Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 185089, 6 pages
doi:10.1155/2008/185089

Research Article

Jensen's Inequality for Convex-Concave Antisymmetric Functions and Applications

S. Hussain,[1] J. Pečarić,[1, 2] and I. Perić[3]

[1] *Abdus Salam School of Mathematical Sciences, GC University Lahore, Gulberge, Lahore 54660, Pakistan*
[2] *Faculty of Textile Technology, University of Zagreb, 10000 Zagreb, Croatia*
[3] *Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*

Correspondence should be addressed to S. Hussain, sabirhus@gmail.com

Received 21 February 2008; Accepted 9 September 2008

Recommended by Lars-Erik Persson

The weighted Jensen inequality for convex-concave antisymmetric functions is proved and some applications are given.

1. Introduction

The famous Jensen inequality states that

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n}p_i f(x_i), \tag{1.1}$$

where $f : I \to \mathbb{R}$ is a convex function, I is interval in $\mathbb{R}$, $x_i \in I$, $p_i > 0$, $i = 1, \dots, n$, and $P_n = \sum_{i=1}^{n} p_i$. Recall that a function $f : I \to \mathbb{R}$ is convex if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \tag{1.2}$$

holds for every $x, y \in I$ and every $t \in [0,1]$ (see [1, Chapter 2]).

The natural problem in this context is to deduce Jensen-type inequality weakening some of the above assumptions. The classical case is the case of Jensen-convex (or mid-convex) functions. A function $f : I \to \mathbb{R}$ is Jensen-convex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.3}$$

holds for every $x, y \in I$. It is clear that every convex function is Jensen-convex. To see that the class of convex functions is a proper subclass of Jensen-convex functions, see [2, page 96]. Jensen's inequality for Jensen-convex functions states that if $f : I \to \mathbb{R}$ is a Jensen-convex function, then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i), \tag{1.4}$$

where $x_i \in I,\ i = 1, \dots, n$. For the proof, see [2, page 71] or [1, page 53].

A class of functions which is between the class of convex functions and the class of Jensen-convex functions is the class of Wright-convex functions. A function $f : I \to \mathbb{R}$ is Wright-convex if

$$f(x+h) - f(x) \le f(y+h) - f(y) \tag{1.5}$$

holds for every $x \le y,\ h \ge 0$, where $x, y + h \in I$ (see [1, page 7]).

The following theorem was the main motivation for this paper (see [3] and [1, pages 55-56]).

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be Wright-convex on $[a, (a+b)/2]$ and $f(x) = -f(a+b-x)$. If $x_i \in [a,b]$ and $(x_i + x_{n-i+1})/2 \in [a, (a+b)/2]$ for $i = 1, 2, \dots, n$, then* (1.4) *is valid.*

Another way of weakening the assumptions for (1.1) is relaxing the assumption of positivity of weights $p_i,\ i = 1, \dots, n$. The most important result in this direction is the Jensen-Steffensen inequality (see, e.g., [1, page 57]) which states that (1.1) holds also if $x_1 \le x_2 \le \cdots \le x_n$ and $0 \le P_k \le P_n,\ P_n > 0$, where $P_k = \sum_{i=1}^{k} p_i$.

The main purpose of this paper is to prove the weighted version of Theorem 1.1. For some related results, see [4, 5]. In Section 3, to illustrate the applicability of this result, we give a generalization of the famous Ky-Fan inequality.

2. Main results

Theorem 2.1. *Let $f : (a,b) \to \mathbb{R}$ be a convex function on $(a, (a+b)/2]$ and $f(x) = -f(a+b-x)$ for every $x \in (a,b)$. If $x_i \in (a,b),\ p_i > 0,\ (x_i + x_{n-i+1})/2 \in (a, (a+b)/2]$, and $(p_i x_i + p_{n-i+1} x_{n-i+1})/(p_i + p_{n-i+1}) \in (a, (a+b)/2]$ for $i = 1, 2, \dots, n$, then* (1.1) *holds.*

Proof. Without loss of generality, we can suppose that $(a,b) = (-1,1)$. So, f is an odd function. First we consider the case $n = 2$. If $x_1, x_2 \in (-1, 0]$, then we have the known case of Jensen inequality for convex functions. Thus, we will assume that $x_1 \in (-1, 0)$ and $x_2 \in (0,1)$. The equation of the straight line through points $(x_1, f(x_1)),\ (0,0)$ is

$$y = \frac{f(x_1)}{x_1} x. \tag{2.1}$$

Since f is convex on $(-1, 0]$ and $x_1 < (p_1 x_1 + p_2 x_2)/(p_1 + p_2) \le 0$, it follows that

$$f\left(\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}\right) \le \frac{f(x_1)}{x_1} \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}. \tag{2.2}$$

It is enough to prove that

$$\frac{f(x_1)}{x_1}\frac{p_1x_1+p_2x_2}{p_1+p_2} \le \frac{p_1f(x_1)+p_2f(x_2)}{p_1+p_2} \tag{2.3}$$

which is obviously equivalent to the inequality

$$\frac{f(x_1)}{x_1} \le \frac{f(x_2)}{x_2} = \frac{f(-x_2)}{-x_2}. \tag{2.4}$$

Since the function f is convex on $(-1,0]$ and $f(0)=0$, by Galvani's theorem it follows that the function $x \mapsto (f(x)-f(0))/(x-0) = f(x)/x$ is increasing on $(-1,0)$. Therefore, from $(x_1+x_2)/2 \le 0$ and $x_2 > 0$ we have $x_1 \le -x_2 < 0$; so (2.4) holds.

