

Research Article

A New Subclass of Analytic Functions Defined by Generalized Ruscheweyh Differential Operator

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Received 1 July 2008; Accepted 3 September 2008

Recommended by Narendra Kumar Govil

We investigate a new subclass of analytic functions in the open unit disk \mathbb{U} which is defined by generalized Ruscheweyh differential operator. Coefficient inequalities, extreme points, and the integral means inequalities for the fractional derivatives of order $p + \eta$ ($0 \leq p \leq n$, $0 \leq \eta < 1$) of functions belonging to this subclass are obtained.

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1. Introduction

Throughout this paper, we use the following notations:

$$\begin{aligned}\mathbb{N} &:= \{1, 2, 3, \dots\}, \\ \mathbb{N}_0 &:= \mathbb{N} \cup \{0\}, \\ \mathbb{R}_{-1} &:= \{u \in \mathbb{R} : u > -1\}, \\ \mathbb{R}_{-1}^0 &:= \mathbb{R}_{-1} \setminus \{0\}.\end{aligned}\tag{1.1}$$

Let \mathcal{A} denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\tag{1.2}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

For $f_j \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2),\tag{1.3}$$

the Hadamard product (or convolution) $f_1 * f_2$ of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \quad (1.4)$$

Using the convolution (1.4), Shaqsi and Darus [1] introduced the generalization of the Ruscheweyh derivative as follows.

For $f \in \mathcal{A}$, $\lambda \geq 0$, and $u \in \mathbb{R}_{-1}$, we consider

$$R_{\lambda}^u f(z) = \frac{z}{(1-z)^{u+1}} * R_{\lambda} f(z) \quad (z \in \mathbb{U}), \quad (1.5)$$

where $R_{\lambda} f(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $z \in \mathbb{U}$.

If $f \in \mathcal{A}$ is of the form (1.2), then we obtain the power series expansion of the form

$$R_{\lambda}^u f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda] C(u, n) a_n z^n, \quad (1.6)$$

where

$$C(u, n) = \frac{(1+u)_{n-1}}{(n-1)!} \quad (n \in \mathbb{N}), \quad (1.7)$$

and where $(a)_n$ is the Pochhammer symbol (or shifted factorial) defined (in terms of the Gamma function) by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0, a \in \mathbb{C} \setminus \{0\}, \\ a(a+1) \cdots (a+n-1), & \text{if } n \in \mathbb{N}, a \in \mathbb{C}. \end{cases} \quad (1.8)$$

In the case $m \in \mathbb{N}_0$, we have

$$R_{\lambda}^m f(z) = \frac{z(z^{m-1} f(z))^{(m)}}{m!}, \quad (1.9)$$

and for $\lambda = 0$, we obtain u th Ruscheweyh derivative introduced in [2], $R_0^m = R^m$.

Using the generalized Ruscheweyh derivative operator R_{λ}^u , we define the following classes.

Definition 1.1. Let $\mathcal{S}_{\lambda}(u, v; \alpha)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left\{ \frac{R_{\lambda}^u f(z)}{R_{\lambda}^v f(z)} \right\} > \alpha \quad (1.10)$$

for some $0 \leq \alpha < 1$, $u \in \mathbb{R}_{-1}^0$, $v \in \mathbb{R}_{-1}$, $\lambda \geq 0$, and all $z \in \mathbb{U}$.

In this paper, basic properties of the class $\mathcal{S}_\lambda(u, v; \alpha)$ are studied, such as coefficient bounds, extreme points, and integral means inequalities for the fractional derivative.

2. Coefficient inequalities

Theorem 2.1. *Let $0 \leq \alpha < 1$, $u \in \mathbb{R}_{-1}^0$, $v \in \mathbb{R}_{-1}$, and $\lambda \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \mathcal{B}_n(u, v, \alpha) |a_n| \leq 2(1 - \alpha), \quad (2.1)$$

where

$$\mathcal{B}_n(u, v, \alpha) := [1 + (n - 1)\lambda] \{|C(u, n) - (1 + \alpha)C(v, n)| + C(u, n) + (1 - \alpha)C(v, n)\}, \quad (2.2)$$

then $f \in \mathcal{S}_\lambda(u, v; \alpha)$.

