INEQUALITIES INVOLVING THE MEAN AND THE STANDARD DEVIATION OF NONNEGATIVE REAL NUMBERS

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Received 22 December 2005; Revised 18 August 2006; Accepted 21 September 2006

Let $m(\mathbf{y}) = \sum_{j=1}^{n} y_j/n$ and $s(\mathbf{y}) = \sqrt{m(\mathbf{y}^2) - m^2(\mathbf{y})}$ be the mean and the standard deviation of the components of the vector $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1}, y_n)$, where $\mathbf{y}^q = (y_1^q, y_2^q, \ldots, y_{n-1}^q, y_n^q)$ with q a positive integer. Here, we prove that if $\mathbf{y} \geq \mathbf{0}$, then $m(\mathbf{y}^{2^p}) + (1/\sqrt{n-1})s(\mathbf{y}^{2^p}) \leq \sqrt{m(\mathbf{y}^{2^{p+1}}) + (1/\sqrt{n-1})s(\mathbf{y}^{2^{p+1}})}$ for $p = 0,1,2,\ldots$. The equality holds if and only if the $(n-1)$ largest components of $\mathbf{y}$ are equal. It follows that $(l_{2^p}(\mathbf{y}))_{p=0}^{\infty}$, $l_{2^p}(\mathbf{y}) = (m(\mathbf{y}^{2^p}) + (1/\sqrt{n-1})s(\mathbf{y}^{2^p}))^{2^{-p}}$, is a strictly increasing sequence converging to y_1, the largest component of $\mathbf{y}$, except if the $(n-1)$ largest components of $\mathbf{y}$ are equal. In this case, $l_{2^p}(\mathbf{y}) = y_1$ for all p.

1. Introduction

Let

$$m(\mathbf{x}) = \frac{\sum_{j=1}^{n} x_j}{n}, \qquad s(\mathbf{x}) = \sqrt{m(\mathbf{x}^2) - m^2(\mathbf{x})} \tag{1.1}$$

be the mean and the standard deviation of the components of $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}, x_n)$, where $\mathbf{x}^q = (x_1^q, x_2^q, \ldots, x_{n-1}^q, x_n^q)$ for a positive integer q.

The following theorem is due to Wolkowicz and Styan [3, Theorem 2.1.].

Theorem 1.1. *Let*

$$x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq x_n. \tag{1.2}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 43465, Pages 1–15
DOI 10.1155/JIA/2006/43465

Then

$$m(\mathbf{x})+\frac{1}{\sqrt{n-1}}s(\mathbf{x})\le x_1, \tag{1.3}$$

$$x_1\le m(\mathbf{x})+\sqrt{n-1}s(\mathbf{x}). \tag{1.4}$$

Equality holds in (1.3) if and only if $x_1=x_2=\cdots=x_{n-1}$. *Equality holds in (1.4) if and only if* $x_2=x_3=\cdots=x_n$.

Let $x_1,x_2,\dots,x_{n-1},x_n$ be complex numbers such that x_1 is a positive real number and

$$x_1\ge |x_2|\ge\cdots\ge |x_{n-1}|\ge |x_n|. \tag{1.5}$$

Then,

$$x_1^p\ge |x_2|^p\ge\cdots\ge |x_{n-1}|^p\ge |x_n|^p \tag{1.6}$$

for any positive integer p. We apply Theorem 1.1 to (1.6) to obtain

$$\begin{gathered} m(|\mathbf{x}|^p)+\frac{1}{\sqrt{n-1}}s(|\mathbf{x}|^p)\le x_1^p,\\ x_1^p\le m(|\mathbf{x}|^p)+\sqrt{n-1}s(|\mathbf{x}|^p), \end{gathered} \tag{1.7}$$

where $|\mathbf{x}|=(|x_1|,|x_2|,\dots,|x_{n-1}|,|x_n|)$.

Then,

$$l_p(\mathbf{x})=\left(m(|\mathbf{x}|^p)+\frac{1}{\sqrt{n-1}}s(|\mathbf{x}|^p)\right)^{1/p} \tag{1.8}$$

is a sequence of lower bounds for x_1 and

$$u_p(\mathbf{x})=\left(m(|\mathbf{x}|^p)+\sqrt{n-1}s(|\mathbf{x}|^p)\right)^{1/p} \tag{1.9}$$

is a sequence of upper bounds for x_1.

We recall that the *p-norm* and the *infinity-norm* of a vector $\mathbf{x}=(x_1,x_2,\dots,x_n)$ are

$$\begin{gathered} \|\mathbf{x}\|_p=\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p},\quad 1\le p<\infty,\\ \|\mathbf{x}\|_\infty=\max_i |x_i|. \end{gathered} \tag{1.10}$$

It is well known that $\lim_{p\to\infty}\|\mathbf{x}\|_p=\|\mathbf{x}\|_\infty$.

Then,

$$l_p(\mathbf{x}) = \left(\frac{\|\mathbf{x}\|_p^p}{n} + \frac{1}{\sqrt{n(n-1)}} \sqrt{\|\mathbf{x}\|_{2p}^{2p} - \frac{\|\mathbf{x}\|_p^{2p}}{n}} \right)^{1/p},$$
$$u_p(\mathbf{x}) = \left(\frac{\|\mathbf{x}\|_p^p}{n} + \sqrt{\frac{n-1}{n}} \sqrt{\|\mathbf{x}\|_{2p}^{2p} - \frac{\|\mathbf{x}\|_p^{2p}}{n}} \right)^{1/p}. \tag{1.11}$$

In [2, Theorem 11], we proved that if $y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n \geq 0$, then

$$m(\mathbf{y}^{2^p}) + \sqrt{n-1}s(\mathbf{y}^{2^p}) \geq \sqrt{m(\mathbf{y}^{2^{p+1}}) + \sqrt{n-1}s(\mathbf{y}^{2^{p+1}})} \tag{1.12}$$

for $p = 0,1,2,\ldots$. The equality holds if and only if $y_2 = y_3 = \cdots = y_n$. Using this inequality, we proved in [2, Theorems 14 and 15] that if $y_2 = y_3 = \cdots = y_n$, then $u_p(\mathbf{y}) = y_1$ for all p, and if $y_i < y_j$ for some $2 \leq j < i \leq n$, then $(u_{2^p}(\mathbf{y}))_{p=0}^{\infty}$ is a strictly decreasing sequence converging to y_1.

