Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 373050, 17 pages
doi:10.1155/2008/373050

Research Article

A Class of Commutators for Multilinear Fractional Integrals in Nonhomogeneous Spaces

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Received 3 March 2008; Accepted 16 July 2008

Recommended by Nikolaos Papageorgiou

Let μ be a nondoubling measure on $\mathbb{R}^d$. A class of commutators associated with multilinear fractional integrals and RBMO(μ) functions are introduced and shown to be bounded on product of Lebesgue spaces with μ.

1. Introduction

In recent years, the study of multilinear operators and their commutator has been attracting many researchers. Many results which parallel to the linear theory of classical integral operators are obtained. For details, one can see [1–4], and so forth. Meanwhile, as a further development, harmonic analysis on $\mathbb{R}^d$ with nondoubling measures has been developed rapidly. Many results of singular integrals and the related operators on Euclidean spaces with Lebesgue measure have been generalized to the Lebesgue spaces with nondoubling measures (see [5–10], etc.). Motivated by [5, 8], we will consider the commutators generated by a class of multilinear fractional integrals and RBMO functions with nondoubling measure, which were introduced by Tolsa in [11].

Before stating our results, we recall some definitions and notations. Let μ be a Radon measure on $\mathbb{R}^d$ satisfying the following growth condition; there exist constants $C > 0$ and $n \in (0, d]$, such that

$$\mu(Q) \le Cl(Q)^n, \tag{1.1}$$

for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, where $l(Q)$ stands for the side length of Q. For $r > 0$, rQ will denote the cube with the same center as Q and with $l(rQ) = rl(Q)$.

Let $0 \le \beta < n$, given two cubes $Q \subset R$ in $\mathbb{R}^d$, we set

$$K_{Q,R}^{(\beta)} = 1 + \sum_{k=1}^{N_{Q,R}} \left[\frac{\mu(2^k Q)}{l(2^k Q)^n} \right]^{1-\beta/n}, \tag{1.2}$$

where $N_{Q,R}$ is the first integer k such that $l(2^k Q) \ge l(R)$. If $\beta = 0$, then $K_{Q,R}^{(0)} = K_{Q,R}$. The later quantity was introduced by Tolsa in [11].

Given β_d (depending on d) large enough (e.g., $\beta_d > 2^n$), we say that a cube $Q \subset \mathbb{R}^d$ is doubling if $\mu(2Q) \le \beta_d \mu(Q)$.

Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest nonnegative integer such that $2^N Q$ is doubling. We denote this cube by $\widetilde{Q}$.

Let $\eta > 1$ be a fixed constant. We say that $b \in L^1_{\text{loc}}(\mu)$ is in RBMO(μ) if there exists a constant C_1 such that for any cube Q

$$\begin{gathered} \frac{1}{\mu(\eta Q)} \int_Q \left| b(y) - m_{\widetilde{Q}} b \right| d\mu(y) \le C_1, \\ \left| m_Q b - m_R b \right| \le C_1 K_{Q,R}, \quad \text{for any two doubling cubes } Q \subset R, \end{gathered} \tag{1.3}$$

where $m_Q b = \mu(Q)^{-1} \int_Q b(y) d\mu(y)$. The minimal constant C_1 is the RBMO(μ) norm of b, and it will be denoted by $\|b\|_*$. In [11], Tolsa obtained equivalent norm in the space RBMO(μ) with different parameters $\eta > 1$ and $\beta_d > 2^n$.

We consider the following multilinear fractional integral operator

$$I_{\alpha,m}(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^d)^m} \frac{f_1(x-y_1) f_2(x-y_2) \cdots f_m(x-y_m)}{\left|(y_1, y_2, \dots, y_m)\right|^{mn-\alpha}} d\mu(y_1) \cdots d\mu(y_m). \tag{1.4}$$

For $m = 1$, we denote $I_{\alpha,1}$ by I_α, which is the Riesz potential operator related to μ.

Given $m \in \mathbb{N}$, for all $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, 2, \dots, m\}$ of j different elements. For any $\sigma \in C_j^m$, we denote $\sigma' = \{1, 2, \dots, m\} \setminus \sigma = \{\sigma'(j+1), \dots, \sigma'(m)\}$. Moreover, for $b_j \in$ RBMO(μ), $j = 1, 2, \dots, m$, let $\vec{b} = (b_1, b_2, \dots, b_m)$ and denote by $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ and by $b_\sigma(x) = b_{\sigma(1)}(x) \cdots b_{\sigma(j)}(x)$. Also, we denote $\vec{f} = (f_1, \dots, f_m)$, $\vec{f_\sigma} = (f_{\sigma(1)}, \dots, f_{\sigma(j)})$, $\vec{b_{\sigma'}} \vec{f_{\sigma'}} = (b_{\sigma'(j+1)} f_{\sigma(j+1)}, \dots, b_{\sigma'(m)} f_{\sigma'(m)})$. We define a kind of commutator of $I_{\alpha,m}$ as follows:

$$\left[\vec{b}, I_{\alpha,m}\right](\vec{f})(x) = \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} b_\sigma(x) I_{\alpha,m}\left(\vec{f_\sigma}, \vec{b_{\sigma'}} \vec{f_{\sigma'}}\right)(x). \tag{1.5}$$

In particular, for $m = 2$, we define

$$\begin{aligned} \left[b_1, b_2, I_{\alpha,2}\right](f_1, f_2)(x) = {} & b_1(x) b_2(x) I_{\alpha,2}(f_1, f_2)(x) - b_1(x) I_{\alpha,2}(f_1, b_2 f_2)(x) \\ & - b_2(x) I_{\alpha,2}(b_1 f_1, f_2)(x) + I_{\alpha,2}(b_1 f_1, b_2 f_2)(x). \end{aligned} \tag{1.6}$$

Obviously, for $m = 1$, the operator defined in (1.5) is the Coifman-Rochberg-Weiss type commutator of fractional integral, $[b, I_\alpha]$. Under the assumption that μ is a nondoubling measure, Chen and Sawyer [5] established the (L^p, L^q)-boundedness of $[b, I_\alpha]$ (also see [9] for the more general case). In this paper, we will extend the result of [5] as follows.

Theorem 1.1. *Let μ be defined as above and $\|\mu\| = \infty$, $b_j \in \mathrm{RBMO}(\mathbb{R}^d)$, $j = 1, 2$, $0 < \alpha < 2n$. Then $[b_1, b_2, I_{\alpha,2}]$ is a bounded operator from $L^{q_1} \times L^{q_2}$ to L^q with $1/q = 1/q_1 + 1/q_2 - \alpha/2n > 0$ and $1 < q_1, q_2 < \infty$.*

Remark 1.2. By Lemma 2.2 in Section 2, Theorem 1.1 for the case $\|\mu\| < \infty$ also holds provided $I_{\alpha,2}$, $[b_1, b_2, I_{\alpha,2}]$, $[b_1, I_{\alpha,2}]$, and $[b_2, I_{\alpha,2}]$ satisfy certain $T(1)$ type conditions. For instance, if $I_{\alpha,2}$ satisfies the $T(1)$ condition, that is, $I^{*1}_{\alpha,2} = 0$, then we can easily obtain $\int I_{\alpha,2}(f_1, f_2)(x)d\mu(x) = 0$ (see [3] for the notation $I^{*1}_{\alpha,2}$).

More generally, we have the following theorem.

