

## Research Article

# Recurring Mean Inequality of Random Variables

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A multidimensional recurring mean inequality is shown. Furthermore, we prove some new inequalities, which can be considered to be the extensions of those established inequalities, including, for example, the Polya-Szegö and Kantorovich inequalities.

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## 1. Introduction

The theory of means and their inequalities is fundamental to many fields including mathematics, statistics, physics, and economics. This is certainly true in the area of probability and statistics. There are large amounts of work available in the literature. For example, some useful results have been given by Shaked and Tong [1], Shaked and Shanthikumar [2], Shaked et al. [3], and Tong [4, 5]. Motivated by different concerns, there are numerous ways to introduce mean values. In probability and statistics, the most commonly used mean is expectation. In [6], the author proves the mean inequality of two random variables. The purpose of the present paper is to establish a recurring mean inequality, which generalizes the mean inequality of two random variables to  $n$  random variables. This result can, in turn, be extended to establish other new inequalities, which include generalizations of the Polya-Szegö and Kantorovich inequalities [7].

We begin by introducing some preliminary concepts and known results which can also be found in [6].

*Definition 1.1.* The supremum and infimum of the random variable  $\xi$  are defined as  $\inf_x \{x : P(\xi \leq x) = 1\}$  and  $\sup_x \{x : P(\xi \geq x) = 1\}$ , respectively, and denoted by  $\sup \xi$  and  $\inf \xi$ .

*Definition 1.2.* If  $\xi$  is bounded, the arithmetic mean of the random variable  $\xi$ ,  $A(\xi)$ , is given by

$$A(\xi) = \frac{\sup \xi + \inf \xi}{2}. \quad (1.1)$$

In addition, if  $\inf \xi \geq 0$ , one defines the geometric mean of the random variable  $\xi$ ,  $G(\xi)$ , to be

$$G(\xi) = \sqrt{\sup \xi \cdot \inf \xi}. \quad (1.2)$$

*Definition 1.3.* If  $\xi_1, \dots, \xi_n$  are bounded random variables, the independent arithmetic mean of the product of random variables  $\xi_1, \dots, \xi_n$ ,  $\bar{A}(\xi_1, \dots, \xi_n)$  is given by

$$\bar{A}(\xi_1, \dots, \xi_n) = \frac{\prod_{i=1}^n \sup \xi_i + \prod_{i=1}^n \inf \xi_i}{2}. \quad (1.3)$$

*Definition 1.4.* If  $\xi_1, \dots, \xi_n$  are bounded random variables with  $\inf \xi_i \geq 0$ ,  $i = 1, \dots, n$ , one defines the independent geometric mean of the product of random variables  $\xi_1, \dots, \xi_n$  to be

$$\bar{G}(\xi_1, \dots, \xi_n) = \sqrt{\prod_{i=1}^n \sup \xi_i \inf \xi_i}. \quad (1.4)$$

*Remark 1.5.* If  $\xi_1, \dots, \xi_n$  are independent, then

$$\begin{aligned} \bar{A}(\xi_1, \dots, \xi_n) &= A\left(\prod_{i=1}^n \xi_i\right), \\ \bar{G}(\xi_1, \dots, \xi_n) &= G\left(\prod_{i=1}^n \xi_i\right). \end{aligned} \quad (1.5)$$

The mean inequality of two random variables [6].

**Theorem 1.6.** Let  $\xi$  and  $\eta$  be bounded random variables. If  $\inf \xi > 0$  and  $\inf \eta > 0$ , then

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} \leq \frac{\bar{A}^2(\xi, \eta)}{\bar{G}^2(\xi, \eta)}. \quad (1.6)$$

Equality holds if and only if

$$\begin{aligned} P\left\{\left(\frac{\xi}{\eta} = \frac{a}{b}\right) \cup \left(\frac{\xi}{\eta} = \frac{A}{b}\right)\right\} &= 1, \\ G(\eta^2)E\xi^2 &= G(\xi^2)E\eta^2 \end{aligned} \quad (1.7)$$

for  $A = \sup \xi$ ,  $B = \sup \eta$ ,  $a = \inf \xi$ ,  $b = \inf \eta$ .

## 2. Main results

Our main results are given by the following theorem.