The main purpose of this paper is to prove that if $y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n \geq 0$, then

$$m(\mathbf{y}^{2^p}) + \frac{1}{\sqrt{n-1}}s(\mathbf{y}^{2^p}) \leq \sqrt{m(\mathbf{y}^{2^{p+1}}) + \frac{1}{\sqrt{n-1}}s(\mathbf{y}^{2^{p+1}})} \tag{1.13}$$

for $p = 0,1,2,\ldots$. The equality holds if and only if $y_1 = y_2 = \cdots = y_{n-1}$. Using this inequality, we prove that if $y_1 = y_2 = \cdots = y_{n-1}$, then $u_p(\mathbf{y}) = y_1$ for all p, and if $y_i < y_j$ for some $1 \leq j < i \leq n-1$, then $(l_{2^p}(\mathbf{y}))_{p=0}^{\infty}$ is a strictly increasing sequence converging to y_1.

2. New inequalities involving $m(\mathbf{x})$ and $s(\mathbf{x})$

Theorem 2.1. *Let $\mathbf{x} = (x_1,x_2,\ldots,x_{n-1},x_n)$ be a vector of complex numbers such that x_1 is a positive real number and*

$$x_1 \geq |x_2| \geq \cdots \geq |x_{n-1}| \geq |x_n|. \tag{2.1}$$

The sequence $(l_p(\mathbf{x}))_{p=1}^{\infty}$ converges to x_1.

Proof. From (1.11),

$$l_p(\mathbf{x}) \geq \frac{\|\mathbf{x}\|_p}{\sqrt[p]{n}} \quad \forall p. \tag{2.2}$$

Then, $0 \leq |l_p(\mathbf{x}) - x_1| = x_1 - l_p(\mathbf{x}) \leq x_1 - \|\mathbf{x}\|_p/\sqrt[p]{n}$ for all p. Since $\lim_{p\to\infty} \| \mathbf{x} \|_p = x_1$ and $\lim_{p\to\infty} \sqrt[p]{n} = 1$, it follows that the sequence $(l_p(\mathbf{x}))$ converges and $\lim_{p\to\infty} l_p(\mathbf{x}) = x_1$. □

We introduce the following notations:

(i) $\mathbf{e} = (1,1,\ldots,1)$,
(ii) $\mathcal{D} = \mathbb{R}^n - \{\lambda\mathbf{e} : \lambda \in \mathbb{R}\}$,
(iii) $\mathcal{C} = \{\mathbf{x} = (x_1,x_2,\ldots,x_n) : 0 \leq x_k \leq 1, k = 1,2,\ldots,n\}$,

(iv) $\mathscr{E} = \{\mathbf{x} = (1, x_2, \ldots, x_n) : 0 \leq x_n \leq x_{n-1} \leq \cdots \leq x_2 \leq 1\}$,
(v) $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
(vi) $\nabla g(\mathbf{x}) = (\partial_1 g(\mathbf{x}), \partial_2 g(\mathbf{x}), \ldots, \partial_n g(\mathbf{x}))$ denotes the gradient of a differentiable function g at the point $\mathbf{x}$, where $\partial_k g(\mathbf{x})$ is the partial derivative of g with respect to x_k, evaluated at $\mathbf{x}$.

Clearly, if $\mathbf{x} \in \mathscr{E}$, then $\mathbf{x}^q \in \mathscr{E}$ with q a positive integer.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the points

$$\begin{aligned} \mathbf{v}_1 &= (1, 0, \ldots, 0), \\ \mathbf{v}_2 &= (1, 1, 0, \ldots, 0), \\ \mathbf{v}_3 &= (1, 1, 1, 0, \ldots, 0), \\ &\vdots \\ \mathbf{v}_{n-2} &= (1, 1, \ldots, 1, 0, 0), \\ \mathbf{v}_{n-1} &= (1, 1, \ldots, 1, 1, 0), \\ \mathbf{v}_n &= (1, 1, \ldots, 1, 1) = \mathbf{e}. \end{aligned} \tag{2.3}$$

Observe that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ lie in $\mathscr{E}$. For any $\mathbf{x} = (1, x_2, x_3, \ldots, x_{n-1}, x_n) \in \mathscr{E}$, we have

$$\begin{aligned} \mathbf{x} = {} & (1 - x_2)\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_3 - x_4)\mathbf{v}_3 \\ & + \cdots + (x_{n-2} - x_{n-1})\mathbf{v}_{n-2} + (x_{n-1} - x_n)\mathbf{v}_{n-1} + x_n \mathbf{v}_n. \end{aligned} \tag{2.4}$$

Therefore, $\mathscr{E}$ is a convex set. We define the function

$$f(\mathbf{x}) = m(\mathbf{x}) + \frac{1}{\sqrt{n-1}} s(\mathbf{x}), \tag{2.5}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We observe that

$$\begin{aligned} n s^2(\mathbf{x}) &= \sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{j=1}^{n} x_j\right)^2}{n} = \sum_{k=1}^{n} (x_k - m(\mathbf{x}))^2 \\ &= \|\mathbf{x} - m(\mathbf{x})\mathbf{e}\|_2^2. \end{aligned} \tag{2.6}$$

Then,

$$\begin{aligned} f(\mathbf{x}) &= m(\mathbf{x}) + \frac{1}{\sqrt{n(n-1)}} \|\mathbf{x} - m(\mathbf{x})\mathbf{e}\|_2 \\ &= \frac{\sum_{j=1}^{n} x_j}{n} + \frac{1}{\sqrt{n(n-1)}} \sqrt{\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{j=1}^{n} x_j\right)^2}{n}}. \end{aligned} \tag{2.7}$$

Next, we give properties of f. Some of the proofs are similar to those in [2].