Now, for an arbitrary $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{i=1}^{n} p_i f(x_i) &= \frac{1}{2}\sum_{i=1}^{n}\left[p_i f(x_i) + p_{n-i+1} f(p_{n-i+1})\right] \\
&\ge \frac{1}{2}\sum_{i=1}^{n}(p_i+p_{n-i+1}) f\left(\frac{p_ix_i+p_{n-i+1}x_{n-i+1}}{p_i+p_{n-i+1}}\right) \\
&= P_n \cdot \frac{1}{2P_n}\sum_{i=1}^{n}(p_i+p_{n-i+1}) f\left(\frac{p_ix_i+p_{n-i+1}x_{n-i+1}}{p_i+p_{n-i+1}}\right) \\
&\ge P_n f\left(\frac{1}{2P_n}\sum_{i=1}^{n}(p_i+p_{n-i+1})\frac{p_ix_i+p_{n-i+1}x_{n-i+1}}{p_i+p_{n-i+1}}\right) \\
&= P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right);
\end{aligned} \tag{2.5}$$

so the proof is complete. □

Remark 2.2. In fact, we have proved that

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) &\ge \frac{1}{2P_n}\sum_{i=1}^{n}(p_i+p_{n-i+1}) f\left(\frac{p_ix_i+p_{n-i+1}x_{n-i+1}}{p_i+p_{n-i+1}}\right) \\
&\ge f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right).
\end{aligned} \tag{2.6}$$

Remark 2.3. Neither condition $(x_i+x_{n-i+1})/2 \in (a,(a+b)/2]$, $i=1,\dots,n$, nor condition $(p_ix_i+p_{n-i+1})/(p_i+p_{n-i+1}) \in (a,(a+b)/2]$, $i=1,\dots,n$, can be removed from the assumptions of Theorem 2.1. To see this, consider the function $f(x)=-x^3$ on $(-2,2)$. That the first condition cannot be removed can be seen by considering $x_1=-1/2$, $x_2=1$, $p_1=7/8$, and $p_2=1/8$. That the second condition cannot be removed can be seen by considering $x_1=-1$, $x_2=3/4$, $p_1=1/8$, and $p_2=7/8$. In both cases, (1.1) does not hold.

Remark 2.4. Using Jensen and Jensen-Steffensen inequalities, it is easy to prove the following inequalities (see also [6, 7]):

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)&\le f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\\&\le f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i),\end{aligned}\tag{2.7}$$

where f is a convex function on $(a-\varepsilon,b+\varepsilon)$, $\varepsilon>0$, $x_i\in(a,b)$, and $p_i>0$ for $i=1,\dots,n$. If f is concave, the reverse inequalities hold in (2.7).

Now, suppose the conditions in Theorem 2.1 are fulfilled except that the function f satisfies $f(x)+f(a+b-x)=2f((a+b)/2)$. It is immediate (consider the function $g(x)=f(x)-f((a+b)/2)$) that inequality (1.1) still holds. Using $f(x)=2f((a+b)/2)-f(a+b-x)$, the inequality (1.1) gives

$$2f\left(\frac{a+b}{2}\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)\le f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right);\tag{2.8}$$

so the left-hand side of inequality (2.7) is valid also in this case. On the other hand, if $f((a+b)/2)=0$ (so $f(a)+f(b)=0$), the previous inequality can be written as

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\ge f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\tag{2.9}$$

which is the reverse of the right-hand side inequality of (2.7); so the concavity properties of the function f are prevailing in this case.

3. Applications

In the following corollary, we give a simple proof of a known generalization of the Levinson inequality (see [8] and [1, pages 71-72]).

Recall that a function $f: I\to\mathbb{R}$ is 3-convex if $[x_0,x_1,x_2,x_3]f\ge 0$ for $x_i\ne x_j$, $i\ne j$, and $x_i\in I$, where $[x_0,x_1,x_2,x_3]f$ denotes third-order divided difference of f. It is easy to prove, using properties of divided differences or using classical case of the Levinson inequality, that if $f:(0,2a)\to\mathbb{R}$ is a 3-convex function, then the function $g(x)=f(2a-x)-f(x)$ is convex on $(0,a]$ (see [1, pages 71-72]).