**Theorem 2.1.** *Suppose that  $\xi_1, \dots, \xi_n, \xi_{n+1}$  are bounded random variables,  $\inf \xi_i > 0$ ,  $i = 1, \dots, n+1$ . Let  $\{U(n)\}$  be a sequence of real numbers. If*

$$\frac{\prod_{i=1}^n E\xi_i^2}{E^2(\prod_{i=1}^n \xi_i)} \leq U(n), \quad (2.1)$$

then

$$\frac{\prod_{i=1}^{n+1} E\xi_i^2}{E^2(\prod_{i=1}^{n+1} \xi_i)} \leq \frac{\bar{A}^2(\xi_1, \dots, \xi_{n+1})}{\bar{G}^2(\xi_1, \dots, \xi_{n+1})} U(n). \quad (2.2)$$

*Proof.* Let  $A_i = \sup \xi_i$ ,  $a_i = \inf \xi_i$ ,  $i = 1, \dots, n+1$ . We have

$$P\{(\xi_1 \cdots \xi_n A_{n+1} - a_1 \cdots a_n \xi_{n+1})(A_1 \cdots A_n \xi_{n+1} - \xi_1 \cdots \xi_n a_{n+1}) \geq 0\} = 1. \quad (2.3)$$

So

$$P\{(A_1 \cdots A_{n+1} + a_1 \cdots a_{n+1})\xi_1 \cdots \xi_{n+1} \geq A_1 a_1 \cdots A_n a_n \xi_{n+1}^2 + A_{n+1} a_{n+1} \xi_1^2 \cdots \xi_n^2\} = 1, \quad (2.4)$$

which implies that

$$(A_1 \cdots A_{n+1} + a_1 \cdots a_{n+1})E(\xi_1 \cdots \xi_{n+1}) \geq A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) + A_{n+1} a_{n+1} E(\xi_1^2 \cdots \xi_n^2). \quad (2.5)$$

Using the Jensen inequality [7] and assumption (2.1), we get

$$\begin{aligned} (A_1 \cdots A_{n+1} + a_1 \cdots a_{n+1})E(\xi_1 \cdots \xi_{n+1}) &\geq A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) + A_{n+1} a_{n+1} E^2(\xi_1 \cdots \xi_n) \\ &\geq A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) + A_{n+1} a_{n+1} \frac{E\xi_1^2 \cdots E\xi_n^2}{U(n)} \\ &\geq 2 \left[ A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) A_{n+1} a_{n+1} \frac{E\xi_1^2 \cdots E\xi_n^2}{U(n)} \right]^{1/2}. \end{aligned} \quad (2.6)$$

Hence,

$$\left[ \frac{\bar{G}^2(\xi_1, \dots, \xi_{n+1}) E\xi_1^2 \cdots E\xi_{n+1}^2}{U(n)} \right]^{1/2} \leq \bar{A}(\xi_1, \dots, \xi_{n+1}) E(\xi_1 \cdots \xi_{n+1}), \quad (2.7)$$

from which (2.2) follows.  $\square$

Combining this result with Theorem 1.6, the following recurring inequalities are immediate.

**Corollary 2.2.** *Let  $\xi_1, \dots, \xi_n$  be bounded random variables. If  $\inf \xi_i > 0$ ,  $i = 1, \dots, n$ , then*

$$\begin{aligned} \frac{E\xi_1^2 E\xi_2^2}{E^2(\xi_1 \xi_2)} &\leq \frac{\bar{A}^2(\xi_1, \xi_2)}{\bar{G}^2(\xi_1, \xi_2)}, \\ \frac{E\xi_1^2 E\xi_2^2 E\xi_3^2}{E^2(\xi_1 \xi_2 \xi_3)} &\leq \frac{\bar{A}^2(\xi_1, \xi_2, \xi_3)}{\bar{G}^2(\xi_1, \xi_2, \xi_3)} \frac{\bar{A}^2(\xi_1, \xi_2)}{\bar{G}^2(\xi_1, \xi_2)}, \\ &\vdots \\ \frac{\prod_{k=1}^n E\xi_k^2}{E^2(\prod_{k=1}^n \xi_k)} &\leq \prod_{k=2}^n \frac{\bar{A}^2(\xi_1, \dots, \xi_k)}{\bar{G}^2(\xi_1, \dots, \xi_k)}. \end{aligned} \quad (2.8)$$

### 3. Some applications

In this section, we exhibit some of the applications of the inequalities just obtained. We make use of the following known lemma which we state here without proof.