Lemma 2.2. *The function f has continuous first partial derivatives on $\mathcal{D}$, and for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{D}$ and $1 \le k \le n$,*

$$\partial_k f(\mathbf{x}) = \frac{1}{n} + \frac{1}{n(n-1)} \frac{x_k - m(\mathbf{x})}{f(\mathbf{x}) - m(\mathbf{x})}, \tag{2.8}$$

$$\sum_{k=1}^{n} \partial_k f(\mathbf{x}) = 1, \tag{2.9}$$

$$\langle \nabla f(\mathbf{x}), \mathbf{x} \rangle = f(\mathbf{x}). \tag{2.10}$$

Proof. From (2.7), it is clear that f is differentiable at every point $\mathbf{x} \neq m(\mathbf{x})\mathbf{e}$, and for $1 \le k \le n$,

$$\begin{aligned} \partial_k f(\mathbf{x}) &= \frac{1}{n} + \frac{1}{\sqrt{n(n-1)}} \frac{x_k - \sum_{j=1}^{n} x_j/n}{\sqrt{\sum_{i=1}^{n} x_i^2 - \left(\sum_{j=1}^{n} x_j\right)^2/n}} \\ &= \frac{1}{n} + \frac{1}{n(n-1)} \frac{x_k - m(\mathbf{x})}{f(\mathbf{x}) - m(\mathbf{x})}, \end{aligned} \tag{2.11}$$

which is a continuous function on $\mathcal{D}$. Then, $\sum_{k=1}^{n} \partial_k f(\mathbf{x}) = 1$. Finally,

$$\begin{aligned} \langle \nabla f(\mathbf{x}), \mathbf{x} \rangle &= \sum_{k=1}^{n} x_k \partial_k f(\mathbf{x}) \\ &= \frac{\sum_{k=1}^{n} x_k}{n} + \frac{1}{n(n-1)} \frac{\sum_{k=1}^{n} x_k^2 - m(\mathbf{x}) \sum_{k=1}^{n} x_k}{f(\mathbf{x}) - m(\mathbf{x})} \\ &= m(\mathbf{x}) + \frac{1}{\sqrt{n(n-1)}} \|\mathbf{x} - a(\mathbf{x})\mathbf{e}\|_2 = f(\mathbf{x}). \end{aligned} \tag{2.12}$$

This completes the proof. □

Lemma 2.3. *The function f is convex on $\mathcal{C}$. More precisely, for $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and $t \in [0,1]$,*

$$f((1-t)\mathbf{x} + t\mathbf{y}) \le (1-t) f(\mathbf{x}) + t f(\mathbf{y}) \tag{2.13}$$

with equality if and only if

$$\mathbf{x} - m(\mathbf{x})\mathbf{e} = \alpha(\mathbf{y} - m(\mathbf{y})\mathbf{e}) \tag{2.14}$$

for some $\alpha \ge 0$.

Proof. Clearly $\mathcal{C}$ is a convex set. Let $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and $t \in [0,1]$. Then,

$$\begin{aligned} f((1-t)\mathbf{x} + t\mathbf{y}) &= m((1-t)\mathbf{x} + t\mathbf{y}) + \frac{1}{\sqrt{n(n-1)}} \|(1-t)\mathbf{x} + t\mathbf{y} - m((1-t)\mathbf{x} + t\mathbf{y})\mathbf{e}\|_2 \\ &= (1-t)m(\mathbf{x}) + t m(\mathbf{y}) + \frac{1}{\sqrt{n(n-1)}} \|(1-t)(\mathbf{x} - m(\mathbf{x})\mathbf{e}) + t(\mathbf{y} - m(\mathbf{y})\mathbf{e})\|_2. \end{aligned} \tag{2.15}$$

Moreover,

$$\begin{aligned}&\|(1-t)(\mathbf{x}-m(\mathbf{x})\mathbf{e})+t(\mathbf{y}-m(\mathbf{y})\mathbf{e})\|_2^2\\&\quad=(1-t)^2\|\mathbf{x}-m(\mathbf{x})\mathbf{e}\|_2^2+2(1-t)t\langle\mathbf{x}-m(\mathbf{x})\mathbf{e},\mathbf{y}-m(\mathbf{y})\mathbf{e}\rangle+t^2\|\mathbf{y}-m(\mathbf{y})\mathbf{e}\|_2^2.\end{aligned}\tag{2.16}$$

We recall the Cauchy-Schwarz inequality to obtain

$$\langle\mathbf{x}-m(\mathbf{x})\mathbf{e},\mathbf{y}-m(\mathbf{y})\mathbf{e}\rangle\le\|\mathbf{x}-m(\mathbf{x})\mathbf{e}\|_2\|\mathbf{y}-m(\mathbf{y})\mathbf{e}\|_2\tag{2.17}$$

with equality if and only if (2.14) holds. Thus,

$$\|(1-t)(\mathbf{x}-m(\mathbf{x})\mathbf{e})+t(\mathbf{y}-m(\mathbf{y})\mathbf{e})\|_2\le(1-t)\|\mathbf{x}-m(\mathbf{x})\mathbf{e}\|_2+t\|\mathbf{y}-m(\mathbf{y})\mathbf{e}\|_2\tag{2.18}$$

with equality if and only if (2.14) holds. Finally, from (2.15) and (2.18), the lemma follows. □

Lemma 2.4. *For* $\mathbf{x},\mathbf{y}\in\mathscr{E}-\{\mathbf{e}\}$,

$$f(\mathbf{x})\ge\langle\nabla f(\mathbf{y}),\mathbf{x}\rangle\tag{2.19}$$

with equality if and only if (2.14) holds for some $\alpha>0$.

Proof. $\mathscr{E}$ is a convex subset of $\mathscr{C}$ and f is a convex function on $\mathscr{E}$. Moreover, f is a differentiable function on $\mathscr{E}-\{\mathbf{e}\}$. Let $\mathbf{x},\mathbf{y}\in\mathscr{E}-\{\mathbf{e}\}$. For all $t\in[0,1]$,

$$f(t\mathbf{x}+(1-t)\mathbf{y})\le tf(\mathbf{x})+(1-t)f(\mathbf{y}).\tag{2.20}$$

Thus, for $0<t\le 1$,

$$\frac{f(\mathbf{y}+t(\mathbf{x}-\mathbf{y}))-f(\mathbf{y})}{t}\le f(\mathbf{x})-f(\mathbf{y}).\tag{2.21}$$

Letting $t\to 0^+$ yields

$$\lim_{t\to 0^+}\frac{f(\mathbf{y}+t(\mathbf{x}-\mathbf{y}))-f(\mathbf{y})}{t}=\langle\nabla f(\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle\le f(\mathbf{x})-f(\mathbf{y}).\tag{2.22}$$