Theorem 1.3. *Let $m \in \mathbb{N}$, μ be defined as above, and $\|\mu\| = \infty$, $b_j \in \mathrm{RBMO}(\mathbb{R}^d)$, $j = 1, 2, \ldots, m$, $0 < \alpha < mn$. Then*

$$\left\| [\vec{b}, I_{\alpha,m}](\vec{f}) \right\|_{L^q(\mu)} \leq C \prod_{j=1}^{m} \|b_j\|_* \|f_j\|_{L^{q_j}(\mu)}, \tag{1.7}$$

where $1/q = 1/q_1 + 1/q_2 + \cdots + 1/q_m - \alpha/mn > 0$ and $1 < q_j < \infty$, $j = 1, 2, \ldots, m$.

Clearly, [5, Theorem 1] is the special case of our Theorem 1.3 for $m = 1$. Throughout this paper, we always use the letter C to denote a positive constant that may vary at each occurrence but is independent of the essential variable.

2. Proofs of theorems

We only prove Theorem 1.1 since Theorem 1.3 can follow from the same arguments and an analogous version of the following Lemma 2.5, which can be deduced by induction on m. Before proving our results, we need to recall some notation and establish some lemmas which play important roles in the proofs.

Let f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we define the noncentered maximal operator

$$M^{(\beta)}_{p,(\eta)} f(x) = \sup_{Q \ni x} \left[\frac{1}{\mu(\eta Q)^{(1-\beta p/n)}} \int_Q |f(y)|^p d\mu(y) \right]^{1/p}, \tag{2.1}$$

and the sharp maximal function

$$M^{\#,(\beta)} f(x) = \sup_{Q \ni x} \frac{1}{\mu((3/2)Q)} \int_Q \left| f(y) - m_{\tilde{Q}} f \right| d\mu(y) + \sup_{\substack{R \supset Q \ni x \\ Q,R \text{ doubling}}} \frac{|m_Q f - m_R f|}{K^{(\beta)}_{Q,R}}, \tag{2.2}$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes, $m_Q(f)$ is the mean value of f on the cube Q. When $\beta = 0$, we denote $M^{(0)}_{p,(\eta)} f$ by $M_{p,(\eta)} f$ and $M^{\#,(0)} f$ by $M^{\#} f$.

We also consider the noncentered doubling maximal operator $\mathcal{N}$, defined by

$$\mathcal{N} f(x) = \sup_{\substack{Q \ni x \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y). \tag{2.3}$$

Lemma 2.1 (see [11]). *Let $1 \le p < \infty$ and $1 < \rho < \infty$. Then $b \in \mathrm{RBMO}(\mu)$, if and only if for any cube $Q \subset \mathbb{R}^d$,*

$$\frac{1}{\mu(\rho Q)} \int_Q \left| b(x) - m_{\widetilde{Q}}(b) \right|^p d\mu(x) \le C \|b\|_*^p, \tag{2.4}$$

and for any doubling cubes $Q \subset R$,

$$\left| m_Q(b) - m_R(b) \right| \le C K_{Q,R} \|b\|_*. \tag{2.5}$$

Lemma 2.2 (see [5]). *Let $f \in L^1_{\mathrm{loc}}(\mu)$ with $\int f \, d\mu = 0$ if $\|\mu\| < \infty$. For $1 < p < \infty$, if $\inf(1, \mathcal{N} f) \in L^p(\mu)$, then for $0 \le \beta < n$ we have*

$$\|\mathcal{N} f\|_{L^p(\mu)} \le C \left\| M^{\#,(\beta)} f \right\|_{L^p(\mu)}. \tag{2.6}$$

Lemma 2.3 (see [5]). *Let $p < r < n/\alpha$ and $1/q = 1/r - \alpha/n$. Then*

$$\left\| M^{(\alpha)}_{p,(\eta)} f \right\|_{L^q(\mu)} \le C \|f\|_{L^r(\mu)}, \tag{2.7}$$

where $\eta > 1$ and $0 \le \alpha < n/p$.

Lemma 2.4. *Suppose μ is a Radon measure satisfying* (1.1). *Let $m \in \mathbb{N}$ and $1/s = 1/r_1 + \cdots + 1/r_m - \alpha/n > 0$ with $0 < \alpha < mn$, $1 \le r_j \le \infty$. Then,*

(a) *if each $r_j > 1$,*

$$\left\| I_{\alpha,m}(f_1, \ldots, f_m) \right\|_{L^s(\mu)} \le C \prod_{j=1}^m \left\| f_j \right\|_{L^{r_j}(\mu)}; \tag{2.8}$$

(b) *if $r_j = 1$ for some j,*

$$\left\| I_{\alpha,m}(f_1, \ldots, f_m) \right\|_{L^{s,\infty}(\mu)} \le C \prod_{j=1}^m \left\| f_j \right\|_{L^{r_j}(\mu)}. \tag{2.9}$$

Proof. The proof follows the idea that, for the classical setting, can be found in [4]. For the sake of completeness, we will show it again.

Since $\alpha > 0$, some $r_i < \infty$. If say, $r_{l+1} = \cdots = r_m = \infty$, $1 \le l < m$, because $\alpha/n < 1/r_1 + \cdots + 1/r_l \le l$, so that $mn - \alpha > (m-l)n$, integration in $y_{l+1}, \ldots, y_m$ reduces matters to the case when all r_i are finite (and $m = l$). Thus, we may assume that all $r_i < \infty$. Now, observe that if $0 < c_i$, $i = 1, \ldots, m$, and $0 < \alpha < \sum_{i=1}^m c_i$, we can find $0 < \alpha_i < c_i$ such that $\alpha = \sum_{i=1}^m \alpha_i$. Apply this observation to $c_i = n/r_i$, and $1/s_i = 1/r_i - \alpha_i/n$. Since $\sum_{i=1}^m 1/s_i = 1/s$, $0 < \alpha_i/n \le 1$, $1 < s_i < \infty$, and

$$\left| y_1 \right|^{n-\alpha_1} \left| y_2 \right|^{n-\alpha_2} \cdots \left| y_m \right|^{n-\alpha_m} \le \left| (y_1, \ldots, y_m) \right|^{nm-\alpha}, \tag{2.10}$$

where $\alpha = \sum_{i=1}^m \alpha_i$. It follows that

$$I_{\alpha,m}(f_1, \ldots, f_m)(x) \le \prod_{i=1}^m I_{\alpha_i}(f_i)(x). \tag{2.11}$$

Then, by [5, Lemma 1, page 1289] (or see [6, page 1269]) and Hölder's inequality (see [12, page 15] for weak spaces when some $r_i = 1$), we can get Lemma 2.4. □

Lemma 2.5. *Let* $[b_1, b_2, I_{\alpha,2}]$ *be as in* (1.6), $0 < \alpha < 2n$, $\tau > 1$, $b_1, b_2 \in \mathrm{RBMO}(\mu)$. *Then there exists a constant* $C > 0$ *such that for all* $f_1 \in L^{q_1}(\mu)$, $f_2 \in L^{q_2}(\mu)$, *and* $x \in \mathbb{R}^d$,

$$\begin{aligned} M^{\#,(\alpha)}([b_1, b_2, I_{\alpha,2}](f_1, f_2))(x) \le C\big[&\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}(I_{\alpha,2}(f_1, f_2))(x) \\ &+ \|b_1\|_* M_{\tau,(3/2)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_2\|_* M_{\tau,(3/2)}([b_1, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_1\|_*\|b_2\|_* M^{(\alpha)}_{p_1,(9/8)} f_1(x) M^{(\alpha)}_{p_2,(9/8)} f_2(x)\big], \end{aligned} \tag{2.12}$$

$$\begin{aligned} M^{\#,(\alpha)}([b_1, I_{\alpha,2}](f_1, f_2))(x) \le C\|b_1\|_*\big[&M_{\tau,(3/2)}(I_{\alpha,2}(f_1, f_2))(x) \\ &+ M^{(\alpha)}_{p_1,(9/8)}(f_1)(x) M^{(\alpha)}_{p_2,(9/8)}(f_2)(x)\big], \end{aligned} \tag{2.13}$$

$$\begin{aligned} M^{\#,(\alpha)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \le C\|b_2\|_*\big[&M_{\tau,(3/2)}(I_{\alpha,2}(f_1, f_2))(x) \\ &+ M^{(\alpha)}_{p_1,(9/8)}(f_1)(x) M^{(\alpha)}_{p_2,(9/8)}(f_2)(x)\big], \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} [b_1, I_{\alpha,2}](f_1, f_2)(x) &= b_1(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(b_1 f_1, f_2)(x), \\ [b_2, I_{\alpha,2}](f_1, f_2)(x) &= b_2(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(f_1, b_2 f_2)(x). \end{aligned} \tag{2.15}$$