**Lemma 3.1.** *If  $0 < m_2 \leq m_1 \leq M_1 \leq M_2$ , then*

$$\frac{(1/2)(m_1 + M_1)}{\sqrt{m_1 M_1}} \leq \frac{(1/2)(m_2 + M_2)}{\sqrt{m_2 M_2}}. \quad (3.1)$$

**Theorem 3.2** (the extensions of the inequality of *Polya-Szegö*). *Let  $a_{ij} > 0$ ,  $a_i = \min_j a_{ij}$ ,  $A_i = \max_j a_{ij}$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . Then,*

$$\prod_{i=1}^n \sum_{j=1}^m a_{ij}^2 \leq \frac{m^{n-2}}{4^{n-1}} \prod_{k=2}^n \frac{(a_1 \cdots a_k + A_1 \cdots A_k)^2}{a_1 \cdots a_k A_1 \cdots A_k} \left( \sum_{j=1}^m \prod_{i=1}^n a_{ij} \right)^2. \quad (3.2)$$

*Proof.* This result is a consequence of inequality (2.8). Let  $\xi_1$  have the distribution

$$P(\xi_1 = a_{1j}) = \frac{1}{m}, \quad j = 1, \dots, m. \quad (3.3)$$

We define  $n - 1$  functions as follows:

$$f_i(a_{1j}) = a_{ij}, \quad i = 2, \dots, n, \quad j = 1, \dots, m. \quad (3.4)$$

Let  $\xi_i = f_i(\xi_1)$ ,  $i = 2, \dots, n$ . Then,

$$E\xi_i^2 = \frac{1}{m} \sum_{j=1}^m a_{ij}^2, \quad i = 1, \dots, n, \\ E(\xi_1 \cdots \xi_n) = \frac{1}{m} \sum_{j=1}^m \prod_{i=1}^n a_{ij}, \quad (3.5)$$

$$\bar{A}(\xi_1, \dots, \xi_k) = \frac{1}{2}(a_1 \cdots a_k + A_1 \cdots A_k), \quad \bar{G}(\xi_1, \dots, \xi_k) = \sqrt{a_1 \cdots a_k A_1 \cdots A_k}.$$

Inequality (2.8) then becomes

$$\frac{\prod_{i=1}^n (1/m) \sum_{j=1}^m a_{ij}^2}{((1/m) \sum_{j=1}^m \prod_{i=1}^n a_{ij})^2} \leq \prod_{k=2}^n \frac{[(1/2)(a_1 \cdots a_k + A_1 \cdots A_k)]^2}{[\sqrt{a_1 \cdots a_k A_1 \cdots A_k}]^2}, \quad (3.6)$$

from which our result follows. □

*Remark 3.3.* For  $n = 2$ , we can get the inequality of *Polya-Szegö* [7]:

$$\left( \sum_{k=1}^m a_k^2 \right) \left( \sum_{k=1}^m b_k^2 \right) \leq \frac{1}{4} \left( \sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}} \right)^2 \left( \sum_{k=1}^m a_k b_k \right)^2, \quad (3.7)$$

where  $a_k, b_k > 0$ ,  $k = 1, \dots, m$ ,  $a = \min a_k$ ,  $A = \max a_k$ ,  $b = \min b_k$ , and  $B = \max b_k$ .

**Theorem 3.4** (the extensions of Kantorovich's inequality). Let  $A$  be an  $m \times m$  positive Hermitian matrix. Denote by  $\lambda_1$  and  $\lambda_m$  the maximum and minimum eigenvalues of  $A$ , respectively. For real  $\beta_1, \dots, \beta_n$  and  $\beta = \beta_1 + \dots + \beta_n$ , and any vector  $0 \neq x \in \mathbb{R}^m$ , the following inequality is satisfied:

$$\frac{\prod_{i=1}^n x^* A^{\beta_i} x}{(x^* A^{\beta/2} x)^2} \leq \frac{(x^* x)^{n-2}}{4^{n-1}} \prod_{k=2}^n \frac{[l_1 \cdots l_k + L_1 \cdots L_k]^2}{l_1 \cdots l_k L_1 \cdots L_k}, \quad (3.8)$$

where

$$l_i = \begin{cases} \lambda_m^{\beta_i/2}, & \beta_i \geq 0, \\ \lambda_1^{\beta_i/2}, & \beta_i < 0, \end{cases} \quad L_i = \begin{cases} \lambda_1^{\beta_i/2}, & \beta_i \geq 0, \\ \lambda_m^{\beta_i/2}, & \beta_i < 0, \end{cases} \quad i = 1, \dots, n. \quad (3.9)$$