Hence,

$$f(\mathbf{x})-f(\mathbf{y})\ge\langle\nabla f(\mathbf{y}),\mathbf{x}\rangle-\langle\nabla f(\mathbf{y}),\mathbf{y}\rangle.\tag{2.23}$$

Now, we use the fact that $\langle\nabla f(\mathbf{y}),\mathbf{y}\rangle=f(\mathbf{y})$ to conclude that

$$f(\mathbf{x})\ge\langle\nabla f(\mathbf{y}),\mathbf{x}\rangle.\tag{2.24}$$

The equality in all the above inequalities holds if and only if $\mathbf{x}-a(\mathbf{x})\mathbf{e}=\alpha(\mathbf{y}-m(\mathbf{y})\mathbf{e})$ for some $\alpha\ge 0$. □

Corollary 2.5. *For* $\mathbf{x} \in \mathcal{E} - \{\mathbf{e}\}$,

$$f(\mathbf{x}) \geq \langle \nabla f(\mathbf{x}^2), \mathbf{x} \rangle, \tag{2.25}$$

where $\nabla f(\mathbf{x}^2)$ *is the gradient of* f *with respect to* $\mathbf{x}$ *evaluated at* $\mathbf{x}^2$*. The equality in (2.25) holds if and only if* $\mathbf{x}$ *is one of the following convex combinations:*

$$\mathbf{x}_i(t) = t\mathbf{e} + (1-t)\mathbf{v}_i, \quad i = 1, 2, \ldots, n-1, \text{ some } t \in [0,1). \tag{2.26}$$

Proof. Let $\mathbf{x} = (1, x_2, x_3, \ldots, x_m) \in \mathcal{E} - \{\mathbf{e}\}$. Then, $\mathbf{x}^2 \in \mathcal{E} - \{\mathbf{e}\}$. Using Lemma 2.4, we obtain

$$f(\mathbf{x}) \geq \langle \nabla f(\mathbf{x}^2), \mathbf{x} \rangle \tag{2.27}$$

with equality if and only if

$$\mathbf{x} - m(\mathbf{x})\mathbf{e} = \alpha(\mathbf{x}^2 - m(\mathbf{x}^2)\mathbf{e}) \tag{2.28}$$

for some $\alpha \geq 0$. Thus, we have proved (2.25). In order to complete the proof, we observe that condition (2.28) is equivalent to

$$\mathbf{x} - \alpha\mathbf{x}^2 = m(\mathbf{x} - \alpha\mathbf{x}^2)\mathbf{e} \tag{2.29}$$

for some $\alpha \geq 0$. Since $x_1 = 1$, (2.29) is equivalent to

$$1 - \alpha = x_2 - \alpha x_2^2 = x_3 - \alpha x_3^2 = \cdots = x_n - \alpha x_n^2 \tag{2.30}$$

for some $\alpha \geq 0$. Hence, (2.28) is equivalent to (2.30).

Suppose that (2.30) is true. If $\alpha = 0$, then $1 = x_2 = \cdots = x_n$. This is a contradiction because $\mathbf{x} \neq \mathbf{e}$, thus $\alpha > 0$.

If $x_2 = 0$, then $x_3 = x_4 = \cdots = x_n = 0$, and thus $\mathbf{x} = \mathbf{v}_1$. Let $0 < x_2 < 1$. Suppose $x_3 < x_2$. From (2.30),

$$\begin{aligned} 1 - x_2 &= \alpha(1 + x_2)(1 - x_2), \\ x_2 - x_3 &= \alpha(x_2 + x_3)(x_2 - x_3). \end{aligned} \tag{2.31}$$

From these equations, we obtain $x_3 = 1$, which is a contradiction. Hence, $0 < x_2 < 1$ implies $x_3 = x_2$. Now, if $x_4 < x_3$, from $x_2 = x_3$ and the equations

$$\begin{aligned} 1 - x_2 &= \alpha(1 + x_2)(1 - x_2), \\ x_3 - x_4 &= \alpha(x_3 + x_4)(x_3 - x_4), \end{aligned} \tag{2.32}$$

we obtain $x_4 = 1$, which is a contradiction. Hence, $x_4 = x_3$ if $0 < x_2 < 1$. We continue in this fashion to conclude that $x_n = x_{n-1} = \cdots = x_3 = x_2$. We have proved that $x_1 = 1$ and $0 \leq x_2 < 1$ imply that $\mathbf{x} = (1, t, \ldots, t) = t\mathbf{e} + (1-t)\mathbf{v}_1$ for some $t \in [0,1)$. Let $x_2 = 1$.

If $x_3 = 0$, then $x_4 = x_5 = \cdots = x_m = 0$, and thus $\mathbf{x} = \mathbf{v}_2$. Let $0 < x_3 < 1$ and $x_4 < x_3$. From (2.30),

$$\begin{aligned} 1 - x_3 &= \alpha(1 + x_3)(1 - x_3), \\ x_3 - x_4 &= \alpha(x_3 + x_4)(x_3 - x_4). \end{aligned} \tag{2.33}$$

From these equations, we obtain $x_4 = 1$, which is a contradiction. Hence, $0 < x_3 < 1$ implies $x_4 = x_3$. Now, if $x_5 < x_4$, from $x_3 = x_4$ and the equations

$$\begin{aligned} 1 - x_3 &= \alpha(1 + x_3)(1 - x_3), \\ x_4 - x_5 &= \alpha(x_4 + x_5)(x_4 - x_5), \end{aligned} \tag{2.34}$$

we obtain $x_5 = 1$, which is a contradiction. Therefore, $x_5 = x_4$. We continue in this fashion to get $x_n = x_{n-1} = \cdots = x_3$. Thus, $x_1 = x_2 = 1$, and $0 \le x_3 < 1$ implies that $\mathbf{x} = (1, 1, t, \ldots, t) = t\mathbf{e} + (1 - t)\mathbf{v}_2$ for some $t \in [0, 1)$.

For $3 \le k \le n - 2$, arguing as above, it can be proved that $x_1 = x_2 = \cdots = x_k = 1$ and $0 \le x_{k+1} < 1$ implies that $\mathbf{x} = (1, \ldots, 1, t, \ldots, t) = t\mathbf{e} + (1 - t)\mathbf{v}_k$. Finally, for $x_1 = x_2 = \cdots = x_{n-1} = 1$ and $0 \le x_n < 1$, we have $\mathbf{x} = t\mathbf{e} + \mathbf{v}_{n-1}$.

