ON THE MEAN SUMMABILITY BY CESARO METHOD OF FOURIER TRIGONOMETRIC SERIES IN TWO-WEIGHTED SETTING

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The Cesaro summability of trigonometric Fourier series is investigated in the weighted Lebesgue spaces in a two-weight case, for one and two dimensions. These results are applied to the prove of two-weighted Bernstein's inequalities for trigonometric polynomials of one and two variables.

1. Introduction

It is well known that (see [9]) Cesaro means of 2π-periodic functions $f \in L^p(\mathbb{T})$ $(1 \le p \le \infty)$ converges by norms. Hereby $\mathbb{T}$ is denoted the interval $(-\pi,\pi)$. The problem of the mean summability in weighted Lebesgue spaces has been investigated in [6].

A 2π-periodic nonnegative integrable function $w : \mathbb{T} \to \mathbb{R}^1$ is called a weight function. In the sequel by $L^p_w(\mathbb{T})$, we denote the Banach function space of all measurable 2π-periodic functions f, for which

$$\|f\|_{p,w} = \left(\int_{\mathbb{T}} |f(x)|^p w(x)dx\right)^{1/p} < \infty. \tag{1.1}$$

In the paper [6] it has been done the complete characterization of that weights w, for which Cesaro means converges to the initial function by the norm of $L^p_w(\mathbb{T})$. Later on Muckenhoupt (see [3]) showed that the condition referred in [6] is equivalent to the condition A_p, that is,

$$\sup \frac{1}{|I|}\int_I w(x)dx\left(\frac{1}{|I|}\int_I w^{1-p'}(x)dx\right)^{p-1} < \infty, \tag{1.2}$$

where $p' = p/(p-1)$ and the supremum is taken over all one-dimensional intervals whose lengths are not greater than 2π.

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The problem of mean summability by linear methods of multiple Fourier trigonometric series in $L^p_w(\mathbb{T})$ in the frame of A_p classes has been studied in [5].

In the present paper we investigate the situation when the weight w can be outside of A_p class. Precisely, we prove the necessary and sufficient condition for the pair of weights (v,w) which governs the (C,α) summability in $L^p_v(\mathbb{T})$ for arbitrary function f from $L^p_w(\mathbb{T})$. This result is applied to the prove of two-weighted Bernstein's inequality for trigonometric polynomials. It should be noted that for monotonic pairs of weights for $(C,1)$ summability was studied in [7].

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.3}$$

be the Fourier series of function $f \in L^1(\mathbb{T})$.

Let

$$\sigma_n^\alpha(x,f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_n^\alpha(t)dt, \quad \alpha > 0 \tag{1.4}$$

when

$$K_n^\alpha = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha-1} D_k(t)}{A_n^\alpha}, \tag{1.5}$$

with

$$D_k(t) = \sum_{\nu=0}^{k} \frac{\sin(\nu+1/2)t}{2\sin(1/2)t},$$
$$A_n^\alpha = \binom{n+\alpha}{\alpha} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}. \tag{1.6}$$

In the sequel we will need the following well-known estimates for Cesaro kernel (see [9, pages 94–95]):

$$K_n^\alpha(t) \le 2n, \qquad K_n^\alpha(t) \le c_\alpha n^{-\alpha}|t|^{-(\alpha+1)} \tag{1.7}$$

when $0 < |t| < \pi$.

2. Two-weight boundedness and mean summability (one-dimensional case)

Let us introduce the certain class of pairs of weight functions.

Definition 2.1. A pair of weights (v,w) is said to be of class $\mathcal{A}_p(\mathbb{T})$, if

$$\sup \frac{1}{|I|}\int_I v(x)dx\left(\frac{1}{|I|}\int_I w^{1-p'}(x)dx\right)^{p-1} < \infty, \tag{2.1}$$

where the least upper bound is taken over all one-dimensional intervals by lengths not more than 2π.

The following statement is true.

Theorem 2.2. *Let $1 < p < \infty$. Then*

$$\lim_{n\to\infty} \|\sigma_n^\alpha(\cdot, f) - f\|_{p,v} = 0 \tag{2.2}$$

for arbitrary f from $L_w^p(\mathbb{T})$ if and only if $(v,w) \in \mathcal{A}_p(\mathbb{T})$.

The proof is based on the following statement.

Theorem 2.3. *Let $1 < p < \infty$. For the validity of the inequality*

$$\|\sigma_n^\alpha(\cdot, f)\|_{p,v} \le c\|f\|_{p,w} \tag{2.3}$$

for arbitrary $f \in L_w^p(\mathbb{T})$, where the constant c does not depend on n and f, it is necessary and sufficient that $(v,w) \in \mathcal{A}_p(\mathbb{T})$.

Note that the condition $(v,w) \in \mathcal{A}_p(\mathbb{T})$ is also necessary and sufficient for boundedness of the Abel-Poisson means from $L_w^p(\mathbb{T})$ to $L_v^p(\mathbb{T})$ [4].