Proof. Let (2.1) be true for $0 \leq \alpha < 1$, $u \in \mathbb{R}_{-1}^0$, $v \in \mathbb{R}_{-1}$, and $\lambda \geq 0$. For $f \in \mathcal{A}$, define the function F by

$$F(z) := \frac{R_\lambda^u f(z)}{R_\lambda^v f(z)} - \alpha. \quad (2.3)$$

It is sufficient to show that

$$\left| \frac{F(z) - 1}{F(z) + 1} \right| < 1 \quad (2.4)$$

for $z \in \mathbb{U}$.

So, we have

$$\begin{aligned} \left| \frac{F(z) - 1}{F(z) + 1} \right| &= \left| \frac{R_\lambda^u f(z) - (1 + \alpha)R_\lambda^v f(z)}{R_\lambda^u f(z) + (1 - \alpha)R_\lambda^v f(z)} \right| \\ &= \left| \frac{\alpha - \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] [C(u, n) - (1 + \alpha)C(v, n)] a_n z^{n-1}}{(2 - \alpha) + \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] [C(u, n) + (1 - \alpha)C(v, n)] a_n z^{n-1}} \right| \\ &\leq \frac{\alpha + \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] |C(u, n) - (1 + \alpha)C(v, n)| |a_n| |z|^{n-1}}{(2 - \alpha) - \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] [C(u, n) + (1 - \alpha)C(v, n)] |a_n| |z|^{n-1}} \\ &< \frac{\alpha + \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] |C(u, n) - (1 + \alpha)C(v, n)| |a_n|}{(2 - \alpha) - \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] [C(u, n) + (1 - \alpha)C(v, n)] |a_n|} \\ &< 1 \quad (\text{by (2.1)}). \end{aligned} \quad (2.5)$$

Therefore, $f \in \mathcal{S}_\lambda(u, v; \alpha)$. □

Theorem 2.2. *If $f \in \mathcal{S}_\lambda(u, v; \alpha)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{[1+(n-1)\lambda]|C(u, n) - C(v, n)|} \sum_{u=1}^{n-1} [1+(n-u-1)\lambda]C(v, n-u)|a_{n-u}| \quad (2.6)$$

for $n \geq 2$, with $a_1 = 1$.

Proof. Define the function

$$G(z) := \frac{1}{1-\alpha} \left(\frac{R_\lambda^u f(z)}{R_\lambda^v f(z)} - \alpha \right) := 1 + \sum_{n=1}^{\infty} \hat{a}_n z^n. \quad (2.7)$$

Since $\operatorname{Re}\{G(z)\} > 0$, we get

$$|\hat{a}_n| \leq 2 \quad (2.8)$$

for $n = 1, 2, \dots$

From the definition of $G(z)$, we obtain

$$\frac{R_\lambda^u f(z) - \alpha R_\lambda^v f(z)}{1-\alpha} = R^v f(z) \left[1 + \sum_{n=1}^{\infty} \hat{a}_n z^n \right]. \quad (2.9)$$

So, by (1.6), we have

$$\begin{aligned} z + \frac{1+\lambda}{1-\alpha} [C(u, 2) - \alpha C(v, 2)] a_2 z^2 + \frac{1+2\lambda}{1-\alpha} [C(u, 3) - \alpha C(v, 3)] a_3 z^3 + \dots \\ = z + \hat{a}_1 z^2 + \hat{a}_2 z^3 + \hat{a}_3 z^4 + \dots \\ + (1+\lambda)C(v, 2) a_2 z^2 + (1+\lambda)C(v, 2) a_2 \hat{a}_1 z^3 + (1+\lambda)C(v, 2) a_2 \hat{a}_2 z^4 + \dots \\ + (1+2\lambda)C(v, 3) a_3 z^3 + (1+2\lambda)C(v, 3) a_3 \hat{a}_1 z^4 + \dots \end{aligned} \quad (2.10)$$

or

$$\begin{aligned} z + \frac{1+\lambda}{1-\alpha} [C(u, 2) - C(v, 2)] a_2 z^2 + \frac{1+2\lambda}{1-\alpha} [C(u, 3) - C(v, 3)] a_3 z^3 + \dots \\ = z + \hat{a}_1 z^2 + [(1+\lambda)C(v, 2) a_2 \hat{a}_1 + \hat{a}_2] z^3 \\ + [(1+2\lambda)C(v, 3) a_3 \hat{a}_1 + (1+\lambda)C(v, 2) a_2 \hat{a}_2 + \hat{a}_3] z^4 + \dots \end{aligned} \quad (2.11)$$