Conversely, if $\mathbf{x}$ is any of the convex combinations in (2.26), then (2.30) holds by choosing $\alpha = 1/(1 + t)$. □

Let us define the following optimization problem.

Problem 2.6. Let

$$F : \mathbb{R}^n \longrightarrow \mathbb{R} \tag{2.35}$$

be given by

$$F(\mathbf{x}) = f(\mathbf{x}^2) - (f(\mathbf{x}))^2. \tag{2.36}$$

We want to find $\min_{\mathbf{x} \in \mathscr{E}} F(\mathbf{x})$. That is, find

$$\min F(\mathbf{x}) \tag{2.37}$$

subject to the constraints

$$\begin{aligned} h_1(\mathbf{x}) &= x_1 - 1 = 0, \\ h_i(\mathbf{x}) &= x_i - x_{i-1} \le 0, \quad 2 \le i \le n, \\ h_{n+1}(\mathbf{x}) &= -x_n \le 0. \end{aligned} \tag{2.38}$$

Lemma 2.7. (1) *If $\mathbf{x} \in \mathscr{E} - \{\mathbf{e}\}$, then $\sum_{k=1}^{n} \partial_k F(\mathbf{x}) \le 0$ with equality if and only if $\mathbf{x}$ is one of the convex combinations $\mathbf{x}_k(t)$ in (2.26).*

(2) *If $\mathbf{x} = \mathbf{x}_N(t)$ with $1 \le N \le n - 2$, then*

$$\partial_1 F(\mathbf{x}) = \cdots = \partial_N F(\mathbf{x}) > 0, \tag{2.39}$$

$$\partial_{N+1} F(\mathbf{x}) = \cdots = \partial_n F(\mathbf{x}) < 0. \tag{2.40}$$

Proof. (1) The function F has continuous first partial derivatives on $\mathcal{D}$, and for $\mathbf{x} \in \mathcal{D}$ and $1 \leq k \leq n$,

$$\partial_k F(\mathbf{x}) = 2x_k \partial_k f(\mathbf{x}^2) - 2f(\mathbf{x})\partial_k f(\mathbf{x}). \tag{2.41}$$

By (2.9),

$$\begin{aligned}\sum_{k=1}^{n} \partial_k F(\mathbf{x}) &= 2\sum_{k=1}^{n} x_k \partial_k f(\mathbf{x}^2) - 2f(\mathbf{x})\sum_{k=1}^{n} \partial_k f(\mathbf{x}) \\ &= 2\langle \nabla f(\mathbf{x}^2), \mathbf{x}\rangle - 2f(\mathbf{x}).\end{aligned} \tag{2.42}$$

It follows from Corollary 2.5 that $\sum_{k=1}^{n} \partial_k F(\mathbf{x}) \leq 0$ with equality if and only if $\mathbf{x}_i = t\mathbf{e} + (1-t)\mathbf{v}_i$, $i = 1,\ldots,n-1$.

(2) Let $\mathbf{x} = \mathbf{x}_N(t)$ with $1 \leq N \leq n-2$ fixed. Then, $\mathbf{x} = t\mathbf{e} + (1-t)\mathbf{v}_N$, some $t \in [0,1)$. Thus, $x_1 = x_2 = \cdots = x_N = 1$, $x_{N+1} = x_{N+2} = \cdots = x_n = t$. From Theorem 1.1, $f(\mathbf{x}) < 1$. Moreover,

$$\begin{aligned} f(\mathbf{x}) - m(\mathbf{x}) &= \sqrt{\frac{1}{n(n-1)}}\sqrt{N + (n-N)t^2 - \frac{(N+(n-N)t)^2}{n}} \\ &= \sqrt{\frac{1}{n(n-1)}}\sqrt{\frac{nN + n(n-N)t^2 - N^2 - 2N(n-N)t - (n-N)^2t^2}{n}} \\ &= \frac{1}{n\sqrt{n-1}}\sqrt{N(n-N)}(1-t). \end{aligned} \tag{2.43}$$

Replacing this result in (2.8), we obtain

$$\begin{aligned} \partial_1 f(\mathbf{x}) = \partial_2 f(\mathbf{x}) &= \cdots = \partial_N f(\mathbf{x}) \\ &= \frac{1}{n} + \frac{1}{n(n-1)}\frac{1-m(\mathbf{x})}{f(\mathbf{x}) - m(\mathbf{x})} \\ &= \frac{1}{n} + \frac{1}{\sqrt{n-1}}\frac{1-(N+(n-N)t)/n}{\sqrt{N(n-N)}(1-t)} \\ &= \frac{1}{n} + \frac{1}{\sqrt{n-1}n}\frac{\sqrt{n-N}}{\sqrt{N}} > 0. \end{aligned} \tag{2.44}$$

Similarly,

$$\begin{aligned} f(\mathbf{x}^2) - m(\mathbf{x}^2) &= \frac{1}{n\sqrt{n-1}}\sqrt{N(n-N)}(1-t^2), \\ \partial_1 f(\mathbf{x}^2) = \partial_2 f(\mathbf{x}^2) &= \cdots = \partial_N f(\mathbf{x}^2) \\ &= \frac{1}{n} + \frac{1}{n\sqrt{n-1}}\frac{\sqrt{n-N}}{\sqrt{N}} > 0. \end{aligned} \tag{2.45}$$

Therefore,

$$\begin{aligned}\partial_1 F(\mathbf{x}) &= \partial_2 F(\mathbf{x}) = \cdots = \partial_N F(\mathbf{x}) \\ &= 2\partial_1 f(\mathbf{x}^2) - 2f(\mathbf{x})\partial_1 f(\mathbf{x}) = 2(1-f(\mathbf{x}))\partial_1 f(\mathbf{x}) > 0.\end{aligned} \tag{2.46}$$

We have thus proved (2.39). We easily see that

$$\partial_{N+1} F(\mathbf{x}) = \partial_{N+2} F(\mathbf{x}) = \cdots = \partial_n F(\mathbf{x}). \tag{2.47}$$

We have $\sum_{k=1}^{n} \partial_k F(\mathbf{x}) = 0$. Hence,

$$\sum_{k=N+1}^{n} \partial_k F(\mathbf{x}) = (n-N)\partial_{N+1} F(\mathbf{x}) = -\sum_{k=1}^{N} \partial_k F(\mathbf{x}) < 0. \tag{2.48}$$

Thus, (2.40) follows. □

We recall the following necessary condition for the existence of a minimum in nonlinear programming.