First of all let us prove two-weighted inequality for the average

$$f_h^\beta(x) = \frac{1}{h^{1-\beta}} \int_{x-h}^{x+h} |f(t)|\,dt, \quad h > 0,\ 0 \le \beta < 1. \tag{2.4}$$

The last functions are an extension of Steklov means.

Theorem 2.4. *Let $1 < p < q < \infty$ and let $1/q = 1/p - \beta$. If the condition*

$$\sup_I \left(\frac{1}{|I|}\int_I v(x)dx\right)^{1/q} \left(\frac{1}{|I|}\int_I w^{1-p'}(x)dx\right)^{1/p'} < \infty \tag{2.5}$$

is satisfied for all intervals I, $|I| \le 2\pi$, then there exists a positive constant c such that for arbitrary $f \in L_w^p(\mathbb{T})$ and $h > 0$ the following inequality holds:

$$\left(\int_{-\pi}^{\pi} |f_h^\beta(x)|^q v(x)dx\right)^{1/q} \le c\left(\int_{-\pi}^{\pi} |f(x)|^p w(x)dx\right)^{1/p}. \tag{2.6}$$

Proof. Let $h \le \pi$ and N be the least natural number for which $Nh \ge \pi$. Then we have

$$\begin{aligned}
&\int_{\mathbb{T}} [f_h^\beta(x)]^q v(x)dx \\
&\quad \le \sum_{k=-N}^{N-1} \int_{kh}^{(k+1)h} h^{-q(1-\beta)} \left[\int_{x-h}^{x+h} |f(t)|\,dt\right]^q v(x)dx \\
&\quad \le \sum_{k=-N}^{N-1} \int_{kh}^{(k+1)h} h^{-q(1-\beta)} \left[\int_{(k-1)h}^{(k+2)h} |f(t)|\,dt\right]^q v(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=-N}^{N-1}\int_{kh}^{(k+1)h} h^{-q(1-\beta)}\left[\int_{(k-1)h}^{(k+2)h} |f(t)|^p w(t)dt\right]^{q/p}\left[\int_{(k-1)h}^{(k+2)h} w^{1-p'}(t)dt\right]^{q/p'} v(x)dx \\
&= \sum_{k=-N}^{N-1}\left(\int_{kh}^{(k+1)h} v(x)dx\right)\left(\int_{(k-1)h}^{(k+2)h} w^{1-p'}(t)dt\right)^{q/p'} h^{-q(1-\beta)} \\
&\qquad \times\left(\int_{(k-1)h}^{(k+2)h} |f(t)|^p w(t)dt\right)^{q/p} \\
&= \sum_{k=-N}^{N-1}\left(\frac{1}{h}\int_{kh}^{(k+1)h} v(x)dx\right)\left(\frac{1}{h}\int_{(k-1)h}^{(k+2)h} w^{1-p'}(t)dt\right)^{q/p'}\left(\int_{(k-1)h}^{(k+2)h} |f(t)|^p w(t)dt\right)^{q/p}.
\end{aligned}
\tag{2.7}
$$

Arguing to the condition (2.5) we conclude that

$$
\int_{-\pi}^{\pi}[f_h^{\beta}(x)]^q v(x)dx \leq c\sum_{k=-N}^{N-1}\left(\int_{(k-1)h}^{(k+2)h} |f(t)|^p w(t)dt\right)^{q/p}. \tag{2.8}
$$

Using [2, Proposition 5.1.3] we obtain that

$$
\int_{-\pi}^{\pi} |f_h^{\beta}(x)|^q v(x)dx \leq c_1\|f\|_{p,w}^q. \tag{2.9}
$$

Theorem is proved. □

Note that Theorem 2.4 is proved in [4] in the case $\beta = 0$.

Proof of Theorem 2.3. Let us show that

$$
|\sigma_n^{\alpha}(x,f)| \leq c_0\int_{1/n}^{2\pi}\frac{1}{n^{\alpha}}h^{-1-\alpha}f_h(x)dh, \tag{2.10}
$$

where the constant c_0 does not depend on f and h. By reversing the order of integration in the right side integral of (2.10), we get that it is more than or equal to

$$
\begin{aligned}
I &= \int_{x-\pi}^{x+\pi} |f(t)|\left[\int_{\max(|x-t|,1/n)}^{2\pi}\frac{1}{n^{\alpha}}h^{-2-\alpha}dh\right]dt \\
&\geq c\int_{x-\pi}^{x+\pi} |f(t)|\frac{1}{n^{\alpha}}\left[\max\left(|x-t|,\frac{1}{n}\right)\right]^{-1-\alpha}dt
\end{aligned}
\tag{2.11}
$$

since $|x-t| \leq \pi$.

Indeed, let us show that for $|x-t| \leq \pi$, the inequality

$$
\int_{\max\{|x-t|,1/n\}}^{2\pi} h^{-2-\alpha}dh > c\left(\max\{|x-t|,1/n\}\right)^{-\alpha-1}, \tag{2.12}
$$

where c does not depend on x, t, and n.

