ON THE DERIVATIVE AND MAXIMUM MODULUS OF A POLYNOMIAL

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If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, having all its zeros in $|z| \le 1$, then it was proved by Turán that $|p'(z)| \ge (n/2) \max_{|z|=1} |p(z)|$. This result of Turán was generalized by Govil, who proved that if $p(z)$ has all its zeros in $|z| \le K$, $K \ge 1$, then $\max_{|z|=1} |p'(z)| \ge (n/(1+K^n)) \max_{|z|=1} |p(z)|$, $K \ge 1$. In this paper, we sharpen this, and some other related results.

1. Introduction and statement of results

If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then it is well known that

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The above inequality, which is an immediate consequence of Bernstein's inequality on the derivative of a trigonometric polynomial, is best possible with equality holding for the polynomial $p(z) = \lambda z^n$, λ being a complex number.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then the above inequality can be sharpened. In fact Erdös conjectured and later Lax [7] proved that if $p(z) \ne 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

If the polynomial $p(z)$ of degree n has all its zeros in $|z| \le 1$, then it was proved by Turán [9], that

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

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The inequalities (1.2) and (1.3) are also best possible, and become equality for polynomials which have all its zeros on $|z| = 1$.

The above inequality (1.3) of Turán [9] was generalized by Govil [3], who proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K} \max_{|z|=1} |p(z)|, \quad \text{if } K \leq 1, \tag{1.4}$$

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p(z)|, \quad \text{if } K \geq 1. \tag{1.5}$$

Both the above inequalities are best possible, with equality in (1.4) holding for $p(z) = (z+K)^n$, while in (1.5) the equality holds for the polynomial $p(z) = z^n + K^n$. The inequality (1.4) was also proved by Malik [8].

The inequality (1.5) was later sharpened by Govil [4, page 67], who proved the following theorem.

Theorem 1.1. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$, is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then*

$$\begin{aligned} \max_{|z|=1} |p'(z)| \geq{}& \frac{n}{1+K^n} \max_{|z|=1} |p(z)| \\ &+ \frac{n|a_{n-1}|}{K(1+K^n)} \left(\frac{K^n - 1}{n} - \frac{K^{n-2} - 1}{n-2} \right) + |a_1| \left(1 - \frac{1}{K^2} \right) \end{aligned} \tag{1.6}$$

if $n > 2$, and

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p(z)| + \frac{K^n - 1}{K^n + 1} |a_1| \tag{1.7}$$

if $n = 2$.

The above inequalities are best possible and are attained for the polynomial $p(z) = z^n + K^n$.

In this paper, we prove the following refinement of Theorem 1.1, which in turn gives the refinements of inequalities (1.3), and (1.5).

Theorem 1.2. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$, is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then*

$$\begin{aligned} \max_{|z|=1} |p'(z)| \geq{}& \frac{n}{1+K^n} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=K} |p(z)| \right\} + |a_1| \left(1 - \frac{1}{K^2} \right) \\ &+ \frac{n|a_{n-1}|}{K(1+K^n)} \left(\frac{K^n - 1}{n} - \frac{K^{n-2} - 1}{n-2} \right) \end{aligned} \tag{1.8}$$

if $n > 2$, and

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n}\left\{\max_{|z|=1} |p(z)| + \min_{|z|=K} |p(z)|\right\} + \frac{K^n-1}{K^n+1}|a_1| \tag{1.9}$$

if $n = 2$.

Both the above inequalities are best possible and are attained for the polynomial $p(z) = z^n + K^n$.

If we take $K = 1$ in the above theorem, we get the following result, which was proved by Aziz and Dawood [1].

Corollary 1.3. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$, is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)|\right\}. \tag{1.10}$$

2. Lemmas

We will need the following lemmas.

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)|\right\}. \tag{2.1}$$

The result is best possible and the equality holds for the polynomial $p(z) = (z+1)^n$.

The above result is due to Aziz and Dawood [1] (also see Govil [5, Theorem 2, inequality (1.7)]).

Lemma 2.2. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, having no zeros on $|z| < 1$, then for $R \geq 1$,*

$$\begin{aligned} \max_{|z|=R\geq 1} |p(z)| \leq {} & \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|p(z)| \\ & - |a_1|\left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right), \quad \text{if } n > 2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \max_{|z|=R\geq 1} |p(z)| \leq {} & \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|p(z)| \\ & - |a_1|\frac{(R-1)^n}{2}, \quad \text{if } n = 2. \end{aligned} \tag{2.3}$$

The above result is a special case, with $s = 1$ and $K = 1$, of a result due to Govil [6, page 625].

Lemma 2.3. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $n \geq 1$, then for all $R \geq 1$,*

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2}) |p(0)|, \quad \text{if } n \geq 2, \tag{2.4}$$

$$\max_{|z|=R} |p(z)| \leq R \max_{|z|=1} |p(z)| - (R-1) |p(0)|, \quad \text{if } n = 1. \tag{2.5}$$

The inequality (2.4) is due to Frappier et al. [2, Theorem 2], while (2.5) follows trivially.