THEOREM 2.8 (see [1, Theorem 9.2-4(1)]). *Let $J : \Omega \subseteq V \to \mathbb{R}$ be a function defined over an open, convex subset Ω of a Hilbert space V and let*

$$U = \{\mathbf{v} \in \Omega : \varphi_i(\mathbf{v}) \leq 0,\ 1 \leq i \leq m\} \tag{2.49}$$

be a subset of Ω, the constraints $\varphi_i : \Omega \to \mathbb{R}$, $1 \leq i \leq m$, being assumed to be convex. Let $\mathbf{u} \in U$ be a point at which the functions φ_i, $1 \leq i \leq m$, and J are differentiable. If the function J has at $\mathbf{u}$ a relative minimum with respect to the set U and if the constraints are qualified, then there exist numbers $\lambda_i(\mathbf{u})$, $1 \leq i \leq m$, such that the Kuhn-Tucker conditions

$$\begin{gathered}\nabla J(\mathbf{u}) + \sum_{i=1}^{m} \lambda_i(\mathbf{u}) \nabla \varphi_i(\mathbf{u}) = \mathbf{0}, \\ \lambda_i(\mathbf{u}) \geq 0, \quad 1 \leq i \leq m, \quad \sum_{i=1}^{m} \lambda_i(\mathbf{u}) \varphi_i(\mathbf{u}) = 0\end{gathered} \tag{2.50}$$

are satisfied.

The convex constraints φ_i in the above necessary condition are said to be qualified if either all the functions φ_i are affine and the set U is nonempty, or there exists a point $\mathbf{w} \in \Omega$ such that for each i, $\varphi_i(\mathbf{w}) \leq 0$ with strict inequality holding if φ_i is not affine.

The solution to Problem 2.6 is given in the following theorem.

THEOREM 2.9. *One has*

$$\min_{\mathbf{x} \in \mathscr{E}} F(\mathbf{x}) = 0 = F(1, 1, 1, \ldots, 1, t) \tag{2.51}$$

for any $t \in [0, 1]$.

Proof. We observe that $\mathscr{E}$ is a compact set and F is a continuous function on $\mathscr{E}$. Then, there exists $\mathbf{x}_0 \in \mathscr{E}$ such that $F(\mathbf{x}_0) = \min_{\mathbf{x}\in\mathscr{E}} F(\mathbf{x})$. The proof is based on the application of the necessary condition given in the preceding theorem. In Problem 2.6, we have $\Omega = V = \mathbb{R}^n$ with the inner product $\langle \mathbf{x},\mathbf{y}\rangle = \sum_{k=1}^{n} x_k y_k$, $\varphi_i(\mathbf{x}) = h_i(\mathbf{x})$, $1 \le i \le n+1$, $U = \mathscr{E}$ and $J = F$. The functions h_i, $2 \le i \le n+1$, are linear. Therefore, they are convex and affine. In addition, the function $h_1(\mathbf{x}) = x_1 - 1$ is affine and convex and $\mathscr{E}$ is nonempty. Consequently, the functions h_i, $1 \le i \le n+1$, are qualified. Moreover, these functions and the objective function F are differentiable at any point in $\mathscr{E} - \{\mathbf{e}\}$. The gradients of the constraint functions are

$$\begin{aligned}
\nabla h_1(\mathbf{x}) &= (1,0,0,0,\dots,0) = \mathbf{e}_1,\\
\nabla h_2(\mathbf{x}) &= (-1,1,0,0,\dots,0),\\
\nabla h_3(\mathbf{x}) &= (0,-1,1,0,\dots,0),\\
&\ \ \vdots\\
\nabla h_{n-1}(\mathbf{x}) &= (0,0,\dots,0,-1,1,0),\\
\nabla h_n(\mathbf{x}) &= (0,0,\dots,0,-1,1),\\
\nabla h_{n+1}(\mathbf{x}) &= (0,0,\dots,0,-1).
\end{aligned} \tag{2.52}$$

Suppose that F has a relative minimum at $\mathbf{x} \in \mathscr{E}-\{\mathbf{e}\}$ with respect to the set $\mathscr{E}$. Then, there exist $\lambda_i(\mathbf{x}) \ge 0$ (for brevity $\lambda_i = \lambda_i(\mathbf{x})$), $1 \le i \le n+1$, such that the Kuhn-Tucker conditions

$$\begin{aligned}
\nabla F(\mathbf{x}) + \sum_{i=1}^{n+1} \lambda_i \nabla h_i(\mathbf{x}) &= \mathbf{0},\\
\sum_{i=1}^{n+1} \lambda_i h_i(\mathbf{x}) &= 0
\end{aligned} \tag{2.53}$$

hold. Hence,

$$\nabla F(\mathbf{x}) + (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \dots, \lambda_n - \lambda_{n+1}) = \mathbf{0}, \tag{2.54}$$

$$\lambda_2(x_2 - 1) + \lambda_3(x_3 - x_2) + \cdots + \lambda_n(x_n - x_{n-1}) + \lambda_{n+1}(-x_n) = 0. \tag{2.55}$$

From (2.55), as $\lambda_i \ge 0$, $1 \le i \le n+1$, and $0 \le x_n \le x_{n-1} \le \cdots \le x_2 \le 1$, we have

$$\lambda_k(x_{k-1} - x_k) = 0, \quad 2 \le k \le n, \quad \lambda_{n+1} x_n = 0. \tag{2.56}$$

Now, from (2.54),

$$\sum_{k=1}^{n} \partial_k F(\mathbf{x}) + \lambda_1 - \lambda_{n+1} = 0. \tag{2.57}$$

We will conclude that $\lambda_1 = 0$ by showing that the cases $\lambda_1 > 0$, $x_n > 0$ and $\lambda_1 > 0$, $x_n = 0$ yield contradictions.