It is obvious that

$$I_1 = \int_{\max\{|x-t|,1/n\}}^{2\pi} h^{-2-\alpha}dh = \frac{1}{1+\alpha}\left(\frac{1}{\left(\max\{|x-t|,1/n\}\right)^{1+\alpha}} - \frac{1}{(2\pi)^{1+\alpha}}\right). \tag{2.13}$$

To prove the latter inequality we consider two cases.

(a) Let $|x-t| < 1/n$. Then

$$I_1 = \frac{1}{1+\alpha}\left(n^{1+\alpha} - \frac{1}{(2\pi)^{1+\alpha}}\right) > \frac{1}{1+\alpha}\left(1-(2\pi)^{-1-\alpha}\right)n^{1+\alpha}. \tag{2.14}$$

(b) Let now $|x-t| \geq 1/n$. Then for the sake of the fact $|x-t| \leq \pi$, we conclude that

$$\begin{aligned} I_1 &= \frac{1}{1+\alpha}\left(\frac{1}{|x-t|^{1+\alpha}} - \frac{1}{(2\pi)^{1+\alpha}}\right) = \frac{1}{2(1+\alpha)}\left(\frac{1}{|x-t|^{1+\alpha}} + \frac{1}{|x-t|^{1+\alpha}} - \frac{2}{(2\pi)^{1+\alpha}}\right) \\ &> \frac{1}{2(1+\alpha)}\left(\frac{1}{|x-t|^{1+\alpha}} + \frac{1}{\pi^{1+\alpha}} - \frac{2}{(2\pi)^{1+\alpha}}\right) \geq \frac{1}{2(1+\alpha)}\left(\frac{1}{|x-t|^{1+\alpha}} + \frac{1}{\pi^{1+\alpha}} - \frac{1}{2^{\alpha}\pi^{1+\alpha}}\right) \\ &> \frac{1}{2(1+\alpha)}\frac{1}{|x-t|^{1+\alpha}} \end{aligned} \tag{2.15}$$

which implies the desired result.

Using the estimates (1.7) we obtain that

$$I \geq c\int_{x-\pi}^{x+\pi} |f(t)| K_n^{\alpha}(x-t)dt \geq c\left|\int_{-\pi}^{\pi} f(t)K_n^{\alpha}(x-t)dt\right| = c\left|\sigma_n^{\alpha}(x,f)\right|. \tag{2.16}$$

Thus we obtain (2.10). Passing to the norms in (2.10), then applying Theorem 2.4 by Minkowski's integral inequality we obtain that

$$\begin{aligned} \int_{\mathbb{T}} \left|\sigma_n^{\alpha}(x,f)\right|^p v(x)dx &\leq c\int_{\mathbb{T}} |f(x)|^p w(x)\left(\frac{1}{n^{\alpha}}\int_{1/n} h^{-1-\alpha}dh\right)^p dx \\ &\leq c_1\int_{\mathbb{T}} |f(x)|^p w(x)dx. \end{aligned} \tag{2.17}$$

Now we will prove that from (2.3) it follows that $(v,w) \in \mathcal{A}_p(\mathbb{T})$. If the length of the interval I is more than $\pi/4$, the validness of the condition (2.1) is clear.

Let now $|I| \leq \pi/4$. Let m be the greatest integer for which

$$m \leq \frac{\pi}{2|I|} - 1. \tag{2.18}$$

Then we have

$$\left|\left(k+\frac{1}{2}\right)(x-t)\right| \leq (m+1)|x-t| \leq \frac{\pi}{2}. \tag{2.19}$$

Then applying Abel's transform we get that for x and t from I, the following estimates are true:

$$\begin{aligned} K_m^\alpha(x-t) &\geq \sum_{k=0}^{m} \frac{A_{m-k}^\alpha}{A_m^\alpha}(2k+1) \geq c(m+2)\frac{1}{(m+1)A_m^\alpha}\sum_{k=0}^{m} A_{m-k}^{\alpha-1}(k+1) \\ &\geq \frac{c}{|I|}\frac{1}{(m+1)A_m^\alpha}\sum_{k=0}^{m} A_{m-k}^\alpha = \frac{c}{|I|}\frac{A_m^{\alpha+1}}{(m+1)A_m^\alpha} \geq \frac{c}{|I|}. \end{aligned} \tag{2.20}$$

Let us put in (2.3) the function

$$f_0(x) = w^{1-p'}(x)\chi_I(x) \tag{2.21}$$

for m which was indicated above. Then we obtain

$$\int_I\left(\int_I w^{1-p'}(t)K_m^\alpha(x-t)dt\right)^p v(x)dx \leq c\int_I w^{1-p'}(x)dx. \tag{2.22}$$

From the last inequality by (2.20) we conclude that

$$\int_I\left(\frac{1}{|I|}\int_I w^{1-p'}(t)dt\right)^p v(x)dx \leq c\int_I w^{1-p'}(x)dx. \tag{2.23}$$

Thus from (2.3) it follows that $(v,w) \in \mathcal{A}_p(\mathbb{T})$. □

Proof of Theorem 2.2. Let us show that if $(v,w) \in \mathcal{A}_p(\mathbb{T})$, then

$$\lim_{n\to\infty} \|\sigma_n^\alpha(\cdot,f) - f\|_{p,v} = 0 \tag{2.24}$$

for arbitrary $f \in L_w^p(\mathbb{T})$.