Proof. By the definition, to obtain (2.12), it suffices to prove that for any $x \in \mathbb{R}^d$ and a cube $Q \ni x$,

$$\begin{aligned} \frac{1}{\mu((3/2)Q)} \int_Q \big| [b_1, b_2, I_{\alpha,2}](f_1, f_2)(z) - h_Q \big| d\mu(z) \le C\big[&\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}(I_{\alpha,2}(f_1, f_2))(x) \\ &+ \|b_1\|_* M_{\tau,(3/2)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_2\|_* M_{\tau,(3/2)}([b_1, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_1\|_*\|b_2\|_* M^{(\alpha)}_{p_1,(9/8)} f_1(x) M^{(\alpha)}_{p_2,(9/8)} f_2(x)\big], \end{aligned} \tag{2.16}$$

and for any cubes $Q \subset R$, where Q is an arbitrary cube and R is doubling,

$$\begin{aligned} |h_Q - h_R| \le C K^2_{Q,R} K^{(\alpha)}_{Q,R}\big[&\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}(I_{\alpha,2}(f_1, f_2))(x) \\ &+ \|b_1\|_* M_{\tau,(3/2)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_2\|_* M_{\tau,(3/2)}([b_1, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_1\|_*\|b_2\|_* M^{(\alpha)}_{p_1,(9/8)} f_1(x) M^{(\alpha)}_{p_2,(9/8)} f_2(x)\big], \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} h_Q &= m_Q\big(I_{\alpha,2}\big((m_{\widetilde{Q}}(b_1) - b_1) f_1 \chi_{\mathbb{R}^d \setminus (4/3)Q}, (m_{\widetilde{Q}}(b_2) - b_2) f_2 \chi_{\mathbb{R}^d \setminus (4/3)Q}\big)\big), \\ h_R &= m_R\big(I_{\alpha,2}\big((m_{\widetilde{R}}(b_1) - b_1) f_1 \chi_{\mathbb{R}^d \setminus (4/3)R}, (m_{\widetilde{R}}(b_2) - b_2) f_2 \chi_{\mathbb{R}^d \setminus (4/3)R}\big)\big). \end{aligned} \tag{2.18}$$

First of all, it is easy to see that

$$\begin{aligned}\left|[b_1,b_2,I_{\alpha,2}](f_1,f_2)(z)-h_Q\right| \le{}& \left|(b_1(z)-m_{\tilde{Q}}(b_1))(b_2(z)-m_{\tilde{Q}}(b_2))I_{\alpha,2}(f_1,f_2)(z)\right| \\ &+\left|(b_1(z)-m_{\tilde{Q}}(b_1))I_{\alpha,2}(f_1,(b_2(z)-b_2)f_2)(z)\right| \\ &+\left|(b_2(z)-m_{\tilde{Q}}(b_2))I_{\alpha,2}(b_1(z)-b_2)f_1,f_2)(z)\right| \\ &+\left|I_{\alpha,2}((b_1-m_{\tilde{Q}}(b)_1)f_1,(b_1-m_{\tilde{Q}}(b)_1)f_2)(z)-h_Q\right| \\ :={}& \mathrm{I}(z)+\mathrm{II}(z)+\mathrm{III}(z)+\mathrm{IV}(z).\end{aligned} \tag{2.19}$$

Consequently,

$$\frac{1}{\mu((3/2)Q)}\int_Q\left|[b_1,b_2,I_{\alpha,2}](f_1,f_2)(z)-h_Q\right|d\mu(z)\le C[\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}], \tag{2.20}$$

where $\mathrm{I}=\mu((3/2)Q)^{-1}\int_Q \mathrm{I}(z)d\mu(z)$, and II, III, IV are defined in the same way.

In what follows, we estimate I–IV, respectively. For I, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}\mathrm{I}={}&\frac{1}{\mu((3/2)Q)}\int_Q \mathrm{I}(z)d\mu(z) \\ \le{}& C\left(\frac{1}{\mu((3/2)Q)}\int_Q\left|b_1(z)-m_{\tilde{Q}}(b_1)\right|^{\tau_1}d\mu(z)\right)^{1/\tau_1} \\ &\times\left(\frac{1}{\mu((3/2)Q)}\int_Q\left|b_2(z)-m_{\tilde{Q}}(b_2)\right|^{\tau_2}d\mu(z)\right)^{1/\tau_2} \\ &\times\left(\frac{1}{\mu((3/2)Q)}\int_Q\left|I_{\alpha,2}(f_1,f_2)\right|^{\tau}d\mu(z)\right)^{1/\tau} \\ \le{}& C\|b_1\|_*\|b_2\|_*M_{\tau,(3/2)}(I_{\alpha,2}(f_1,f_2))(x),\end{aligned} \tag{2.21}$$

where $\tau_1>1$, $\tau_2>1$ and $1/\tau+1/\tau_1+1/\tau_2=1$.

For II, we have

$$\begin{aligned}\mathrm{II}={}&\frac{1}{\mu((3/2)Q)}\int_Q \mathrm{II}(z)d\mu(z) \\ \le{}& C\left(\frac{1}{\mu((3/2)Q)}\int_Q\left|b_1(z)-m_{\tilde{Q}}(b_1)\right|^{s}d\mu(z)\right)^{1/s} \\ &\times\left(\frac{1}{\mu((3/2)Q)}\int_Q\left|[b_2,I_{\alpha,2}](f_1,f_2)(z)\right|^{\tau}d\mu(z)\right)^{1/\tau} \\ \le{}& C\|b_1\|_*M_{\tau,(3/2)}([b_2,I_{\alpha,2}](f_1,f_2))(x),\end{aligned} \tag{2.22}$$

where $s>1$ and $1/s+1/\tau=1$.

Similarly, we have

$$\mathrm{III} \le C\|b_2\|_* M_{\tau,(3/2)}\left(\left[b_2, I_{\alpha,2}\right](f_1, f_2)\right)(x). \tag{2.23}$$

It remains to estimate IV. For convenience, we set $f_j^0 = f_j\chi_{4/3Q}$, $f_j = f_j^0 + f_j^\infty$, $j = 1, 2$. Then,

$$\begin{aligned}
|\mathrm{IV}(z)| &\le \left|I_{\alpha,2}\left(\left(b_1 - m_{\tilde{Q}}(b_1)\right)f_1^0, \left(b_2 - m_{\tilde{Q}}(b_2)\right)f_2^0\right)(z)\right| \\
&\quad + \left|I_{\alpha,2}\left(\left(b_1 - m_{\tilde{Q}}(b_1)\right)f_1^0, \left(b_2 - m_{\tilde{Q}}(b_2)\right)f_2^\infty\right)(z)\right| \\
&\quad + \left|I_{\alpha,2}\left(\left(b_1 - m_{\tilde{Q}}(b_1)\right)f_1^\infty, \left(b_2 - m_{\tilde{Q}}(b_2)\right)f_2^0\right)(z)\right| \\
&\quad + \left|I_{\alpha,2}\left(\left(b_1 - m_{\tilde{Q}}(b_1)\right)f_1^\infty, \left(b_2 - m_{\tilde{Q}}(b_2)\right)f_2^\infty\right)(z) - h_Q\right| \\
&= \mathrm{IV}_1(z) + \mathrm{IV}_2(z) + \mathrm{IV}_3(z) + \mathrm{IV}_4(z),
\end{aligned} \tag{2.24}$$

and so we have

$$\frac{1}{\mu((3/2)Q)}\int_Q |\mathrm{IV}(z)|d\mu(z) \le \sum_{j=1}^4 \frac{1}{\mu((3/2)Q)}\int_Q \mathrm{IV}_j(z)d\mu(z) := \sum_{j=1}^4 \mathrm{IV}_j. \tag{2.25}$$