Suppose $\lambda_1 > 0$ and $x_n > 0$. In this case, $\lambda_{n+1}x_n = 0$ implies $\lambda_{n+1} = 0$. Thus, (2.57) becomes

$$\sum_{k=1}^{n} \partial_k F(\mathbf{x}) = -\lambda_1 < 0. \tag{2.58}$$

We apply Lemma 2.7 to conclude that **x** is not one of the convex combinations in (2.26). From (2.4),

$$\begin{aligned} \mathbf{x} = {} & (1 - x_2)\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_3 - x_4)\mathbf{v}_3 \\ & + \cdots + (x_{n-2} - x_{n-1})\mathbf{v}_{n-2} + (x_{n-1} - x_n)\mathbf{v}_{n-1} + x_n\mathbf{v}_n. \end{aligned} \tag{2.59}$$

Then, there are at least two indexes i, j such that

$$1 = \cdots = x_i > x_{i+1} = \cdots = x_j > x_{j+1}. \tag{2.60}$$

Therefore,

$$\begin{aligned} \partial_1 F(\mathbf{x}) &= \cdots = \partial_i F(\mathbf{x}), \\ \partial_{i+1} F(\mathbf{x}) &= \cdots = \partial_j F(\mathbf{x}). \end{aligned} \tag{2.61}$$

From (2.56), we get $\lambda_{i+1} = 0$ and $\lambda_{j+1} = 0$. Now, from (2.54),

$$\begin{aligned} \partial_i F(\mathbf{x}) &= -\lambda_i \le 0, \\ \partial_{i+1} F(\mathbf{x}) &= \lambda_{i+2} \ge 0, \\ \partial_j F(\mathbf{x}) &= -\lambda_j \le 0, \\ \partial_n F(\mathbf{x}) &= -\lambda_n \le 0. \end{aligned} \tag{2.62}$$

The above equalities and inequalities together with (2.8) and (2.41) give

$$\frac{1}{n}(1 - f(\mathbf{x})) + \frac{1}{n(n-1)}\left(\frac{1 - m(\mathbf{x}^2)}{f(\mathbf{x}^2) - m(\mathbf{x}^2)} - \frac{1 - m(\mathbf{x})}{f(\mathbf{x}) - m(\mathbf{x})}\right) \le 0, \tag{2.63}$$

$$\frac{1}{n}(1 - f(\mathbf{x})) + \frac{1}{n(n-1)}\left(\frac{x_j^2 - m(\mathbf{x}^2)}{f(\mathbf{x}^2) - m(\mathbf{x}^2)} - \frac{x_j - m(\mathbf{x})}{f(\mathbf{x}) - m(\mathbf{x})}\right) = 0, \tag{2.64}$$

$$\frac{1}{n}(1 - f(\mathbf{x})) + \frac{1}{n(n-1)}\left(\frac{x_n^2 - m(\mathbf{x}^2)}{f(\mathbf{x}^2) - m(\mathbf{x}^2)} - \frac{x_n - m(\mathbf{x})}{f(\mathbf{x}) - m(\mathbf{x})}\right) \le 0. \tag{2.65}$$

Subtracting (2.64) from (2.63) and (2.65), we obtain

$$\begin{aligned} \frac{1 - x_j^2}{f(\mathbf{x}^2) - m(\mathbf{x}^2)} &\le \frac{1 - x_j}{f(\mathbf{x}^2) - m(\mathbf{x}^2)}, \\ \frac{x_n^2 - x_j^2}{f(\mathbf{x}^2) - m(\mathbf{x}^2)} &\le \frac{x_n - x_j}{f(\mathbf{x}^2) - m(\mathbf{x}^2)}. \end{aligned} \tag{2.66}$$

Dividing these inequalities by $(1-x_j)$ and (x_n-x_j), respectively, we get

$$\frac{1+x_j}{f(\mathbf{x}^2)-m(\mathbf{x}^2)} \leq \frac{1}{f(\mathbf{x}^2)-m(\mathbf{x}^2)}, \qquad \frac{x_n+x_j}{f(\mathbf{x}^2)-a(\mathbf{x}^2)} \geq \frac{1}{f(\mathbf{x}^2)-a(\mathbf{x}^2)}. \tag{2.67}$$

The last two inequalities imply $x_n \geq x_j$, which is contradiction.

Suppose now that $\lambda_1 > 0$ and $x_n = 0$. Let l be the largest index such that $x_l > 0$. Thus, $x_{l+1} = 0$. From (2.55),

$$\lambda_2(x_2-1)+\lambda_3(x_3-x_2)+\cdots+\lambda_l(x_l-x_{l-1})+\lambda_{l+1}(-x_l)=0. \tag{2.68}$$

Then,

$$\lambda_k(x_{k-1}-x_k)=0, \quad 2\leq k\leq l, \qquad \lambda_{l+1}x_l=0. \tag{2.69}$$

Hence, $\lambda_{l+1}=0$. If $l=n-1$, then $\lambda_n=0$ and $\partial_n F(\mathbf{x})=\lambda_{n+1}\geq 0$. If $l\leq n-2$, then $\partial_l F(\mathbf{x}) = -\lambda_l \leq 0$. In both situations, we conclude that $\mathbf{x}$ is not one of the convex combinations in (2.26). Therefore, there are at least two indexes i, j such that

$$1=\cdots=x_i>x_{i+1}=\cdots=x_j>x_{j+1}. \tag{2.70}$$

Now, we repeat the argument used above to get that $x_l \geq x_j$, which is a contradiction.

Consequently, $\lambda_1 = 0$. From (2.57),

$$\sum_{k=1}^{n}\partial_k F(\mathbf{x})=\lambda_{n+1}\geq 0. \tag{2.71}$$

We apply now Lemma 2.7 to conclude that $\mathbf{x}$ is one of the convex combinations in (2.26). Let $\mathbf{x}=\mathbf{x}_N(t)=t\mathbf{e}+(1-t)\mathbf{v}_N$, $1\leq N\leq n-2$, and $t\in[0,1)$. Then, $x_1=x_2=\cdots=x_N=1$, $x_{N+1}=x_{N+2}=\cdots=x_n=t$, and $h_{N+1}(\mathbf{x})=t-1<0$. From (2.56), we obtain $\lambda_{N+1}=0$. Thus, from (2.54), $\partial_{N+1}F(\mathbf{x})=\lambda_{N+2}\geq 0$. This contradicts (2.40). Thus, $\mathbf{x}\neq\mathbf{x}_N(t)$ for $N = 1,2,\ldots,n-2$ and $t\in[0,1)$. Consequently, $\mathbf{x}=\mathbf{x}_{n-1}(t)=(1,1,\ldots,1,t)$ for some $t\in[0,1)$.