or, equivalently,

$$\begin{aligned} z + \sum_{n=2}^{\infty} \frac{1 + (n-1)\lambda}{1-\alpha} [C(u, j) - C(v, j)] a_n z^n \\ = z + \sum_{n=2}^{\infty} \left(\sum_{u=1}^{n-1} [1 + (n-u-1)\lambda] C(v, n-u) a_{n-u} \hat{a}_u \right) z^n. \end{aligned} \quad (2.12)$$

When we consider the coefficients of z^n of both series in the above equality, we have

$$a_n = \frac{1-\alpha}{[1 + (n-1)\lambda][C(u, n) - C(v, n)]} \sum_{u=1}^{n-1} [1 + (n-u-1)\lambda] C(v, n-u) a_{n-u} \hat{a}_u. \quad (2.13)$$

Therefore,

$$\begin{aligned} |a_n| &\leq \frac{1-\alpha}{[1 + (n-1)\lambda]|C(u, n) - C(v, n)|} \sum_{u=1}^{n-1} [1 + (n-u-1)\lambda] C(v, n-u) |a_{n-u}| |\hat{a}_u| \\ &\leq \frac{2(1-\alpha)}{[1 + (n-1)\lambda]|C(u, n) - C(v, n)|} \sum_{u=1}^{n-1} [1 + (n-u-1)\lambda] C(v, n-u) |a_{n-u}|, \end{aligned} \quad (2.14)$$

since $|\hat{a}_u| \leq 2$, ($u = 1, 2, \dots$). □

3. Extreme points

Definition 3.1. Let $\tilde{\mathcal{S}}_\lambda(u, v; \alpha)$ be the subclass of $\mathcal{S}_\lambda(u, v; \alpha)$ which consists of function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \quad (3.1)$$

whose coefficients satisfy inequality (2.1).

Theorem 3.2. Let $f_1(z) = z$ and

$$f_k(z) = z + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \quad (k = 2, 3, \dots), \quad (3.2)$$

where $\mathcal{B}_k(u, v, \alpha)$ is given by (2.2).

Then $f \in \tilde{\mathcal{S}}_\lambda(u, v; \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z), \quad (3.3)$$

where $\delta_k \geq 0$ and $\sum_{k=1}^{\infty} \delta_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z). \quad (3.4)$$

Then

$$\begin{aligned} f(z) &= \delta_1 f_1(z) + \sum_{k=2}^{\infty} \delta_k f_k(z) \\ &= \delta_1 z + \sum_{k=2}^{\infty} \delta_k \left(z + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \right) \\ &= \left(\sum_{k=1}^{\infty} \delta_k \right) z + \sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \\ &= z + \sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k. \end{aligned} \quad (3.5)$$

Thus

$$\sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \mathcal{B}_k(u, v, \alpha) = 2(1-\alpha) \sum_{k=2}^{\infty} \delta_k = 2(1-\alpha)(1-\delta_1) \leq 2(1-\alpha). \quad (3.6)$$

Therefore, we have $f \in \tilde{\mathcal{S}}_\lambda(u, v; \alpha)$.

Conversely, suppose that $f \in \tilde{\mathcal{S}}_\lambda(u, v; \alpha)$. Since

$$a_k \leq \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \quad (k = 2, 3, \dots), \quad (3.7)$$

we can set

$$\begin{aligned} \delta_k &:= \frac{\mathcal{B}_k(u, v, \alpha)}{2(1-\alpha)} a_k \quad (k = 2, 3, \dots), \\ \delta_1 &:= 1 - \sum_{k=2}^{\infty} \delta_k. \end{aligned} \quad (3.8)$$

Then

$$\begin{aligned}
 f(z) &= z + \sum_{k=2}^{\infty} a_k z^k \\
 &= \left(\sum_{k=1}^{\infty} \delta_k \right) z + \sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \\
 &= \delta_1 z + \sum_{k=2}^{\infty} \delta_k \left(z + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \right) \\
 &= \delta_1 f_1(z) + \sum_{k=2}^{\infty} \delta_k f_k(z) \\
 &= \sum_{k=1}^{\infty} \delta_k f_k(z).
 \end{aligned} \tag{3.9}$$