To estimate IV_1, set $s_1 = \sqrt{p_1}$, $s_2 = \sqrt{p_2}$, and $1/v = 1/s_1 + 1/s_2 - \alpha/n$. It follows from Hölder's inequality and Lemma 2.4 that

$$\begin{aligned}
\mathrm{IV}_1 &\le \frac{\mu(Q)^{1-1/v}}{\mu((3/2)Q)}\left\|I_{\alpha,2}\left(\left(b_1 - m_{\tilde{Q}}(b_1)\right)f_1^0, \left(b_2 - m_{\tilde{Q}}(b_2)\right)f_2^0\right)\right\|_{L^v(\mu)} \\
&\le C\mu((3/2)Q)^{-1/v}\left\|\left(b_1 - m_{\tilde{Q}}(b_1)\right)f_1^0\right\|_{L^{s_1}(\mu)}\left\|\left(b_2 - m_{\tilde{Q}}(b_2)\right)f_2^0\right\|_{L^{s_2}(\mu)} \\
&\le \frac{C}{\mu((3/2)Q)^{1/v}}\left(\int_{(4/3)Q}|f_1(y_1)|^{p_1}d\mu(y_1)\right)^{1/p_1} \\
&\quad \times \left(\int_{(4/3)Q}\left|b_1(y_1) - m_{\tilde{Q}}(b_1)\right|^{p_1/(\sqrt{p_1}-1)}d\mu(y_1)\right)^{(\sqrt{p_1}-1)/p_1} \\
&\quad \times \left(\int_{(4/3)Q}|f_2(y_2)|^{p_2}d\mu(y_2)\right)^{1/p_2}\left(\int_{(4/3)Q}\left|b_2(y_1) - m_{\tilde{Q}}(b_i)\right|^{p_2/(\sqrt{p_2}-1)}d\mu(y_i)\right)^{(\sqrt{p_2}-1)/p_2} \\
&\le C\prod_{i=1}^2\left(\frac{1}{\mu((3/2)Q)^{1-\alpha p_i/2n}}\int_{(4/3)Q}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{1/p_i} \\
&\quad \times \left(\frac{1}{\mu((3/2)Q)}\int_{(4/3)Q}\left|b_i(y_i) - m_{\tilde{Q}}(b_i)\right|^{p_i/(\sqrt{p_i}-1)}d\mu(y_i)\right)^{(\sqrt{p_i}-1)/p_i} \\
&\le C\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\end{aligned} \tag{2.26}$$

For term IV_2, by Lemma 2.1, we have

$$
\begin{aligned}
\mathrm{IV}_2 &= \frac{1}{\mu((3/2)Q)}\int_Q \mathrm{IV}_2(z)\,d\mu(z) \\
&\le C\frac{1}{\mu((3/2)Q)}\int_Q\int_{\mathbb{R}^d\setminus(4/3)Q}\int_{(4/3)Q} \\
&\quad\times \frac{\left|(b_1(y_1)-m_{\widetilde{Q}}(b_1))f_1^0(y_1)\right|\left|(b_2(y_2)-m_{\widetilde{Q}}(b_2))f_2^\infty(y_2)\right|}{\left|(z-y_1,z-y_2)\right|^{2n-\alpha}}d\mu(y_1)\,d\mu(y_2)\,d\mu(z) \\
&\le \frac{C}{\mu((3/2)Q)}\int_Q\int_{(4/3)Q}\left|(b_1(y)-m_{\widetilde{Q}}(b_1))f_1^0(y_1)\right|d\mu(y_1) \\
&\quad\times\int_{\mathbb{R}^d\setminus(4/3)Q}\frac{\left|(b_2(y_2)-m_{\widetilde{Q}}(b_2))f_2^\infty(y_2)\right|}{\left|z-y_2\right|^{2n-\alpha}}d\mu(y_2)\,d\mu(z) \\
&\le C\left(\frac{1}{\mu((3/2)Q)^{1-\alpha p_1/2n}}\int_{(4/3)Q}\left|f_1(y_1)\right|^{p_1}d\mu(y_1)\right)^{1/p_1} \\
&\quad\times\left(\frac{1}{\mu((3/2)Q)}\int_{(4/3)Q}\left|b_1-m_{\widetilde{Q}}(b_1)\right|^{p_1'}d\mu(y_1)\right)^{1/p_1'} \\
&\quad\times\mu\left(\frac{3}{2}Q\right)^{-\alpha/2n}\mu(Q)\sum_{k=1}^{\infty}\int_{2^k(4/3)Q\setminus2^{k-1}(4/3)Q}\frac{\left|(b_2(y_2)-m_{\widetilde{Q}}(b_2))f_2(y_2)\right|}{2^{k(2n-\alpha)}l(Q)^{2n-\alpha}}d\mu(y_2) \\
&\le C\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)\sum_{k=1}^{\infty}2^{-k(n-\alpha/2)}l\left(2^k\frac{3}{2}Q\right)^{-n+\alpha/2} \\
&\quad\times\int_{2^k(4/3)Q}\left|(b_2(y_2)-m_{\widetilde{Q}}(b_2))f_2(y_2)\right|d\mu(y_2) \\
&\le C\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)\sum_{k=1}^{\infty}2^{-k(n-\alpha/2)}l\left(2^k\frac{3}{2}Q\right)^{-n+\alpha/2} \\
&\quad\times\left[\int_{2^k(4/3)Q}\left|(b_2(y_2)-m_{2^k\widetilde{(4/3)Q}}(b_2))f_2(y_2)\right|d\mu(y_2)\right. \\
&\qquad\left.+\left|m_{2^k\widetilde{(4/3)Q}}(b_2)-m_{\widetilde{Q}}(b_2)\right|\int_{2^k(3/2)Q}\left|f_2(y_2)\right|d\mu(y_2)\right] \\
&\le C\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x) \\
&\quad\times\left[\sum_{k=1}^{\infty}2^{-k(n-\alpha/2)}\left(\frac{1}{l(2^k(3/2)Q)^n}\int_{2^k(4/3)Q}\left|(b_2(y_2)-m_{2^k\widetilde{(4/3)Q}}(b_2))\right|^{p_2'}d\mu(y_2)\right)^{1/p_2'}\right. \\
&\qquad\times\left(\frac{1}{l(2^k(3/2)Q)^{n-\alpha p_2/2}}\int_{2^k(4/3)Q}\left|f_2(y_2)\right|^{p_2}d\mu(y_2)\right)^{1/p_2} \\
&\qquad\left.+\sum_{k=1}^{\infty}k2^{-k(n-\alpha/2)}\|b_2\|_*\frac{1}{l(2^k(3/2)Q)^{n-\alpha/2}}\int_{2^k(4/3)Q}\left|f_2(y_2)\right|d\mu(y_2)\right] \\
&\le C\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x),
\end{aligned}
\tag{2.27}
$$