Consider the sequence of linear operators:

$$U_n : f \longrightarrow \sigma_n^\alpha(\cdot,f). \tag{2.25}$$

It is easy to see that U_n is bounded from $L_w^p(\mathbb{T})$ to $L_v^p(\mathbb{T})$. Indeed applying Hölder's inequality we get

$$\begin{aligned} \int_{\mathbb{T}} |\sigma_n^\alpha(x,f)|^p v(x)dx &\leq 2n\int_{\mathbb{T}}\left(\int_{\mathbb{T}}|f(t)|dt\right)^p v(x)dx \\ &\leq 2n\int_{\mathbb{T}}|f(t)|^p w(t)dt\int_{\mathbb{T}} v(x)dx\left(\int_{\mathbb{T}} w^{1-p'}(x)dx\right)^{p-1}. \end{aligned} \tag{2.26}$$

By our assumptions all these integrals are finite, the constant

$$c = 2n\int_{\mathbb{T}} v(x)dx\left(\int_{\mathbb{T}} w^{1-p'}(x)dx\right)^{p-1} \tag{2.27}$$

does not depend on f.

Then since $(v,w) \in \mathcal{A}_p(\mathbb{T})$ by Theorem 2.3, we have that the sequence of operators norms is bounded. On the other hand, the set of all 2π-periodic continuous on the line functions is dense in $L^p_w(\mathbb{T})$. It is known (see [9]) that the Cesaro means of continuous function uniformly converges to the initial function and since $v \in L^1(\mathbb{T})$ they converge in $L^p_v(\mathbb{T})$ as well. Applying the Banach-Steinhaus theorem (see, [1]) we conclude that the convergence holds for arbitrary $f \in L^p_w(\mathbb{T})$.

Now we prove the necessity part. From the convergence in $L^p_v(\mathbb{T})$ of the Cesaro means by Banach-Steinhaus theorem we conclude that

$$\left\{ ||U_n||_{L^p_w(\mathbb{T}) \to L^p_v(\mathbb{T})} \right\}_{n=1}^{\infty} \tag{2.28}$$

is bounded. It means that (2.3) holds. Then by Theorem 2.3 we conclude that $(v,w) \in \mathcal{A}_p(\mathbb{T})$.

Theorem is proved. □

3. On the mean (C,α,β) summability of the double trigonometric Fourier series

Let $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ and $f(x,y)$ be an integrable function on $\mathbb{T}^2$ which is 2π-periodic with respect to each variable.

Let

$$\begin{aligned} f(x,y) \sim \sum_{m,n=0}^{\infty} \lambda_{mn} (a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin my \\ + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny), \end{aligned} \tag{3.1}$$

where

$$\lambda_{mn} = \begin{cases} \dfrac{1}{4}, & \text{when } m = n = 0, \\ \dfrac{1}{2}, & \text{for } m = 0,\ n > 0 \text{ or } m > 0,\ n = 0, \\ 1, & \text{when } m > 0,\ n > 0. \end{cases} \tag{3.2}$$

Let

$$\sigma_{mn}^{(\alpha,\beta)}(x,y,f) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} S_{ij}(x,y,f)}{A_m^{\alpha} A_n^{\beta}}, \quad (\alpha,\beta > 0) \tag{3.3}$$

be the Cesaro means for the function f, where $S_{ij}(x,y,f)$ are partial sums of (3.1).

We consider the mean summability in weighted space defined by the norm

$$\|f\|_{p,w} = \left(\int_{\mathbb{T}^2} |f(x,y)|^p w(x,y)\,dx\,dy \right)^{1/p}, \tag{3.4}$$

where w is a weight function of two variables.

In this section our goal is to prove the following result and some its converse.

Theorem 3.1. *Let $1 < p < \infty$. Assume that the pair of weights (v, w) satisfies the condition*

$$\sup_J \frac{1}{|J|} \int_J v(x,y)\,dx\,dy \left(\frac{1}{|J|} \int_J w^{1-p'}(x,y)\,dx\,dy \right)^{p-1} < \infty, \tag{3.5}$$

where the least upper bound is taken over all rectangles, with the sides parallel to the coordinate axes. Then for arbitrary $f \in L^p_w(\mathbb{T}^2)$, we have

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \left\| \sigma_{mn}^{(\alpha,\beta)}(\cdot,\cdot,f) - f \right\|_{p,v} \longrightarrow 0. \tag{3.6}$$

In the sequel the set of all pairs with the condition (3.5) will be denoted by $\mathcal{A}_p(\mathbb{T}^2, \mathbb{J})$. Here $\mathbb{J}$ denotes the set of all rectangles with parallel to the coordinate axes.