Finally,

$$F(1,1,\ldots,1,t)=f(1,1,\ldots,1,t^2)-\left(f(1,1,\ldots,1,t)\right)^2=1-1=0 \tag{2.72}$$

for any $t\in[0,1]$. Hence, $\min_{\mathbf{x}\in\mathscr{C}} F(\mathbf{x})=0=F(1,1,\ldots,1,t)$ for any $t\in[0,1]$. Thus, the theorem has been proved. □

Theorem 2.10. *If $y_1\geq y_2\geq y_3\geq\cdots\geq y_n\geq 0$, then*

$$m(\mathbf{y}^{2^p})+\frac{1}{\sqrt{n-1}}s(\mathbf{y}^{2^p})\leq\sqrt{m(\mathbf{y}^{2^{p+1}})+\frac{1}{\sqrt{n-1}}s(\mathbf{y}^{2^{p+1}})}, \tag{2.73}$$

that is,

$$\frac{\sum_{k=1}^{n} y_k^{2^p}}{n} + \frac{1}{\sqrt{n(n-1)}}\sqrt{\sum_{k=1}^{n} y_k^{2^{p+1}} - \frac{\left(\sum_{k=1}^{n} y_k^{2^p}\right)^2}{n}} \leq \left[\frac{\sum_{k=1}^{n} y_k^{2^{p+1}}}{n} + \frac{1}{\sqrt{n(n-1)}}\sqrt{\sum_{k=1}^{n} y_k^{2^{p+2}} - \frac{\left(\sum_{k=1}^{n} y_k^{2^{p+1}}\right)^2}{n}}\right]^{1/2} \tag{2.74}$$

for $p = 0, 1, 2, \ldots$. The equality holds if and only if $y_1 = y_2 = \cdots = y_{n-1}$.

Proof. If $y_1 = 0$, then $y_2 = y_3 = \cdots = y_n = 0$ and the theorem is immediate. Hence, we assume that $y_1 > 0$. Let p be a nonnegative integer and let $x_k = y_k/y_1$ for $k = 1, 2, \ldots, n$. Clearly, $1 = x_1^{2^p} \geq x_2^{2^p} \geq x_3^{2^p} \geq \cdots \geq x_n^{2^p} \geq 0$. From Theorem 2.9, we have

$$\left(f\left(1, x_2^{2^p}, x_3^{2^p}, \ldots, x_m^{2^p}\right)\right)^2 \leq f\left(1, x_2^{2^{p+1}}, x_3^{2^{p+1}}, \ldots, x_m^{2^{p+1}}\right), \tag{2.75}$$

that is,

$$\left(\frac{1 + \sum_{k=2}^{n} x_k^{2^p}}{n} + \frac{1}{\sqrt{n(n-1)}}\sqrt{1 + \sum_{k=2}^{n} x_k^{2^{p+1}} - \frac{\left(1 + \sum_{j=2}^{n} x_j^{2^p}\right)^2}{n}}\right)^2 \leq \frac{1 + \sum_{k=2}^{n} x_k^{2^{p+1}}}{n} + \frac{1}{\sqrt{n(n-1)}}\sqrt{1 + \sum_{k=2}^{n} x_k^{2^{p+2}} - \frac{\left(1 + \sum_{j=2}^{n} x_j^{2^{p+1}}\right)^2}{n}} \tag{2.76}$$

with equality if and only if $x_1 = x_2 = \cdots = x_{n-1}$. Multiplying by $y_1^{2^{p+1}}$, the inequality in (2.74) is obtained with equality if and only if $y_1 = y_2 = \cdots = y_{n-1}$. This completes the proof. □

COROLLARY 2.11. *Let $y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n \geq 0$. Then $(l_{2^p}(\mathbf{y}))_{p=0}^{\infty}$,*

$$l_{2^p}(\mathbf{y}) = \left(\frac{\|\mathbf{y}\|_{2^p}^{2^p}}{n} + \frac{1}{\sqrt{n(n-1)}}\sqrt{\|y\|_{2^{p+1}}^{2^{p+1}} - \frac{\|\mathbf{y}\|_{2^p}^{2^{p+1}}}{n}}\right)^{2^{-p}} = \left(m(\mathbf{y}^{2^p}) + \frac{1}{\sqrt{n-1}} s(\mathbf{y}^{2^p})\right)^{2^{-p}}, \tag{2.77}$$

is an strictly increasing sequence converging to y_1 except if $y_1 = y_2 = \cdots = y_{n-1}$. In this case, $l_{2^p}(\mathbf{y}) = y_1$ for all p.

Proof. We know that $(l_{2^p}(\mathbf{y}))_{p=0}^{\infty}$ is a sequence of lower bounds for y_1. From Theorem 2.1, this sequence converges to y_1. Applying inequality (2.74), we obtain

$$\begin{aligned}&\left(\frac{\sum_{k=1}^{n} y_k^{2^p}}{n}+\frac{1}{\sqrt{n(n-1)}}\sqrt{\sum_{k=1}^{n} y_k^{2^{p+1}}-\frac{\left(\sum_{j=1}^{n} y_j^{2^p}\right)^2}{n}}\right)^2\\&\quad\leq\frac{\sum_{k=1}^{n} y_k^{2^{p+1}}}{n}+\frac{1}{\sqrt{n(n-1)}}\sqrt{\sum_{k=1}^{n} y_k^{2^{p+2}}-\frac{\left(\sum_{j=1}^{n} y_j^{2^{p+1}}\right)^2}{n}}.\end{aligned}\tag{2.78}$$

Therefore, $l_{2^p}^{2^{p+1}}(\mathbf{y}) \leq l_{2^{p+1}}^{2^{p+1}}(\mathbf{y})$, that is, $l_{2^p}(\mathbf{y}) \leq l_{2^{p+1}}(\mathbf{y})$. The equality in all the above inequalities takes place if and only if $\lambda_1 = y_2 = \cdots = y_{n-1}$. In this case, $l_{2^p}(\mathbf{y}) = \lambda_1$ for all p. □

Acknowledgment

This work is supported by Fondecyt 1040218, Chile.

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