where the last inequality follows from the following two facts:

$$\begin{aligned}
&\frac{1}{l(2^k(3/2)Q)^{n-\alpha/2}}\int_{2^k(4/3)Q}|f_2(y_2)|d\mu(y_2)\\
&\quad\le \frac{\mu(2^k(4/3)Q)^{1-1/p_2}}{l(2^k(3/2)Q)^{n-\alpha/2}}\left(\int_{2^k(4/3)Q}|f_2(y_2)|^{p_2}d\mu(y_2)\right)^{1/p_2}\\
&\quad\le C\frac{\mu(2^{k+1}(4/3)Q)^{1-1/p_2+1/p_2-(\alpha/2n)}}{l(2^{k+1}(4/3)Q)^{n-\alpha/2}}\left(\frac{1}{\mu(2^{k+1}(4/3)Q)^{1-(\alpha p_2/2n)}}\int_{2^{k+1}(4/3)Q}|f_2(y_2)|^{p_2}d\mu(y_2)\right)^{1/p_2}\\
&\quad\le CM^{(\alpha)}_{p_2,9/8}f_2(x),
\end{aligned} \tag{2.28}$$

and (see[11])

$$\left|m_{\widetilde{2^k(4/3)Q}}(b_j)-m_{\tilde{Q}}(b_j)\right|\le C\|b_j\|_*K_{\tilde{Q},\widetilde{2^k(4/3)Q}}\le C\|b_j\|_*K_{Q,2^k(4/3)Q}\le Ck\|b_j\|_*,\quad j=1,2. \tag{2.29}$$

Similarly,

$$\mathrm{IV}_3\le C\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x). \tag{2.30}$$

For term IV_4, we have

$$\begin{aligned}
&\left|I_{\alpha,2}\left((b_1-m_{\tilde{Q}}(b_1))f_1^\infty,(b_2-m_{\tilde{Q}}(b_2))f_2^\infty\right)(z)-I_{\alpha,2}\left((b_1-m_{\tilde{Q}}(b_1))f_1^\infty,(b_2-m_{\tilde{Q}}(b_2))f_2^\infty\right)(y)\right|\\
&\quad\le \int_{\mathbb{R}^d\setminus(4/3)Q}\int_{\mathbb{R}^d\setminus(4/3)Q}\left|\frac{1}{|(z-y_1,z-y_2)|^{2n-\alpha}}-\frac{1}{|(y-y_1,y-y_2)|^{2n-\alpha}}\right|\\
&\qquad\times\left|\prod_{i=1}^{2}(b_i(y_i)-m_{\tilde{Q}}(b_i))f_i^\infty(y_i)\right|d\mu(y_1)d\mu(y_2)\\
&\quad\le \int_{\mathbb{R}^d\setminus(4/3)Q}\int_{\mathbb{R}^d\setminus(4/3)Q}\frac{|z-y|}{|(y-y_1,y-y_2)|^{2n-\alpha+1}}\\
&\qquad\times\left|\prod_{i=1}^{2}(b_i(y_i)-m_{\tilde{Q}}(b_i))f_i^\infty(y_i)\right|d\mu(y_1)d\mu(y_2)\\
&\quad\le C\prod_{i=1}^{2}\int_{\mathbb{R}^d\setminus(4/3)Q}\frac{|z-y|^{1/2}}{|y-y_i|^{n-\alpha/2+1/2}}\left|(b_i(y_i)-m_{\tilde{Q}}(b_i))f_i^\infty(y_i)\right|d\mu(y_i)\\
&\quad\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}\int_{2^k(4/3)Q\setminus2^{k-1}(4/3)Q}2^{-k/2}\frac{1}{l(2^kQ)^{n-\alpha/2}}\left|(b_i(y_i)-m_{\tilde{Q}}(b_i))\right||f_i^\infty(y_i)|d\mu(y_i)\\
&\quad\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{l(2^k(3/2)Q)^n}\int_{2^k(4/3)Q}\left|(b_i(y_i)-m_{\tilde{Q}}(b_i))\right|^{p_i'}d\mu(y_i)\right)^{1/p_i'}\\
&\qquad\times\left(\frac{1}{l(2^k(3/2)Q)^{n-(\alpha p_i/2)}}\int_{2^k(4/3)Q}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{1/p_i}
\end{aligned}$$

$$
\begin{aligned}
&\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}2^{-k/2}M^{(\alpha)}_{p_i,(9/8)}f_i(x)\\
&\quad\times\left(\frac{1}{l(2^k(3/2)Q)^n}\int_{2^k(4/3)Q}\left|(b_i(y_i)-m_{\widetilde{2^k(4/3)Q}}(b_i)+m_{\widetilde{2^k(4/3)Q}}(b_i)-m_{\tilde{Q}}(b_i))\right|^{p_i'}d\mu(y_i)\right)^{1/p_i'}\\
&\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}2^{-k/2}k\|b_i\|_*M^{(\alpha)}_{p_i,(9/8)}f_i(x)\\
&\le C\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\end{aligned}
\tag{2.31}
$$

Taking the mean over $y\in Q$, we obtain

$$
\left|I_{\alpha,2}((b_1-m_{\tilde{Q}}(b1))f_1^{\infty},(b_1-m_{\tilde{Q}}(b_1))f_2^{\infty})(z)-h_Q\right|\le C\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\tag{2.32}
$$

Thus,

$$
\mathrm{IV}_4=\frac{1}{\mu((3/2)Q)}\int_Q\mathrm{IV}_4(z)d\mu(z)\le C\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\tag{2.33}
$$

Combing (2.20)–(2.33), we obtain (2.16). □

Now we turn to estimate (2.17). For any cubes, $Q\subset R$ with $x\in Q$, where Q is arbitrary and R is doubling. We denote $N_{Q,R}+1$ simply by N, write

$$
\begin{aligned}
|h_Q-h_R|&=\left|m_Q\left[I_{\alpha,2}((b_1-m_{\tilde{Q}}b_1)f_1^{\infty},(b_2-m_{\tilde{Q}}b_2)f_2^{\infty})\right]\right.\\
&\qquad\left.-m_R\left[I_{\alpha,2}((b_1-m_Rb_1)f_1^{\infty},(b_2-m_Rb_2)f_2^{\infty})\right]\right|\\
&\le\left|m_R\left[I_{\alpha,2}((b_1-m_{\tilde{Q}}b_1)f_1\chi_{\mathbb{R}^d\setminus 2^NQ},(b_2-m_{\tilde{Q}}b_2)f_2\chi_{\mathbb{R}^d\setminus 2^NQ})\right]\right.\\
&\qquad\left.-m_Q\left[I_{\alpha,2}((b_1-m_{\tilde{Q}}b_1)f_1\chi_{\mathbb{R}^d\setminus 2^NQ},((b_2-m_{\tilde{Q}}b_2)f_2\chi_{\mathbb{R}^d\setminus 2^NQ})\right]\right|\\
&\quad+\left|m_R\left[I_{\alpha,2}((b_1-m_Rb_1)f_1\chi_{\mathbb{R}^d\setminus 2^NQ},(b_2-m_Rb_2)f_2\chi_{\mathbb{R}^d\setminus 2^NQ})\right]\right.\\
&\qquad\left.-m_R\left[I_{\alpha,2}((b_1-m_{\tilde{Q}}b_1)f_1\chi_{\mathbb{R}^d\setminus 2^NQ},(b_2-m_{\tilde{Q}}b_2)f_2\chi_{\mathbb{R}^d\setminus 2^NQ})\right]\right|\\
&\quad+\left|m_Q\left[I_{\alpha,2}((b_1-m_{\tilde{Q}}b_1)f_1\chi_{2^NQ\setminus(4/3)Q},(b_2-m_{\tilde{Q}}b_2)f_2\chi_{\mathbb{R}^d\setminus(4/3)Q})\right]\right|\\
&\quad+\left|m_Q\left[I_{\alpha,2}((b_1-m_{\tilde{Q}}b_1)f_1\chi_{\mathbb{R}^d\setminus 2^NQ},(b_2-m_{\tilde{Q}}b_2)f_2\chi_{2^NQ\setminus(4/3)Q})\right]\right|\\
&\quad+\left|m_R\left[I_{\alpha,2}((b_1-m_Rb_1)f_1\chi_{\mathbb{R}^d\setminus(4/3)R},(b_2-m_Rb_2)f_2\chi_{2^NQ\setminus(4/3)R})\right]\right|\\
&\quad+\left|m_R\left[I_{\alpha,2}((b_1-m_Rb_1)f_1\chi_{2^NQ\setminus(4/3)R},(b_2-m_Rb_2)f_2\chi_{\mathbb{R}^d\setminus 2^NQ})\right]\right|\\
&=\sum_{i=1}^{6}A_i.
\end{aligned}
\tag{2.34}
$$