The proof of this theorem is based on the following statement.

Theorem 3.2. *Let $1 < p < \infty$ and $(v,w) \in \mathcal{A}_p(\mathbb{T}^2, \mathbb{J})$, then*

$$\left\| \sigma_{mn}^{(\alpha,\beta)}(\cdot,\cdot,f) \right\|_{p,v} \le c \|f\|_{p,w}, \tag{3.7}$$

with the constant c independent of m, n, and f.

To prove Theorem 3.2 we need the two-dimensional version of Theorem 2.4. Let us consider generalized multiple Steklov means

$$f_{hk}^{\gamma}(x) = \sup_{\substack{h>0\\ k>0}} \frac{1}{(hk)^{\gamma}} \int_{x-h}^{x+h} \int_{y-k}^{y+k} |f(t,\tau)|\,dt\,d\tau, \quad 0 < \gamma \le 1. \tag{3.8}$$

Theorem 3.3. *Let $1 < p < \infty$ and $1/q = 1/p - \gamma$. Let $(v,w) \in \mathcal{A}_p(\mathbb{T}^2, \mathbb{J})$. Then there exists a constant $c > 0$ such that for arbitrary $f \in L^p_w(\mathbb{T}^2)$ and positive h and k, we have*

$$\| f_{hk}^{\gamma} \|_{q,v} \le c \|f\|_{p,w}. \tag{3.9}$$

Proof. Let $h \le \pi$ and $k \le \pi$. Let M and N be the least natural numbers for which $Mh \ge \pi$ and $Nk \ge \pi$. Then

$$\begin{aligned}
\int_{\mathbb{T}^2} [f_{hk}^{\gamma}(x,y)]^q v(x,y)\,dx\,dy &\le \sum_{i=-M}^{M} \sum_{j=-N}^{N} \int_{ih}^{(i+1)h} \int_{jk}^{(j+1)k} (hk)^{-q(1-\gamma)} \\
&\quad \times \left[\int_{x-h}^{x+h} \int_{y-k}^{y+k} |f(t,\tau)|\,dt\,d\tau \right]^q v(x,y)\,dx\,dy \\
&\le \sum_{i=-M}^{M-1} \sum_{j=-N}^{N-1} \int_{ih}^{(i+1)h} \int_{jk}^{(j+1)k} (hk)^{-q(1-\gamma)} \\
&\quad \times \left[\int_{(i-1)h}^{(i+2)h} \int_{(j-1)k}^{(j+1)k} |f(t,\tau)|\,dt\,d\tau \right]^q v(x,y)\,dx\,dy.
\end{aligned} \tag{3.10}$$

Using the Hölder's inequality we get

$$\begin{aligned}\int_{\mathbb{T}^2}&[f_{hk}^{\gamma}(x,y)]^q v(x,y)\,dx\,dy\\ &\le \sum_{i=-M}^{M-1}\sum_{j=-N}^{N-1}\int_{ih}^{(i+1)h}\int_{jk}^{(j+1)k}(hk)^{-q(1-\gamma)}\left[\int_{(i-1)h}^{(i+2)h}\int_{(j-1)k}^{(j+1)k}|f(t,\tau)|^p w(t,\tau)\,dt\,d\tau\right]^{q/p}\\ &\quad\times\left[\int_{(i-1)h}^{(i+2)h}\int_{(j-1)k}^{(j+2)k}w^{1-p'}(x,y)\,dx\,dy\right]^{q/p'}v(x,y)\,dx\,dy.\end{aligned} \tag{3.11}$$

By the condition $\mathcal{A}_p(\mathbb{T}^2,\mathbb{J})$ we derive that

$$\int_{\mathbb{T}^2}[f_{hk}^{\gamma}(x,y)]^q v(x,y)\,dx\,dy \le c\sum_{i=-M}^{M-1}\sum_{j=-N}^{N-1}\left(\int_{(i-1)h}^{(i+2)h}\int_{(j-1)k}^{(j+1)k}|f(t,\tau)|^p w(t,\tau)\,dt\,d\tau\right)^{q/p}. \tag{3.12}$$

Consequently,

$$\int_{\mathbb{T}^2}|f_{hk}^{\gamma}(x,y)|^q v(x,y)\,dx\,dy \le c\|f\|_{p,w}^q. \tag{3.13}$$

Theorem is proved. □

Proof of Theorem 3.2. Let us prove that

$$\left|\sigma_{mn}^{(\alpha,\beta)}(x,y,f)\right| \le c\int_{1/m}^{\pi}\int_{1/n}^{\pi}\frac{1}{m^{\alpha}n^{\beta}}h^{-1-\alpha}k^{-1-\beta}f_{hk}(x,y,f)\,dh\,dk, \tag{3.14}$$

where the constant does not depend on f, x, y, m, and n.

