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*Research Article*

# On $\overline{\lambda}$-Statistically Convergent Double Sequences of Fuzzy Numbers

**Ekrem Savaş**

*Department of Mathematics, Istanbul Ticaret University, Uskudar 36472, Istanbul, Turkey*

Correspondence should be addressed to Ekrem Savaş, ekremsavas@yahoo.com

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We study the notion of $\overline{\lambda}$-statistically convergent for double sequence of fuzzy numbers and also get some inclusion relations.



## 1. Introduction

Nanda [1] studied sequence of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Nuray [2] proved the inclusion relations between the set of statistically convergent and lacunary statistically convergent sequences of fuzzy numbers. Kwon and Shim [3] studied statistical convergence and lacunary statistical convergence of sequences of fuzzy numbers, and they showed that Nuray's conditions are sufficient as well as necessary. Savaş [4] introduced and discussed double convergent sequence of fuzzy numbers and showed that the set of all double convergent sequences of fuzzy numbers is complete. In [5], Savaş generalized the statistical convergence by using de la Vallee-Poussin mean. Quite recently, Savaş and Mursaleen [6] introduced of statistically convergent and statistically Cauchy for double sequence of fuzzy numbers.

In this paper, we continue to study the concepts of strongly double $[V, \overline{\lambda}]$-summable and double $S_{\overline{\lambda}}$-convergent for double sequence of fuzzy numbers.

## 2. Preliminaries

Before continuing with the discussion, we pause to establish some notation. Let $C(R^n) = \{A \subset R^n : \text{A compact and convex}\}$. The spaces $C(R^n)$ have a linear structure induced by the operations

$$A + B = \{a + b,\ a \in A,\ b \in B\},$$
$$\lambda A = \{\lambda a,\ \lambda \in A\} \tag{2.1}$$

for $A$, $B \in C(R^n)$, and $\lambda \in R$. The Hausdorff distance between $A$ and $B$ of $C(R^n)$ is defined as

$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}. \tag{2.2}$$

It is well known that $(C(R^n),\delta_\infty)$ is a complete (not separable) metric space.

A fuzzy number is a function $X$ from $R^n$ to $[0,1]$ satisfying

(1) $X$ is normal, that is, there exists an $x_0 \in R^n$ such that $X(x_0) = 1$;

(2) $X$ is fuzzy convex, that is, for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$X(\lambda x + (1-\lambda)y) \geq \min\{X(x), X(y)\}; \tag{2.3}$$

(3) $X$ is upper semicontinuous;

(4) the closure of $\{x \in R^n : X(x) > 0\}$, denoted by $X^0$, is compact.

These properties imply that for each $0 < \alpha \leq 1$, the $\alpha$-level set

$$X^\alpha = \{x \in R^n : X(x) \geq \alpha\} \tag{2.4}$$

is a nonempty compact convex, subset of $R^n$, as is the support $X^0$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces addition $X+Y$ and scalar multiplication $\lambda X$, $\lambda \in R$, in terms of $\alpha$-level sets by

$$\begin{gathered}[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha,\\ [\lambda X]^\alpha = \lambda[X]^\alpha\end{gathered} \tag{2.5}$$

for each $0 \leq \alpha \leq 1$.

Define for each $1 \leq q < \infty$,

$$d_q(X,Y) = \left\{\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\right\}^{1/q} \tag{2.6}$$

and $d_\infty = \sup_{0\leq\alpha\leq 1}\delta_\infty(X^\alpha, Y^\alpha)$. Clearly, $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover, $d_q$ is a complete, separable, and locally compact metric space [7].

Throughout the paper, $d$ will denote $d^q$ with $1 \leq q \leq \infty$.

We will need the following definitions.

*Definition 2.1.* A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or $P$-convergent to a fuzzy number $X_0$ if for every $\varepsilon > 0$, there exists $N \in \mathcal{N}$ such that

$$d(X_{kl}, X_0) < \epsilon \quad \text{for } k, l > N, \tag{2.7}$$

and we denote $P - \lim X = X_0$. The number $X_0$ is called the Pringsheim limit of $X_{kl}$.

More exactly, we say that a double sequence $(X_{kl})$ converges to a finite number $X_0$ if $X_{kl}$ tend to $X_0$ as both $k$ and $l$ tends to $\infty$ independently of one another.

Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

*Definition 2.2.* A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number $M$ such that $d(X_{kl}, X_0) < M$ for all $k$ and $l$,

$$\|x\|_{(\infty,2)} = \sup_{k,l} d(X_{kl}, X_0) < \infty. \tag{2.8}$$

We will denote the set of all bounded double sequences by $l^2_\infty(F)$.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two-dimensional set of positive integers and let $K_{m,n}$ be the numbers of $(i, j)$ in $K$ such that $i \le n$ and $j \le m$. Then the lower asymptotic density of $K$ is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \tag{2.9}$$

In the case when the sequence $(K_{m,n}/mn)^{\infty,\infty}_{m,n=1,1}$ has a limit, then we say that $K$ has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \tag{2.10}$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \le P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0 \tag{2.11}$$

(i.e., the set $K$ has double natural density zero).