By the similar arguments used in proving (2.33), we obtain that

$$
A_1\le C\left[K_{Q,R}\right]^2\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\tag{2.35}
$$

To estimate A_2, we write

$$\begin{aligned}
&I_{\alpha,2}\left(\left(b_1 - m_R(b_1)\right)f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, \left(b_2 - m_R(b_2)\right)f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z) \\
&- I_{\alpha,2}\left(\left(b_1 - m_{\widetilde{Q}}(b_1)\right)f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, \left(b_2 - m_{\widetilde{Q}}(b_2)\right)f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z) \\
&\quad= \left(m_R(b_2) - m_{\widetilde{Q}}(b_2)\right)I_{\alpha,2}\left(\left(b_1 - m_R(b_1)\right)f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z) \\
&\qquad+ \left(m_R(b_1) - m_{\widetilde{Q}}(b_1)\right)I_{\alpha,2}\left(f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, \left(b_2 - m_R(b_2)\right)f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z) \\
&\qquad+ \left(m_R(b_1) - m_{\widetilde{Q}}(b_1)\right)\left(m_R(b_2) - m_{\widetilde{Q}}(b_2)\right)I_{\alpha,2}\left(f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z).
\end{aligned} \tag{2.36}$$

Then,

$$\begin{aligned}
A_2 &\le \left|m_R(b_2) - m_{\widetilde{Q}}(b_2)\right| \left|\frac{1}{\mu(R)}\int_R I_{\alpha,2}\left(\left(b_1 - m_R(b_1)\right)f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z)\,d\mu(z)\right| \\
&\quad+ \left|m_R(b_1) - m_{\widetilde{Q}}(b_1)\right| \left|\frac{1}{\mu(R)}\int_R I_{\alpha,2}\left(f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, \left(b_2 - m_R(b_2)\right)f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z)\,d\mu(z)\right| \\
&\quad+ \left|m_R(b_1) - m_{\widetilde{Q}}(b_1)\right|\left|m_R(b_2) - m_{\widetilde{Q}}(b_2)\right| \left|\frac{1}{\mu(R)}\int_R I_{\alpha,2}\left(f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z)\,d\mu(z)\right| \\
&= A_{21} + A_{22} + A_{23}.
\end{aligned} \tag{2.37}$$

It is obvious that

$$A_{23} \le CK_{Q,R}^2\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1, f_2)\right)(x). \tag{2.38}$$

In order to estimate term A_{21}, we write

$$\begin{aligned}
&I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{\mathbb{R}^d\setminus 2^N Q}, f_2\chi_{\mathbb{R}^d\setminus 2^N Q}\right)(z) \\
&\quad= I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1, f_2\right)(z) - I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{2^N Q}\chi_{(4/3)R}, f_2\chi_{(4/3)R}\right)(z) \\
&\qquad- I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{(4/3)R}, f_2\chi_{2^N Q}\chi_{(4/3)R}\right)(z) \\
&\qquad+ I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{2^N Q}\chi_{(4/3)R}, f_2\chi_{2^N Q}\chi_{(4/3)R}\right)(z) \\
&\qquad- I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{\mathbb{R}^d\setminus(4/3)R}, f_2\chi_{2^N Q}\right)(z) \\
&\qquad- I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{2^N Q}, f_2\chi_{\mathbb{R}^d\setminus(4/3)R}\right)(z) \\
&\qquad+ I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1\chi_{2^N Q\setminus(4/3)R}, f_2\chi_{2^N Q\setminus(4/3)R}\right)(z) \\
&\quad= \sum_{j=1}^{7} B_j(z).
\end{aligned} \tag{2.39}$$

For $B_1(z)$, we write

$$\left|I_{\alpha,2}\left(\left(b_1 - m_R b_1\right)f_1, f_2\right)(z)\right| \le \left|I_{\alpha,2}\left(\left(b_1 - b_1(z)\right)f_1, f_2\right)(z)\right| + \left|I_{\alpha,2}\left(\left(b_1(z) - m_R b_1\right)f_1, f_2\right)(z)\right|. \tag{2.40}$$

By Hölder's inequality and the fact that R is doubling, we have

$$\begin{aligned}
&\frac{1}{\mu(R)}\int_R I_{\alpha,2}\big((b_1 - m_R b_1)f_1, f_2\big)(z)\,d\mu(z) \le C\|b_1\|_* M_{\tau,(3/2)}\big(I_{\alpha,2}(f_1,f_2)\big)(x),\\
&\frac{1}{\mu(R)}\int_R I_{\alpha,2}\big((b_1 - b_1(z))f_1, f_2\big)(z)\,d\mu(z) \le C M_{\tau,(3/2)}\big([b_1, I_{\alpha,2}](f_1,f_2)\big)(x),
\end{aligned} \tag{2.41}$$

which imply

$$\left|m_R B_1\right| \le C\big(\|b_1\|_* M_{\tau,(3/2)}\big(I_{\alpha,2}(f_1,f_2)\big)(x) + M_{\tau,(3/2)}\big([b_1, I_{\alpha,2}](f_1,f_2)\big)(x)\big). \tag{2.42}$$

For $B_2(z)$, set $s_1 = \sqrt{p_1}$, $s_2 = p_2$, and $1/v = 1/s_1 + 1/s_2 - \alpha/n$. Using Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned}
\frac{1}{\mu(R)}\int_R B_2(z)\,d\mu(z) &\le C\frac{\mu(R)^{1-1/v}}{\mu(R)}\left\|I_{\alpha,2}\big((b_1 - m_R(b_1))f_1\chi_{2^N Q}\chi_{(4/3)R}, f_2\chi_{(4/3)R}\big)\right\|_{L^v(\mu)}\\
&\le C\mu\Big(\frac{3}{2}R\Big)^{-1/v}\left\|(b_1 - m_R(b_1))f_1\chi_{2^N Q}\chi_{(4/3)R}\right\|_{L^{s_1}(\mu)}\left\|f_2\chi_{(4/3)R}\right\|_{L^{s_2}(\mu)}\\
&\le C\mu\Big(\frac{3}{2}R\Big)^{-1/v}\left(\int_{(4/3)R}|f_2(y)|^{p_2}d\mu(y)\right)^{1/p_2}\left(\int_{(4/3)R}|f_1(x)|^{p_1}d\mu(y)\right)^{1/p_1}\\
&\quad\times\left(\int_{(4/3)R}|b_1(y) - m_R(b_1)|^{p_1/(\sqrt{p_1}-1)}d\mu(y)\right)^{(\sqrt{p_1}-1)/p_1}\\
&\le C\left(\frac{1}{\mu((3/2)R)^{1-\alpha p_1/2n}}\int_{(4/3)R}|f_1(y)|^{p_1}d\mu(y)\right)^{1/p_1}\\
&\quad\times\left(\frac{1}{\mu((3/2)R)}\int_{(4/3)R}|b_1(y) - m_R(b_1)|^{p_1/(\sqrt{p_1}-1)}d\mu(y)\right)^{(\sqrt{p_1}-1)/p_1}\\
&\quad\times\left(\frac{1}{\mu((3/2)R)^{1-\alpha p_2/2n}}\int_{(4/3)Q}|f_2(y)|^{p_2}d\mu(y)\right)^{1/p_2}\\
&\le C\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x),
\end{aligned} \tag{2.43}$$

which implies

$$\left|m_R B_2\right| \le C\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x). \tag{2.44}$$