If we reverse the order of integration in right side of (3.14), then by the arguments similar to that of the one-dimensional case we obtain that

$$\begin{aligned}I&=\int_{x-\pi}^{x+\pi}\int_{y-\pi}^{y+\pi}|f(t,s)|\left[\int_{\max(|x-t|,1/m)}^{2\pi}\int_{\max(|y-s|,1/n)}^{2\pi}\frac{1}{m^{\alpha}n^{\beta}}h^{-2-\alpha}k^{-2-\beta}\,dh\,dk\right]dt\,ds\\ &\ge c\int_{x+\pi}^{x-\pi}\int_{y-\pi}^{y+\pi}|f(t,s)|\frac{1}{m^{\alpha}n^{\beta}}\left[\max\left(|x-t|,\frac{1}{m}\right)\right]^{-1-\alpha}\left[\max\left(|y-s|,\frac{1}{n}\right)\right]^{-1-\beta}dt\,ds.\end{aligned} \tag{3.15}$$

Applying the known estimates for Cesaro kernel from the last estimate we derive that

$$I \ge c\int_{\mathbb{T}^2}|f(t,s)|K_m^{\alpha}(x-t)K_n^{\beta}(y-s)\,dt\,ds \ge c\left|\sigma_{mn}^{(\alpha,\beta)}(x,y,f)\right|. \tag{3.16}$$

We proved (3.14).

Taking the norms in (3.14), by Theorem 3.3 and Minkowski's inequality we conclude that

$$\begin{aligned}\int_{\mathbb{T}^2}\left|\sigma_{mn}^{(\alpha,\beta)}(x,y,f)\right|^p v(x,y)d\,dx\,dy \\ \leq c\int_{\mathbb{T}^2}|f(x,y)|^p w(x,y)\left(\frac{1}{m^\alpha n^\beta}\int_{1/m}^{2\pi}\int_{1/n}^{2\pi}h^{-1-\alpha}k^{-1-\beta}dh\,dk\right)^p dx\,dy \\ \leq c_1\int_{\mathbb{T}^2}|f(x,y)|^p w(x,y)dx\,dy.\end{aligned} \tag{3.17}$$

By this we obtain (3.7). □

Proof of Theorem 3.1. Consider the sequence of operators

$$U_{mn}: f \longrightarrow \sigma_{mn}^{(\alpha,\beta)}(\cdot,\cdot,f). \tag{3.18}$$

It is evident that U_{mn} is linear bounded for each (m,n) as

$$\int_{\mathbb{T}^2}v(x,y)dx\,dy<\infty, \qquad \int_{\mathbb{T}^2}w^{1-p'}(x,y)dx\,dy<\infty. \tag{3.19}$$

Then since $(v,w)\in\mathcal{A}_p(\mathbb{T}^2,\mathbb{J})$ by Theorem 3.2, the sequence of operators norms

$$\left\{\|U_{mn}\|_{L_w^p\to L_v^p}\right\}_{m,n=1}^{\infty} \tag{3.20}$$

is bounded. On the other hand, the set of 2π-periodic functions which are continuous on the plane is dense in $L_w^p(\mathbb{T}^2)$. Then it is known that Cesaro means of Lipschitz functions of two variables converges uniformly (see [8, page 181]). Since $v\in L^1(\mathbb{T}^2)$ the last convergence we have by means of L_v^p norms as well. Applying the Banach-Steinhaus theorem (see [1]) we conclude that the norm convergence (3.6) holds for arbitrary $f\in L_w^p(\mathbb{T}^2)$. □

Theorem 3.4. *Let $1<p<\infty$. If the inequality (3.7) is satisfied, then the condition (3.5) holds when the least upper bound is taken over all rectangles $J_0=I_1\times I_2$ and $|I_1|<\pi/4$ and $|I_2|<\pi/4$.*

Proof. Let m and n be that greatest natural numbers with

$$\frac{\pi}{2(m+2)}\leq|I_1|\leq\frac{\pi}{2(m+1)}, \qquad \frac{\pi}{2(n+2)}\leq|I_2|\leq\frac{\pi}{2(n+1)}. \tag{3.21}$$

Then for $(x,y)\in J_0$ and $(t,\tau)\in J_0$, we have

$$K_m^\alpha(x-t)\geq\frac{c}{|I_1|}, \qquad K_n^\beta(y-s)\geq\frac{c}{|I_2|} \tag{3.22}$$

with some constant c nondepending on m, n, (x,y) and (t,s).

Indeed Abel's transform for K_m^α gives

$$K_m^\alpha(x-t) \ge \sum_{k=0}^{m} \frac{A_{m-k}^\alpha}{A_m^\alpha}(2k+1) \ge c(m+2)\frac{1}{(m+1)A_m^\alpha}\sum_{k=0}^{m} A_{m-k}^{\alpha-1}(k+1)$$
$$\ge \frac{c}{|I_1|}\frac{1}{(m+1)A_m^\alpha}\sum_{k=0}^{n} A_k^\alpha = \frac{c}{|I_1|}\frac{A_m^{\alpha+1}}{(m+1)A_m^\alpha} \ge \frac{c}{|I_1|}, \tag{3.23}$$

for $(x,y) \in J_0$ and $(t,s) \in J_0$.