*Proof.* Let  $\lambda_1 \geq \dots \geq \lambda_m$  be eigenvalues of  $A$  and let  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ . There is a Hermitian matrix  $U$  that satisfies

$$A = U^* \Lambda U. \quad (3.10)$$

Let

$$y = Ux = (y_1, y_2, \dots, y_m)^T, \quad p_i = \frac{|y_i|^2}{\sum_{i=1}^m |y_i|^2}, \quad i = 1, \dots, m. \quad (3.11)$$

Then,

$$\begin{aligned} \frac{\prod_{i=1}^n x^* A^{\beta_i} x}{(x^* A^{\beta/2} x)^2} &= \frac{\prod_{i=1}^n x^* U^* \Lambda^{\beta_i} U x}{(x^* U^* \Lambda^{\beta/2} U x)^2} \\ &= \frac{\prod_{i=1}^n y^* \Lambda^{\beta_i} y}{(y^* \Lambda^{\beta/2} y)^2} \\ &= \frac{(y^* y)^{n-2} \prod_{i=1}^n \sum_{k=1}^m \lambda_k^{\beta_i} p_k}{(\sum_{k=1}^m \lambda_k^{\beta/2} p_k)^2} \\ &= \frac{(x^* x)^{n-2} \prod_{i=1}^n \sum_{k=1}^m \lambda_k^{\beta_i} p_k}{(\sum_{k=1}^m \lambda_k^{\beta/2} p_k)^2}. \end{aligned} \quad (3.12)$$

What remains to show is that

$$\frac{\prod_{i=1}^n \sum_{k=1}^m \lambda_k^{\beta_i} p_k}{(\sum_{k=1}^m \lambda_k^{\beta/2} p_k)^2} \leq \frac{1}{4^{n-1}} \prod_{k=2}^n \frac{[l_1 \cdots l_k + L_1 \cdots L_k]^2}{l_1 \cdots l_k L_1 \cdots L_k}, \quad \forall p_i \geq 0, \sum_{i=1}^m p_i = 1. \quad (3.13)$$

We define the random variable  $\zeta$ , and assign  $P(\zeta = \lambda_i) = p_i$ ,  $i = 1, \dots, m$ . Suppose  $\xi_i = \zeta^{\beta_i/2}$ ,  $i = 1, \dots, n$ . Notice that  $\lambda_1$  and  $\lambda_m$  are the upper and lower bounds of the random variable  $\zeta$ , so  $l_i$  and  $L_i$  are the lower and upper bounds of  $\xi_i$ . According to Lemma 3.1, we know that

$$\frac{\overline{A}(\xi_1, \dots, \xi_n)}{\overline{G}(\xi_1, \dots, \xi_n)} \leq \frac{[(1/2)(l_1 \cdots l_n + L_1 \cdots L_n)]^2}{[\sqrt{l_1 \cdots l_n L_1 \cdots L_n}]^2}. \quad (3.14)$$

Noticing that

$$E(\xi_1 \cdots \xi_n) = E\zeta^{\beta/2} = \sum_{k=1}^m \lambda_k^{\beta/2} p_k, \quad (3.15)$$

we can use inequality (2.8) to express inequality (3.13) as

$$\frac{E\xi_1^2 \cdots E\xi_n^2}{E^2(\xi_1 \cdots \xi_n)} \leq \prod_{k=2}^n \frac{[(1/2)(l_1 \cdots l_k + L_1 \cdots L_k)]^2}{[\sqrt{l_1 \cdots l_k L_1 \cdots L_k}]^2}. \quad (3.16)$$

□

*Remark 3.5.* If  $n = 2$ ,  $\beta_1 = 1$ , and  $\beta_2 = -1$ , this inequality takes the form

$$\frac{x^* A x x^* A^{-1} x}{(x^* x)^2} \leq \frac{(\lambda_1 + \lambda_m)^2}{4\lambda_1 \lambda_m} \quad (3.17)$$

which is *Kantorovich's* inequality [7].

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