*Definition 2.3.* A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be statistically convergent to $X_0$ provided that for each $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{nm} \left|\left\{(j, k);\ j \le m,\ k \le n : d(X_{kl}, X_0) \ge \epsilon\right\}\right| = 0. \tag{2.12}$$

In this case, we write $\mathrm{st}_2 - \lim_{k,l} X_{k,l} = X_0$ and we denote the set of all double statistically convergent sequences of fuzzy numbers by $\mathrm{st}^2(F)$.

*Definition 2.4.* $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ could be two nondecreasing sequences of positive real numbers such that each tends to $\infty$ and

$$\begin{aligned} \lambda_{n+1} &\le \lambda_n + 1, \qquad \lambda_1 = 1, \\ \mu_{m+1} &\le \mu_m + 1, \qquad \mu_1 = 1. \end{aligned} \tag{2.13}$$

A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be $\overline{\lambda}$-summable if there is fuzzy number $X_0$ such that

$$P - \lim_{nm} \frac{1}{\overline{\lambda}_{nm}} \sum_{k \in I_n} \sum_{l \in I_m} d(X_{kl}, X_0) = 0, \tag{2.14}$$

where $I_n = [n - \lambda_n + 1, n]$, $I_m = [m - \mu_m + 1, m]$, and $\overline{\lambda}_{nm} = \lambda_n \mu_m$.

In this case, we say that $X$ is strongly double $\overline{\lambda}$-summable to $X_0$ and we denote the set of all strongly double $\overline{\lambda}$-summable sequences by $[V,\overline{\lambda}](F)$. If $\overline{\lambda}_{nm} = nm$, then strongly double $\overline{\lambda}$-summable reduces to $[C,1,1](F)$, the space of strongly double Cesàro summable sequences defined as follows:

$$P-\lim_{nm}\frac{1}{nm}\sum_{k,l=1,1}^{mn} d(X_{kl},X_0) = 0. \tag{2.15}$$

*Definition 2.5.* A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be double $\overline{\lambda}$-statistically convergent or $S_{\overline{\lambda}}$-convergent to $X_0$ if for every $\epsilon > 0$,

$$P-\lim_{n,m}\frac{1}{\overline{\lambda}_{nm}}\left|\left\{k \in I_n, l \in I_m : d(X_{kl},X_0) \geq \epsilon\right\}\right| = 0. \tag{2.16}$$

In this case, we write $S_{\overline{\lambda}}-\lim X = X_0$ or $X_{kl} \xrightarrow{P} X_0(S_{\overline{\lambda}})$ and we denote the set of all double $S_{\overline{\lambda}}$-statistically convergent sequences of fuzzy numbers by $(S_{\overline{\lambda}})(F)$.

If $\overline{\lambda}_{nm} = nm$, for all $n,m$, then the set $S_{\overline{\lambda}}(F)$ of $S_{\overline{\lambda}}$-convergent sequences reduces to the space $\text{st}^2(F)$.

We need the following proposition in future. A metric $d$ on $L(\mathbb{R})$ is said to be a translation invariant if $d(X+Z,Y+Z) = d(X,Y)$ for $X,Y,Z \in L(\mathbb{R})$.

**Proposition 2.6.** *If $d$ is a translation invariant metric on $L(\mathbb{R})$, then*

$$d(X+Y,0) \leq d(X,0) + d(Y,0). \tag{2.17}$$

Proof is clear so we omitted it.

In the next theorem, we give some connections between strongly double $\overline{\lambda}$-summable and double $\overline{\lambda}$-statistical convergences.

## 3. Main results

**Theorem 3.1.** *A double sequence $X = (X_{kl})$ of fuzzy numbers is strongly double $\overline{\lambda}$-summable $X_0$, then it is double $\overline{\lambda}$-statistically convergent to $X_0$.*

*Proof.* Let $\epsilon > 0$ and since

$$\sum_{k\in I_n, l\in I_m} d(X_{kl},X_0) \geq \sum_{k\in I_n, l\in I_m, d(X_{kl},X_0)\geq\epsilon} d(X_{kl},X_0) \geq \epsilon\left|\left\{k \in I_n, l \in I_m : d(X_{kl},X_0) \geq \epsilon\right\}\right|. \tag{3.1}$$

This implies that if a sequence $X = (X_{kl})$ is strongly double $\overline{\lambda}$-summable $X_0$, then $X$ is double $\overline{\lambda}$-statistically convergent to $X_0$.

This completes the proof. □

We have the following theorem.