Analogously we can estimate $K_n^\beta(y-s)$.

Now for indicated m and n, put (3.7) in the function

$$f_0(x,y) = w^{1-p'}(x,y)\chi_{J_0}(x,y). \tag{3.24}$$

Then we get

$$\int_{J_0}\left(\int_{J_0} w^{1-p'}(t,s)K_m^\alpha(x-t)K_n^\beta(y-s)\,dt\,ds\right)^p v(x,y)\,dx\,dy \le c\int_{J_0} w^{1-p'}(x,y)\,dx\,dy. \tag{3.25}$$

By (3.23) from the last inequality we obtain

$$\int_{J_0}\left(\frac{1}{|J_0|}\int_{J_0} w^{1-p'}(t,s)\,dt\,ds\right)^p v(x,y)\,dx\,dy \le c\int_{J_0} w^{1-p'}(x,y)\,dx\,dy, \tag{3.26}$$

which is (3.5) with the least upper bound taken over all rectangles J_0, such that $J_0 = I_1 \times I_2$ and $|I_i| < \pi/4$, $i = 1,2$. □

Theorem 3.5. *Let* $1 < p < \infty$. *If (3.7) holds, then there exist* $k \in \mathbb{N}$ *and a positive* $c > 0$ *such that*

$$\frac{1}{|J|}\int_J v(x,y)\,dx\,dy\left(\frac{1}{|J|}\int_J w^{1-p'}(x,y)\,dx\,dy\right)^{p-1} < c \tag{3.27}$$

for arbitrary $J = I_1 \times I_2$ *with* $|I_i| < \pi/(2k+1)$ $(i = 1,2)$.

Proof. Let us consider the double sequence of operators

$$U_{mn} : f \longrightarrow \sigma_{mn}^{(\alpha,\beta)}(\cdot,\cdot,f). \tag{3.28}$$

Since the sequence is double, following to the proof of Banach-Steinhaus theorem, we can conclude only that there exists some natural number k such that

$$||U_{mn}|| \le M \tag{3.29}$$

when $m \ge k$, $n \ge k$.

Note that, in general the convergence of a double sequence does not imply the boundedness of this sequence. Thus we have that

$$\left\|\sigma_{mn}^{(\alpha,\beta)}(\cdot,\cdot,f)\right\|_{p,v} \le c\|f\|_{p,w} \tag{3.30}$$

when $m \ge k$ and $n \ge k$.

Let us consider such rectangles that $J_0 = I_1 \times I_2$ and

$$|I_1| < \frac{\pi}{2(k+1)}, \qquad |I_2| < \frac{\pi}{2(k+1)}. \tag{3.31}$$

Then choose the greatest m and n such that

$$\frac{\pi}{2(m+2)} < |I_1| < \frac{\pi}{2(m+1)}, \qquad \frac{\pi}{2(n+2)} < |I_2| < \frac{\pi}{2(n+1)}. \tag{3.32}$$

Now it is sufficient to repeat the last part of the proof of previous theorem. □

4. Two-weighted Bernstein's inequalities

Applying the two-norm inequalities for the Cesaro means derived in the previous sections, we are able to prove the two-weighted version of the well-known Bernstein's inequality. For any trigonometric polynomial $T_n(x)$ of order $\le n$, for every p $(1 \le p \le \infty)$, we have

$$\left(\int_0^{2\pi} |T_n'(x)|^p dx\right)^{1/p} \le cn\left(\int_0^{2\pi} |T_n(x)|^p dx\right)^{1/p}. \tag{4.1}$$

The last inequality is known as integral Bernstein's inequality.

The following extension of (4.1) is true.

Theorem 4.1. *Let $1 < p < \infty$ and assume that $(v,w) \in \mathcal{A}_p(\mathbb{T})$. Then the two-weighted inequality*

$$\left(\int_0^{2\pi} |T_n'(x)|^p v(x)dx\right)^{1/p} \le cn\left(\int_0^{2\pi} |T_n(x)|^p w(x)dx\right)^{1/p} \tag{4.2}$$

holds. Also for the conjugate trigonometric polynomial $\widetilde{T}_n$, we have

$$\left(\int_0^{2\pi} |\widetilde{T}_n'(x)|^p v(x)dx\right)^{1/p} \le cn\left(\int_0^{2\pi} |T_n(x)|^p w(x)dx\right)^{1/p}. \tag{4.3}$$