Similarly,

$$\begin{aligned}
\left|m_R B_3\right| &\le C\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x),\\
\left|m_R B_4\right| &\le C\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\end{aligned} \tag{2.45}$$

For $B_5(z)$, since $z \in \mathbb{R}$, we have

$$
\begin{aligned}
\left|B_5(z)\right| &\leq \int_{2^N Q}\int_{\mathbb{R}^d\setminus(4/3)Q} \frac{\left|(b_1 - m_R(b_1))f_1\right|\left|f_2(y_2)\right|}{\left|(z-y_1, z-y_2)\right|^{2n-\alpha}} d\mu(y_1)d\mu(y_2) \\
&\leq \int_{2^N Q}\left|f_2(y_2)\right|d\mu(y_2)\int_{\mathbb{R}^d\setminus(4/3)Q} \frac{\left|(b_1(y_1) - m_R(b_1))f_1\right|}{\left|z-y_1\right|^{2n-\alpha}} d\mu(y_1) \\
&\leq \frac{C}{l(2R)^{n-\alpha/2}}\int_{2^N Q}\left|f_2(y_2)\right|d\mu(y_2) \\
&\quad \times \sum_{k=1}^{\infty}\frac{2^{-kn}}{l(2^k(4/3))^{n-\alpha/2}}\int_{2^k(4/3)R\setminus 2^{k-1}(4/3)R}\left|(b_1(y_1) - m_R(b_1))f_1\right|d\mu(y_1) \\
&\leq C\frac{1}{l(2R)^{n-\alpha/2}}\int_{2^N Q}\left|f_2(y_2)\right|d\mu(y_2) \\
&\quad \times \sum_{k=1}^{\infty}\frac{2^{-kn}}{l(2^k(4/3)R)^{n-\alpha/2}} \\
&\quad \times \left[\int_{2^k(4/3)R}\left|(b_1(y_1) - m_{2^k(4/3)R}(b_1))f_1\right|d\mu(y_1)\right. \\
&\qquad \left. + \int_{2^k(4/3)R}\left|m_{2^k(4/3)R}(b_1) - m_R(b_1)(y_1)\right|\left|f_1(y_1)\right|d\mu(y_1)\right] \\
&\leq C\frac{1}{l(2R)^{n-\alpha/2}}\int_{2^N Q}\left|f_2(y_2)\right|d\mu(y_2) \\
&\quad \times \sum_{k=1}^{\infty}2^{-kn}\left[\frac{1}{l(2^k(4/3)R)^{n}}\int_{2^k(4/3)R}\left|b_1(y_1) - m_{2^k(4/3)R}(b_1)\right|^{p_1'}d\mu(y_1)^{1/p_1'}\right. \\
&\qquad\qquad \times \frac{1}{l(2^k(4/3)R)^{n-\alpha p_1/2}}\int_{2^k(4/3)R}\left|f_1\right|^{p_1}d\mu(y_1)^{1/p_1} \\
&\qquad\qquad \left. + C\|b_1\|_*\frac{1}{l(2^k(4/3)R)^{n-\alpha/2}}\int_{2^k(4/3)R}\left|f_1\right|d\mu(y_1)\right] \\
&\leq C\frac{1}{l(2R)^{n-\alpha/2}}\int_{2^N Q}\left|f_2(y_2)\right|d\mu(y_2)\sum_{k=1}^{\infty}2^{-kn}\|b_1\|_*\left(M^{(\alpha)}_{p_1,(9/8)}f_1(x) + M_{p_1,(9/8)}f_1(x)\right) \\
&\leq C\sum_{k=1}^{N}\frac{1}{l(2R)^{n-\alpha/2}}\int_{2^k Q\setminus 2^{k-1}Q}\left|f_2(y_2)\right|d\mu(y_2)\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x) \\
&\quad + \frac{1}{l(2R)^{n-\alpha/2}}\int_{Q}\left|f_2(y_2)\right|d\mu(y_2)\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x) \\
&\leq C\sum_{k=0}^{N}\frac{1}{l(2R)^{n-\alpha/2}}\int_{2^k Q}\left|f_2(y_2)\right|d\mu(y_2)\|b_1\|_* M^{(\alpha)}_{p_1,(9/8)}f_1(x)
\end{aligned}
$$

$$\begin{aligned}
&\leq C\sum_{k=0}^{N}\frac{\mu(2^{k+1}Q)^{1-(\alpha/2n)}}{l(2^{k+1}Q)^{n-(\alpha/2)}}\frac{l(2^{k+1}Q)^{n-(\alpha/2)}}{l(2R)^{n-(\alpha/2)}}\frac{1}{\mu(2^{k+1}Q)^{1-(\alpha/2n)}}\\
&\quad\times\int_{2^kQ}|f_2(y_2)|d\mu(y_2)\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)\\
&\leq C\sum_{k=0}^{N}\frac{\mu(2^{k+1}Q)^{1-(\alpha/2n)}}{l(2^{k+1}Q)^{n-(\alpha/2)}}\frac{1}{\mu((9/8)2^kQ)^{1-(\alpha/2n)}}\int_{2^kQ}|f_2(y_2)|d\mu(y_2)\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)\\
&\leq CK^{(\alpha)}_{Q,R}\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\end{aligned}\tag{2.46}$$

Taking the mean on z over R, we obtain

$$|m_RB_5|\leq CK^{(\alpha)}_{Q,R}\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).\tag{2.47}$$

Similarly, we have

$$\begin{aligned}
|m_RB_6|&\leq CK^{(\alpha)}_{Q,R}\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x),\\
|m_RB_7|&\leq CK^{(\alpha)}_{Q,R}\|b_1\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).
\end{aligned}\tag{2.48}$$

Summing up the estimates (2.39)–(2.48), we obtain

$$\begin{aligned}
A_{21}\leq CK_{Q,R}K^{(\alpha)}_{Q,R}\big[&\|b_1\|_*\|b_2\|_*M_{\tau,(3/2)}(I_{\alpha,2}(f_1,f_2))(x)\\
&+\|b_2\|_*M_{\tau,(3/2)}([b_1,I_{\alpha,2}](f_1,f_2))(x)\\
&+\|b_1\|_*\|b_2\|_*M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].
\end{aligned}\tag{2.49}$$

By the same arguments, we can get

$$\begin{aligned}
A_{22}\leq CK_{Q,R}K^{(\alpha)}_{Q,R}\big[&\|b_1\|_*\|b_2\|_*M_{\tau,(3/2)}(I_{\alpha,2}(f_1,f_2))(x)\\
&+\|b_1\|_*M_{\tau,(3/2)}([b_2,I_{\alpha,2}](f_1,f_2))(x)\\
&+\|b_1\|_*\|b_2\|_*M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].
\end{aligned}\tag{2.50}$$