This completes the proof of Theorem 3.2. \square

Corollary 3.3. *The extreme points of $\tilde{\mathcal{S}}_{\lambda}(u, v; \alpha)$ are given by*

$$f_1(z) = z, \quad f_k(z) = z + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \quad (k = 2, 3, \dots), \tag{3.10}$$

where $\mathcal{B}_k(u, v, \alpha)$ is given by (2.2).

4. The main integral means inequalities for the fractional derivative

We discuss the integral means inequalities for functions $f \in \tilde{\mathcal{S}}_{\lambda}(u, v; \alpha)$.

The following definitions of fractional derivatives by Owa [3] (also by Srivastava and Owa [4]) will be required in our investigation.

Definition 4.1. The fractional derivative of order η is defined, for a function f , by

$$D_z^{\eta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\eta}} d\xi \quad (0 \leq \eta < 1), \tag{4.1}$$

where the function f is analytic in a simply connected region of the complex z -plane containing the origin, and the multiplicity of $(z-\xi)^{-\eta}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 4.2. Under the hypothesis of Definition 4.1, the fractional derivative of order $p + \eta$ is defined, for a function f , by

$$D_z^{p+\eta} f(z) = \frac{d^p}{dz^p} D_z^{\eta} f(z), \tag{4.2}$$

where $0 \leq \eta < 1$ and $p \in \mathbb{N}_0$.

It readily follows from (4.1) in Definition 4.1 that

$$D_z^\eta z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\eta)} z^{k-\eta} \quad (0 \leq \eta < 1, k \in \mathbb{N}). \quad (4.3)$$

We will also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [5] in our investigation.

Definition 4.3. Given two functions f and g , which are analytic in \mathbb{U} , the function f is said to be subordinate to g in \mathbb{U} if there exists a function w analytic in \mathbb{U} with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}), \quad (4.4)$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \quad (4.5)$$

We denote this subordination by

$$f(z) < g(z). \quad (4.6)$$

Lemma 4.4. *If the functions f and g are analytic in \mathbb{U} with*

$$f(z) < g(z), \quad (4.7)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta. \quad (4.8)$$

Our main theorem is contained in the following.

Theorem 4.5. *Let $f \in \tilde{\mathcal{S}}_\lambda(u, v; \alpha)$ and suppose that*

$$\sum_{n=2}^{\infty} (n-p)_{p+1} a_n \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\eta-p)}{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta-p)\Gamma(2-p)} \quad (4.9)$$

for $0 \leq p \leq n$, $k \geq p$, $0 \leq \eta < 1$, where $(n-p)_{p+1}$ denotes the Pochhammer symbol defined by

$$(n-p)_{p+1} = (n-p)(n-p+1) \cdots n. \quad (4.10)$$

Also let the function f_k be defined by

$$f_k(z) = z + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \quad (k = 2, 3, \dots). \quad (4.11)$$

If there exists an analytic function w defined by

$$(w(z))^{k-1} := \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1} \Psi(n) a_n z^{n-1} \quad (4.12)$$

with

$$\Psi(n) = \frac{\Gamma(n-p)}{\Gamma(n+1-\eta-p)}, \quad (0 \leq \eta < 1, n = 2, 3, \dots), \quad (4.13)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |D_z^{p+\eta} f(z)|^\mu d\theta \leq \int_0^{2\pi} |D_z^{p+\eta} f_k(z)|^\mu d\theta, \quad (0 \leq \eta < 1). \quad (4.14)$$

Proof. By means of (4.3) and Definition 4.2, we find from (3.1) that

$$\begin{aligned} D_z^{p+\eta} f(z) &= \frac{z^{1-\eta-p}}{\Gamma(2-\eta-p)} \left[1 + \sum_{n=2}^{\infty} \frac{\Gamma(2-\eta-p)\Gamma(n+1)}{\Gamma(n+1-\eta-p)} a_n z^{n-1} \right] \\ &= \frac{z^{1-\eta-p}}{\Gamma(2-\eta-p)} \left[1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1} \Psi(n) a_n z^{n-1} \right], \end{aligned} \quad (4.15)$$

where

$$\Psi(n) = \frac{\Gamma(n-p)}{\Gamma(n+1-\eta-p)}, \quad (0 \leq \eta < 1, n = 2, 3, \dots). \quad (4.16)$$