**Theorem 3.2.** *If a bounded $(X_{kl})$ is double $\overline{\lambda}$-statistically convergent to $X_0$, then it is strongly double $\overline{\lambda}$-summable $X_0$.*

*Proof.* Suppose that $(X_{kl})$ is bounded and double $\overline{\lambda}$-statistically convergent to $X_0$. Since $X$ is bounded we write $d(X_{kl}, X_0) \leq M$ for all $k, l$. Also for given $\epsilon > 0$ and $n$ and $m$ large we obtain

$$\frac{1}{\overline{\lambda}_{nm}} \sum_{k\in I_n, l\in I_m} d(X_{kl}, X_0) = \frac{1}{\overline{\lambda}_{nm}} \sum_{k\in I_n, l\in I_m, d(X_{kl},X_0)\geq\epsilon} d(X_{kl}, X_0) + \frac{1}{\overline{\lambda}_{nm}} \sum_{k\in I_n, l\in I_m, d(X_{kl},X_0)<\epsilon} d(X_{kl}, X_0)$$
$$\leq \frac{M}{\overline{\lambda}_{nm}} \left|\left\{k \in I_n,\ l \in I_m : d(X_{kl}, X_0) \geq \epsilon\right\}\right| + \epsilon, \tag{3.2}$$

which implies that $X$ is strongly double $\overline{\lambda}$-summable $X_0$.

This completes the proof. □

**Theorem 3.3.** *If a sequence $X = (X_{kl})$ of fuzzy numbers is double statistically convergent to $X_0$, then it is double $\overline{\lambda}$-statistically convergent to $X_0$ if and only if*

$$P - \lim_{nm} \inf \frac{\overline{\lambda}_{nm}}{nm} > 0. \tag{3.3}$$

*Proof.* For given $\varepsilon > 0$, we have

$$\left\{k \leq n,\ l \leq m : d(X_{kl}, X_0) \geq \epsilon\right\} \supset \left\{k \in I_n,\ l \in I_m : d(X_{kl}, X_0) \geq \epsilon\right\}. \tag{3.4}$$

Therefore,

$$\frac{1}{nm}\left|\left\{k \leq n,\ l \leq m : d(X_{kl}, X_0) \geq \epsilon\right\}\right| \geq \frac{1}{nm}\left|\left\{k \in I_n,\ l \in I_m : d((X_{kl}, X_0) \geq \epsilon\right\}\right|$$
$$\geq \frac{\overline{\lambda}_{nm}}{nm} \frac{1}{\overline{\lambda}_{nm}} \left|\left\{k \in I_n,\ l \in I_m : d((X_{kl}, X_0) \geq \epsilon\right\}\right|. \tag{3.5}$$

Taking the limit as $n, m \to \infty$ and using hypothesis, we get $X$ is double $\overline{\lambda}$-statistically convergent to $X_0$.

Conversely, suppose that $X \in \mathrm{st}_2(F)$ and since $\overline{\lambda}_{nm} = \lambda_n \mu_m$, either $P - \lim_n \inf \lambda_n / n = 0$ or $P - \lim_m \inf (\mu_m / m) = 0$ or both are zero. Then we can choose subsequences $(n(p))_{p=1}^{\infty}$ and $(m(q))_{q=1}^{\infty}$ such that $\lambda_{n_{(p)}} / n(p) < 1/p$ and $\mu_{m_{(q)}} / m(q) < 1/q$. Define a sequence $X = (X_{kl})$ by

$$X_{kl} = \begin{cases} 1 & \text{if } k \in I_{n_{(p)}},\ l \in I_{m(q)}\ (p,\ q = 1, 2, \ldots), \\ 0 & \text{otherwise}. \end{cases} \tag{3.6}$$

Then $X \in [C, 1, 1](F)$ and hence, by [6, Theorem 6(a)], $X \in \mathrm{st}^2(F)$. But on the other hand, $X \notin [V, \overline{\lambda}](F)$ and from Theorem 3.1, $X \notin (S_{\overline{\lambda}})(F)$; a contradiction and hence (3.3) must hold. □

Finally, we conclude this paper by stating a definition which generalizes Definition 2.4.

*Definition 3.4.* Let $X = (X_{kl})$ be a double sequence of fuzzy numbers and let $p$ be positive real numbers. The sequence $X$ is said to be strongly double $\overline{\lambda}_p$-summable if there is fuzzy number $X_0$ such that

$$P - \lim_{nm} \frac{1}{\overline{\lambda}_{nm}} \sum_{k \in I_n} \sum_{l \in I_m} d(X_{kl}, X_0)^p = 0. \tag{3.7}$$

In this case, we say that $X$ is strongly double $\overline{\lambda}_p$-summable to $X_0$. If $\overline{\lambda}_{nm} = nm$, then strongly double $\overline{\lambda}_p$-summable reduces to strongly double $p$-Cesàro summable to $X_0$.

**Theorem 3.5.** (1) *Let $p \in (0, \infty)$. If a double sequence $X = (X_{kl})$ of fuzzy numbers is strongly double $\overline{\lambda}_p$-summable $X_0$, then it is double $\overline{\lambda}$-statistically convergent to $X_0$.*

(2) *Let $p \in (0, \infty)$. If a bounded $(X_{kl})$ is double $\overline{\lambda}$-statistically convergent to $X_0$, then it is strongly double $\overline{\lambda}_p$-summable $X_0$.*

*Proof.* The proof of theorem is similar to that of Theorems 3.1 and 3.2 so we omitted it. □

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