Corollary 3.1. *Let $f:(0,2a)\to\mathbb{R}$ be a 3-convex function; $p_i>0$, $x_i\in(0,2a)$, $x_i+x_{n+1-i}\le 2a$, and*

$$\frac{p_ix_i+p_{n+1-i}x_{n+1-i}}{p_i+p_{n+1-i}}\le a\tag{3.1}$$

for $i = 1, 2, \ldots, n$. *Then,*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right). \tag{3.2}$$

Proof. It is a simple consequence of Theorem 2.1 and the above-mentioned fact that $g(x) = f(2a - x) - f(x)$ is convex on $(0, a]$. □

Remark 3.2. In fact, the following improvement of inequality (3.2) is valid:

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) &\ge \frac{1}{2P_n}\sum_{i=1}^{n} (p_i + p_{n+1-i}) f\left(2a - \frac{p_i x_i + p_{n+1-i}x_{n+1-i}}{p_i + p_{n+1-i}}\right)\\ &\quad - \frac{1}{2P_n}\sum_{i=1}^{n} (p_i + p_{n+1-i}) f\left(\frac{p_i x_i + p_{n+1-i}x_{n+1-i}}{p_i + p_{n+1-i}}\right)\\ &\ge f\left(2a - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right).\end{aligned} \tag{3.3}$$

A famous inequality due to Ky-Fan states that

$$\frac{G_n}{G'_n} \le \frac{A_n}{A'_n}, \tag{3.4}$$

where $G_n,\ G'_n$ and $A_n,\ A'_n$ are the weighted geometric and arithmetic means, respectively, defined by

$$\begin{aligned} G_n &= \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}, & A_n &= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i,\\ G'_n &= \left(\prod_{i=1}^{n} (1 - x_i)^{p_i}\right)^{1/P_n}, & A'_n &= \frac{1}{P_n}\sum_{i=1}^{n} p_i (1 - x_i), \end{aligned} \tag{3.5}$$

where $x_i \in (0, 1/2],\ i = 1, \ldots, n$ (see [6, page 295]).

In the following corollary, we give an improvement of the Ky-Fan inequality.

Corollary 3.3. *Let* $p_i > 0,\ x_i \in (0,1),\ A_2(x_i, x_{n+1-i}) = (p_i x_i + p_{n+1-i}x_{n+1-i})/(p_i + p_{n+1-i})$, *and* $x'_i = 1 - x_i,\ i = 1, \ldots, n$. *If* $x_i + x_{n+1-i} \le 1$ *and* $A_2(x_i, x_{n+1-i}) \le 1/2,\ i = 1, \ldots, n$, *then*

$$\frac{G'_n}{G_n} \ge \left[\prod_{i=1}^{n}\left(\frac{A_2(x'_i, x'_{n+1-i})}{A_2(x_i, x_{n+1-i})}\right)^{p_i + p_{n+1-i}}\right]^{1/2P_n} \ge \frac{A'_n}{A_n}. \tag{3.6}$$

Proof. Set $f(x) = \log x$ and $2a = 1$ in (3.3). It follows that

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n} p_i \log(1-x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \log x_i &\geq \frac{1}{2P_n}\sum_{i=1}^{n}(p_i + p_{n+1-i}) \log \frac{p_i(1-x_i) + p_{n+1-i}(1-x_{n+1-i})}{p_i + p_{n+1-i}} \\ &\quad - \frac{1}{2P_n}\sum_{i=1}^{n}(p_i + p_{n+1-i}) \log \frac{p_i x_i + p_{n+1-i}x_{n+1-i}}{p_i + p_{n+1-i}} \\ &\geq \log\left(1 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \log \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i,\end{aligned} \tag{3.7}$$

which by obvious rearrangement implies (3.6). □

Acknowledgments

The research of J. Pečarić and I. Perić was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grants 117-1170889-0888 (J. Pečarić) and 058-1170889-1050 (I. Perić). S. Hussain and J. Pečarić also acknowledge with thanks the facilities provided to them by Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan. The authors also thank the careful referee for helpful suggestions which have improved the final version of this paper.

References

[1] J. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[2] G. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1967.

[3] D. S. Mitrinović and J. Pečarić, "Generalizations of the Jensen inequality," *Österreichische Akademie der Wissenschaften Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte*, vol. 196, no. 1–3, pp. 21–26, 1987.

[4] P. Czinder, "A weighted Hermite-Hadamard-type inequality for convex-concave symmetric functions," *Publicationes Mathematicae Debrecen*, vol. 68, no. 1-2, pp. 215–224, 2006.

[5] P. Czinder and Z. Páles, "An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 2, article 42, 8 pages, 2004.

[6] P. S. Bullen, *Handbook of Means and Their Inequalities*, vol. 560 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[7] A. McD. Mercer, "A variant of Jensen's inequality," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, article 73, 2 pages, 2003.

[8] J. Pečarić, "On an inequality of N. Levinson," *Publikacije Elektrotehničkog Fakulteta. Serija Matematika*, no. 678–715, pp. 71–74, 1980.