Since Ψ is a decreasing function of n , we get

$$0 < \Psi(n) \leq \Psi(2) = \frac{\Gamma(2-p)}{\Gamma(3-\eta-p)}. \quad (4.17)$$

Similarly, from (4.11), (4.3), and Definition 4.2, we have

$$D_z^{p+\eta} f_k(z) = \frac{z^{1-\eta-p}}{\Gamma(2-\eta-p)} \left[1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} z^{k-1} \right]. \quad (4.18)$$

For $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$), we want to show that

$$\begin{aligned} & \int_0^{2\pi} \left| 1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1} \Psi(n) a_n z^{n-1} \right|^\mu d\theta \\ & \leq \int_0^{2\pi} \left| 1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} z^{k-1} \right|^\mu d\theta. \end{aligned} \quad (4.19)$$

So, by applying Lemma 4.4, it is enough to show that

$$1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1} \Psi(n) a_n z^{n-1} < 1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} z^{k-1}. \quad (4.20)$$

If the above subordination holds true, then we have an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1} \Psi(n) a_n z^{n-1} = 1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} (w(z))^{k-1}. \quad (4.21)$$

By the condition of the theorem, we define the function w by

$$(w(z))^{k-1} = \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1} \Psi(n) a_n z^{n-1}, \quad (4.22)$$

which readily yields $w(0) = 0$. For such a function w , we have

$$\begin{aligned} |w(z)|^{k-1} & \leq \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1} \Psi(n) a_n |z|^{n-1} \\ & \leq |z| \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \Psi(2) \sum_{n=2}^{\infty} (n-p)_{p+1} a_n \\ & = |z| \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \frac{\Gamma(2-p)}{\Gamma(3-\eta-p)} \sum_{n=2}^{\infty} (n-p)_{p+1} a_n \\ & \leq |z| < 1 \end{aligned} \quad (4.23)$$

by means of the hypothesis of the theorem.

Thus the theorem is proved. \square

As a special case $p = 0$, we have the following result from Theorem 4.5.

Corollary 4.6. Let $f \in \tilde{\mathcal{S}}_\lambda(u, v; \alpha)$ and suppose that

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\eta)}{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta)} \quad (k = 2, 3, \dots). \quad (4.24)$$

If there exists an analytic function w defined by

$$(w(z))^{k-1} = \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} n \Psi(n) a_n z^{n-1} \quad (4.25)$$

with

$$\Psi(n) = \frac{\Gamma(n)}{\Gamma(n+1-\eta)}, \quad (0 \leq \eta < 1, n = 2, 3, \dots), \quad (4.26)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |D_z^\eta f(z)|^\mu d\theta \leq \int_0^{2\pi} |D_z^\eta f_k(z)|^\mu d\theta, \quad (0 \leq \eta < 1). \quad (4.27)$$

Letting $p = 1$ in Theorem 4.5, we have the following.

Corollary 4.7. Let $f \in \tilde{\mathcal{S}}_\lambda(u, v; \alpha)$ and suppose that

$$\sum_{n=2}^{\infty} n(n-1) a_n \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(2-\eta)}{\mathcal{B}_k(u, v, \alpha)\Gamma(k-\eta)} \quad (k = 2, 3, \dots). \quad (4.28)$$

If there exists an analytic function w defined by

$$(w(z))^{k-1} = \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k-\eta)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} n(n-1) \Psi(n) a_n z^{n-1} \quad (4.29)$$

with

$$\Psi(n) = \frac{\Gamma(n-1)}{\Gamma(n-\eta)}, \quad (0 \leq \eta < 1, n = 2, 3, \dots), \quad (4.30)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |D_z^{1+\eta} f(z)|^\mu d\theta \leq \int_0^{2\pi} |D_z^{1+\eta} f_k(z)|^\mu d\theta, \quad (0 \leq \eta < 1). \quad (4.31)$$

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