Consequently,

$$\begin{aligned}
A_{2}\leq CK^2_{Q,R}K^{(\alpha)}_{Q,R}\big[&\|b_1\|_*\|b_2\|_*M_{\tau,(3/2)}(I_{\alpha,2}(f_1,f_2))(x)\\
&+\|b_1\|_*M_{\tau,(3/2)}([b_2,I_{\alpha,2}](f_1,f_2))(x)\\
&+\|b_2\|_*M_{\tau,(3/2)}([b_1,I_{\alpha,2}](f_1,f_2))(x)\\
&+\|b_1\|_*\|b_2\|_*M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].
\end{aligned}\tag{2.51}$$

Using the similar arguments to those used in proving $B_5(z)$, we can conclude that

$$A_3+A_4+A_5+A_6\leq CK^2_{Q,R}K^{(\alpha)}_{Q,R}\|b_1\|_*\|b_2\|_*M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2(x).\tag{2.52}$$

Therefore, we obtain

$$\begin{aligned}\left|h_Q-h_R\right| \le C K_{Q,R}^2 K_{Q,R}^{(\alpha)}\big[&\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)(x)\\&+\|b_1\|_* M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)(x)\\&+\|b_2\|_* M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)(x)\\&+\|b_1\|_*\|b_2\|_* M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big],\end{aligned} \tag{2.53}$$

which implies (2.17).

Finally, we show how to derive (2.12) from (2.16) and (2.17). From (2.16), if Q is doubling and $x \in Q$, we have

$$\begin{aligned}\left|m_Q\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right)-h_Q\right| &\le \frac{1}{\mu(Q)}\int_Q\left|[b_1,b_2,I_{\alpha,2}](f_1,f_2)(z)-h_Q\right|d\mu(z)\\&\le C\big[\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)(x)\\&\quad+\|b_1\|_* M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)(x)\\&\quad+\|b_2\|_* M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)(x)\\&\quad+\|b_1\|_*\|b_2\|_* M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].\end{aligned} \tag{2.54}$$

Also, for any cube Q with $x \in Q$, $K_{Q,\tilde{Q}} \le C$ and $K_{Q,\tilde{Q}}^{(\alpha)} \le C$, by (2.16), (2.17), and (2.54), we have

$$\begin{aligned}&\frac{1}{\mu((3/2)Q)}\int_Q\left|[b_1,b_2,I_{\alpha,2}](f_1,f_2)(z)-m_{\tilde{Q}}[b_1,b_2,I_{\alpha,2}](f_1,f_2)\right|d\mu(z)\\&\quad\le \frac{1}{\mu((3/2)Q)}\int\left|[b_1,b_2,I_{\alpha,2}](f_1,f_2)(z)-h_Q\right|d\mu(z)\\&\qquad+\frac{1}{\mu((3/2)Q)}\int_Q\left|m_{\tilde{Q}}\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right)-h_{\tilde{Q}}\right|d\mu(z)\\&\qquad+\frac{1}{\mu((3/2)Q)}\int_Q\left|h_Q-h_{\tilde{Q}}\right|d\mu(z)\\&\quad\le C\big[\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)(x)\\&\qquad+\|b_1\|_* M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)(x)\\&\qquad+\|b_2\|_* M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)(x)\\&\qquad+\|b_1\|_*\|b_2\|_* M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].\end{aligned} \tag{2.55}$$

On the other hand, for all doubling cubes $Q \subset R$ with $x \in Q$, such that $K_{Q,R}^{(\alpha)} \le P'_\alpha$, where P'_α is the constant in [5, Lemma 6], by (2.17), we have

$$\begin{aligned}\left|h_Q-h_R\right| \le C K_{Q,R}^2 P'_\alpha\big[&\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)(x)\\&+\|b_1\|_* M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)(x)\\&+\|b_2\|_* M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)(x)\\&+\|b_1\|_*\|b_2\|_* M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].\end{aligned} \tag{2.56}$$

Hence, by [5, Lemma 6], we get

$$\begin{aligned}\left|h_Q - h_R\right| \le CK_{Q,R}^{(\alpha)}\big[&\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)(x)\\&+\|b_1\|_* M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)(x)\\&+\|b_2\|_* M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)(x)\\&+\|b_1\|_*\|b_2\|_* M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big],\end{aligned} \tag{2.57}$$

for all doubling cubes $Q \subset R$ with $x \in Q$. Invoking (2.55) yields that

$$\begin{aligned}&\left|m_Q\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right) - m_R\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right)\right|\\&\quad\le \left|m_Q\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right) - h_Q\right| + \left|h_R - m_R\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right)\right| + \left|h_Q - h_R\right|\\&\quad\le CK_{Q,R}^{(\alpha)}\big[\|b_1\|_*\|b_2\|_* M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)(x)\\&\qquad+\|b_1\|_* M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)(x)\\&\qquad+\|b_2\|_* M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)(x)\\&\qquad+\|b_1\|_*\|b_2\|_* M_{p_1,(9/8)}f_1(x)M_{p_2,(9/8)}f_2(x)\big].\end{aligned} \tag{2.58}$$

From this estimate and the definition of the sharp maximal function, we complete the proof of (2.12). Similarly, we can deduce (2.13) and (2.14). The details are omitted.

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. By the Lebesgue differentiation theorem, it is easy to see that for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$,

$$\left|f(x)\right| \le \mathcal{N}f(x), \tag{2.59}$$

for μ − a.e. $x \in \mathbb{R}^d$, see [11] for details. By Lemmas 2.2–2.5, we have

$$\begin{aligned}\left\|[b_1,b_2,I_{\alpha,2}](f_1,f_2)\right\|_{L^q} &\le \left\|\mathcal{N}\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right)\right\|_{L^q}\\&\le \left\|M^{\#,(\beta)}\left([b_1,b_2,I_{\alpha,2}](f_1,f_2)\right)\right\|_{L^q}\\&\le C\big[\|b_1\|_*\|b_2\|_*\left\|M_{\tau,(3/2)}\left(I_{\alpha,2}(f_1,f_2)\right)\right\|_{L^q}\\&\qquad+\|b_1\|_*\left\|M_{\tau,(3/2)}\left([b_2,I_{\alpha,2}](f_1,f_2)\right)\right\|_{L^q}\\&\qquad+\|b_2\|_*\left\|M_{\tau,(3/2)}\left([b_1,I_{\alpha,2}](f_1,f_2)\right)\right\|_{L^q}\\&\qquad+\|b_1\|_*\|b_2\|_*\left\|M^{(\alpha)}_{p_1,(9/8)}f_1(x)M^{(\alpha)}_{p_2,(9/8)}f_2\right\|_{L^q}\big]\\&\le C\big[\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}}\|\|f_2\|_{L^{q_2}}\\&\qquad+\|b_1\|_*\left\|[b_2,I_{\alpha,2}](f_1,f_2)\right\|_{L^q}\\&\qquad+\|b_2\|_*\left\|[b_1,I_{\alpha,2}](f_1,f_2)\right\|_{L^q}\big]\end{aligned}$$

$$
\begin{aligned}
&\le C\big(\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}}\|\|f_2\|_{L^{q_2}} \\
&\qquad + \|b_1\|_*\|M^{\#,(\beta)}([b_2, I_{\alpha,2}](f_1, f_2))\|_{L^{p_1}} \\
&\qquad + \|b_2\|_*\|M^{\#,(\beta)}([b_1, I_{\alpha,2}](f_1, f_2))\|_{L^{p_2}}\big) \\
&\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}}\|\|f_2\|_{L^{q_2}}.
\end{aligned}
\tag{2.60}
$$

This proves Theorem 1.1. □

Acknowledgments

The authors would like to express their deep thanks to the referees for their valuable remarks and suggestions. Huoxiong Wu was partially supported by the NSF of China (G10571122).

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