3. Proof of the theorem

We first consider the case when $p(z)$ is degree $n > 2$. Since $p(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, the polynomial $P(z) = p(Kz)$ is of degree n, and has all its zeros in $|z| \leq 1$. Hence if we apply Lemma 2.1 to the polynomial $P(z)$, we will get

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\right\}, \tag{3.1}$$

which is equivalent to

$$K \max_{|z|=K} |p'(z)| \geq \frac{n}{2}\left\{\max_{|z|=K} |p(z)| + \min_{|z|=K} |p(z)|\right\}. \tag{3.2}$$

The polynomial $p(z)$ is of degree $n > 2$, and so the polynomial $p'(z)$ is of degree $n - 1$, where $n - 1 \geq 2$, and hence applying Lemma 2.3 to the polynomial $p'(z)$, we get for $K \geq 1$,

$$\max_{|z|=K} |p'(z)| \leq K^{n-1} \max_{|z|=1} |p'(z)| - (K^{n-1} - K^{n-3}) |a_1|. \tag{3.3}$$

Combining (3.2) and (3.3), we get for $K \geq 1$,

$$K^{n-1} \max_{|z|=1} |p'(z)| - (K^{n-1} - K^{n-3}) |a_1| \geq \frac{n}{2K}\left\{\max_{|z|=K} |p(z)| + \min_{|z|=K} |p(z)|\right\}, \tag{3.4}$$

which is equivalent to

$$K^{n} \max_{|z|=1} |p'(z)| - (K^{n} - K^{n-2}) |a_1| \geq \frac{n}{2}\left\{\max_{|z|=K} |p(z)| + \min_{|z|=K} |p(z)|\right\}. \tag{3.5}$$

Since the polynomial $p(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, the polynomial $q(z) = z^n p(1/z)$ has no zeros in $|z| < 1/K$, hence the polynomial $q(z/K)$ is of degree $n > 2$, and has no zeros in $|z| < 1$. Therefore, on applying Lemma 2.2 to the polynomial $q(z/K)$, we get

$$\begin{aligned} \max_{|z|=K\geq 1} |q(z/K)| \leq{}& \frac{K^n+1}{2} \max_{|z|=1} |q(z/K)| - \frac{K^n-1}{2} \min_{|z|=1} |q(z/K)| \\ &- \frac{|a_{n-1}|}{K}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right), \end{aligned} \tag{3.6}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=1}|p(z)| \leq \frac{K^n+1}{2K^n}\max_{|z|=K}|p(z)| - \frac{K^n-1}{2K^n}\min_{|z|=K}|p(z)| \\ - \frac{|a_{n-1}|}{K}\left(\frac{K^n-1}{n}-\frac{K^{n-2}-1}{n-2}\right).\end{aligned} \tag{3.7}$$

The above inequality easily gives

$$\begin{aligned}\max_{|z|=K}|p(z)| \geq \frac{2K^n}{K^n+1}\max_{|z|=1}|p(z)| + \frac{K^n-1}{K^n+1}\min_{|z|=K}|p(z)| \\ + \frac{2K^{n-1}}{1+K^n}|a_{n-1}|\left(\frac{K^n-1}{n}-\frac{K^{n-2}-1}{n-2}\right),\end{aligned} \tag{3.8}$$

and this when combined with (3.5) gives

$$\begin{aligned}&\frac{2K^n}{n}\max_{|z|=1}|p'(z)| - \frac{2(K^n-K^{n-2})}{n}|a_1| - \min_{|z|=K}|p(z)| \\ &\quad\geq \frac{2K^n}{K^n+1}\max_{|z|=1}|p(z)| + \frac{K^n-1}{K^n+1}\min_{|z|=K}|p(z)| + \frac{2K^{n-1}}{1+K^n}|a_{n-1}|\left(\frac{K^n-1}{n}-\frac{K^{n-2}-1}{n-2}\right).\end{aligned} \tag{3.9}$$

The above inequality (3.9) is clearly equivalent to

$$\begin{aligned}\max_{|z|=1}|p'(z)| \geq |a_1|\left(1-\frac{1}{K^2}\right) + \frac{n}{K^n+1}\left(\max_{|z|=1}|p(z)| + \min_{|z|=K}|p(z)|\right) \\ + \frac{n|a_{n-1}|}{K(1+K^n)}\left(\frac{K^n-1}{n}-\frac{K^{n-2}-1}{n-2}\right),\end{aligned} \tag{3.10}$$

which is inequality (1.8), and thus our theorem, in the case $n > 2$, is proved.

The proof of the theorem in the case $n = 2$ follows on the same lines as above except that instead of inequalities (2.2) and (2.4), we use inequalities (2.3) and (2.5), respectively.

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