h\rangle\right|^2 - \left|\frac{1}{n}\left\langle\left(\sum_{j=1}^{n}T_j\right)h, h\right\rangle\right|^2 = \frac{1}{n}\sum_{j=1}^{n}\left\langle\left(T_j - \frac{1}{n}\sum_{k=1}^{n}T_k\right)h, h\right\rangle\langle h, (T_j - T)h\rangle, \tag{4.9}$$

for any $(T_1, \ldots, T_n) \in B(H)^{(n)}$, T a bounded linear operator on H and $h \in H$ with $\|h\| = 1$.

By the Hölder inequality, we also have

$$\begin{aligned}
&\left|\sum_{j=1}^{n}\left\langle\left(T_j - \frac{1}{n}\sum_{k=1}^{n}T_k\right)h, h\right\rangle\langle h, (T_j - T)h\rangle\right| \\
&\quad \le \begin{cases}
\max\limits_{1\le j\le n}\left|\langle (T_j - T)h, h\rangle\right|\sum\limits_{j=1}^{n}\left|\left\langle\left(T_j - \frac{1}{n}\sum\limits_{k=1}^{n}T_k\right)h, h\right\rangle\right|, \\
\left(\sum\limits_{j=1}^{n}\left|\left\langle\left(T_j - \frac{1}{n}\sum\limits_{k=1}^{n}T_k\right)h, h\right\rangle\right|^p\right)^{1/p}\left(\sum\limits_{j=1}^{n}\left|\langle (T_j - T)h, h\rangle\right|^q\right)^{1/q} & \text{with } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\
\max\limits_{1\le j\le n}\left|\left\langle\left(T_j - \frac{1}{n}\sum\limits_{k=1}^{n}T_k\right)h, h\right\rangle\right|\sum\limits_{j=1}^{n}\left|\langle (T_j - T)h, h\rangle\right|,
\end{cases}
\end{aligned} \tag{4.10}$$

for any $h \in H$, $\|h\| = 1$.

On utilizing (4.9) and (4.10), we thus have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left|\langle T_j h, h\rangle\right|^2 \\
&\quad \le \left|\frac{1}{n}\left\langle\left(\sum_{j=1}^{n}T_j\right)h, h\right\rangle\right|^2 \\
&\qquad + \begin{cases}
\max\limits_{1\le j\le n}\left|\langle (T_j - T)h, h\rangle\right|\sum\limits_{j=1}^{n}\left|\left\langle\left(T_j - \frac{1}{n}\sum\limits_{k=1}^{n}T_k\right)h, h\right\rangle\right|, \\
\left(\sum\limits_{j=1}^{n}\left|\left\langle\left(T_j - \frac{1}{n}\sum\limits_{k=1}^{n}T_k\right)h, h\right\rangle\right|^p\right)^{1/p}\left(\sum\limits_{j=1}^{n}\left|\langle (T_j - T)h, h\rangle\right|^q\right)^{1/q} & \text{with } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\
\max\limits_{1\le j\le n}\left|\left\langle\left(T_j - \frac{1}{n}\sum\limits_{k=1}^{n}T_k\right)h, h\right\rangle\right|\sum\limits_{j=1}^{n}\left|\langle (T_j - T)h, h\rangle\right|,
\end{cases}
\end{aligned} \tag{4.11}$$

for any $h \in H$, $\|h\| = 1$.

Taking the supremum over h, $\|h\| = 1$ in (4.11), we easily deduce the desired result (4.7). □

Remark 4.5. We observe that for $p = q = 2$ in (4.11) we can also get the inequality of interest

$$\begin{aligned}(0 \le)\frac{1}{n}w_e^2(T_1,\dots,T_n) - w^2\left(\frac{1}{n}\sum_{j=1}^{n}T_j\right) \\ \le \inf_{T\in B(H)}\left[w_e(T_1 - T,\dots,T_n - T)\right]w_e\left(T_1 - \frac{1}{n}\sum_{k=1}^{n}T_k,\dots,T_n - \frac{1}{n}\sum_{k=1}^{n}T_k\right).\end{aligned} \tag{4.12}$$

In particular, we have

$$(0 \le)\frac{1}{n}w_e^2(T_1,\dots,T_n) - w^2\left(\frac{1}{n}\sum_{j=1}^{n}T_j\right) \le w_e^2\left(T_1 - \frac{1}{n}\sum_{k=1}^{n}T_k,\dots,T_n - \frac{1}{n}\sum_{k=1}^{n}T_k\right). \tag{4.13}$$

The following particular case of Theorem 4.4 may be of interest for applications.

Corollary 4.6. *Assume that $(T_1,\dots,T_n) \in B(H)^{(n)}$ are such that there exists an operator $T \in B(H)$ and a constant $M > 0$ such that $w(T_j - T) \le M$ for each $j \in \{1,\dots,n\}$. Then*

$$\begin{aligned}(0 \le)\frac{1}{n}w_e^2(T_1,\dots,T_n) - w^2\left(\frac{1}{n}\sum_{j=1}^{n}T_j\right) \\ \le M \times \begin{cases}\displaystyle\sum_{j=1}^{n} w\left(T_j - \frac{1}{n}\sum_{k=1}^{n}T_k\right), \\ \displaystyle n^{1/q}\left[\sum_{j=1}^{n} w^p\left(T_j - \frac{1}{n}\sum_{k=1}^{n}T_k\right)\right]^{1/p} \quad \text{with } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \displaystyle n\max_{1\le j\le n}\left\{w\left(T_j - \frac{1}{n}\sum_{k=1}^{n}T_k\right)\right\}.\end{cases}\end{aligned} \tag{4.14}$$

Remark 4.7. Notice that by the Hölder inequality, the first branch in (4.14) provides a tighter bound for the nonnegative quantity $(1/n)w_e^2(T_1,\dots,T_n) - w^2((1/n)\sum_{j=1}^{n}T_j)$ than the other two.

Remark 4.8. Finally, we observe that the case $n = 2$ in (4.14) provides the simple inequality

$$\begin{aligned}w_e^2(B,C) \le \frac{1}{2}w^2(B+C) + \frac{\sqrt{2}}{2}w(B-C)\cdot\inf_{T\in B(H)} w_e(B-T,C-T) \\ \left(\le \frac{1}{2}\left[w^2(B+C) + w^2(B-C)\right]\right),\end{aligned} \tag{4.15}$$

for any $B, C \in B(H)$, which is a refinement of the inequality (4.6).

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