Proof. It is well known that

$$T_n(x) = \frac{1}{\pi}\int_0^{2\pi} T_n(u)D_n(u-x)du, \tag{4.4}$$

where

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos ku \tag{4.5}$$

is the Dirichlet's kernel of order n. By the derivation, we obtain

$$\begin{aligned} T_n'(x) &= -\frac{1}{\pi}\int_0^{2\pi} T_n(u)D_n'(u-x)du = -\frac{1}{\pi}\int_0^{2\pi} T_n(u+x)D_n'(u)du \\ &= \frac{1}{\pi}\int_0^{2\pi} T_n(u+x)\left\{\sum_{k=1}^{n} k\sin ku\right\}du \\ &= \frac{1}{\pi}\int_0^{2\pi} T_n(u+x)\left\{\sum_{k=1}^{n} k\sin ku+\sum_{k=1}^{n-1} k\sin(2n-k)u\right\}du \\ &= \frac{1}{\pi}\int_0^{2\pi} T_n(u+x)2n\sin nu\left\{\frac{1}{2}+\sum_{k=1}^{n-1}\frac{n-k}{n}\cos ku\right\}du \\ &= 2n\frac{1}{\pi}\int_0^{2\pi} T_n(u+x)\sin nuK_{n-1}(u)du, \end{aligned} \tag{4.6}$$

where K_{n-1} is the Fejer's kernel of order $n-1$. By taking the absolute values, we get (see [9, Volume I, page 85])

$$|T_n'(x)| \leq 2n\frac{1}{\pi}\int_0^{2\pi} |T_n(u+x)|K_{n-1}(u)du = 2n\sigma_{n-1}(x,|T_n|). \tag{4.7}$$

If we use Theorem 2.3, we get that

$$\begin{aligned} \left(\int_0^{2\pi} |T_n'(x)|^p v(x)dx\right)^{1/p} &\leq \left(\int_0^{2\pi} [2n\sigma_{n-1}(x,|T_n|)]^p v(x)dx\right)^{1/p} \\ &= 2n\left(\int_0^{2\pi} [\sigma_{n-1}(x,|T_n|)]^p v(x)dx\right)^{1/p} \\ &\leq cn\left(\int_0^{2\pi} |T_n|^p w(x)dx\right)^{1/p}. \end{aligned} \tag{4.8}$$

For the conjugate of T_n, we have

$$\widetilde{T}_n(x) = \frac{1}{\pi}\int_0^{2\pi} T_n(u)\widetilde{D}_n(u-x)du, \tag{4.9}$$

where

$$\widetilde{D}_n = \sum_{k=1}^{n} \sin ku \tag{4.10}$$

is the conjugate Dirichlet's kernel. By differentiation we get

$$\widetilde{T}_n'(x) = \frac{2n}{\pi}\int_0^{2\pi} T_n(x+u)\cos nuK_{n-1}(u)du \tag{4.11}$$

and hence

$$|\widetilde{T}_n'(x)| \leq 2n\sigma_{n-1}(x, |T_n|). \tag{4.12}$$

From this we obtain

$$\left(\int_0^{2\pi} |\widetilde{T}_n'(x)|^p v(x)dx\right)^{1/p} \leq cn\left(\int_0^{2\pi} |T_n(x)|^p w(x)dx\right)^{1/p}. \tag{4.13}$$

and the theorem is proved. □

The inequality derived in Theorem 4.1 also extended to the case of trigonometric polynomials of several variables. Thus, if $T_{mn}(x,y)$ is a trigonometric polynomial of order $\leq m$ with respect to x and of order $\leq n$ with respect to y, we have the following.

Theorem 4.2. *Let $1 < p < \infty$. Assume that $(v,w) \in \mathcal{A}_p(\mathbb{T}^2, \mathbb{J})$. Then the inequality*

$$\left\|\frac{\partial^2 T_{mn}(x,y)}{\partial x \partial y}\right\|_{p,v} \leq cmn\|T_{mn}(x,y)\|_{p,w} \tag{4.14}$$

holds with a positive constant c independent of T_{mn}.

Proof. It is known that (see [9, Volume II, pages 302–303])

$$\begin{aligned}\sigma_{mn}(x,y) &= \frac{1}{\pi^2}\iint_0^{2\pi} f(x+s,y+t)K_m(s)K_n(t)ds\,dt,\\ T_{mn}(x,y) &= \frac{1}{\pi^2}\iint_0^{2\pi} T_{mn}(s,t)D_m(s-x)D_n(t-y)ds\,dt.\end{aligned} \tag{4.15}$$

If we take the partial derivatives of T_{mn} with respect to x and y from the last relation, we obtain

$$\frac{\partial^2 T_{mn}(x,y)}{\partial x \partial y} = \frac{1}{\pi^2}\iint_0^{2\pi} T_{mn}(s,t)D_m'(s-x)D_n'(t-y)ds\,dt. \tag{4.16}$$

By the process used in the previous theorem, this gives

$$\frac{\partial^2 T_{mn}(x,y)}{\partial x \partial y} = \frac{2m2n}{\pi^2}\iint_0^{2\pi} T_{mn}(x+s,y+t)\sin ms \sin nt K_{m-1}(s)K_{n-1}(t)ds\,dt \tag{4.17}$$

and hence

$$\left|\frac{\partial^2 T_{mn}(x,y)}{\partial x \partial y}\right| \leq \frac{4mn}{\pi^2}\sigma_{(m-1)(n-1)}(x,y,|T_{mn}|). \tag{4.18}$$

If we take the norms and consider Theorem 3.2, we obtain the desired inequality. □

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