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TRACKING CONTROL OF A FLEXIBLE BEAM BY NONLINEAR BOUNDARY FEEDBACK[1]

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ABSTRACT

This paper is concerned with tracking control of a dynamic model consisting of a flexible beam rotated by a motor in a horizontal plane at the one end and a tip body rigidly attached at the free end. The well-posedness of the closed loop systems considering the dissipative nonlinear boundary feedback is discussed and the asymptotic stability about difference energy of the hybrid system is also investigated.

Key words: Flexible Beam, Exponential Decay, Stability, Tracking Control, Boundary Feedback Control.

AMS (MOS) subject classifications: 93D15.

1. Introduction and System Formulation

Mechanical flexibility in motion control systems attracted more attention in recent years. Motivated by [4], in which a hybrid system describing the overhead crane model was studied, we will consider in this paper a flexible beam rotated by a motor in a horizontal plane at one end and a top body rigidly attached at the free end. This model fits a large class of real applications such as links of robot system and space-shuttle arms in which high speed manipulation and long and slender geometrical dimensions are the major factors causing mechanical vibration. To achieve high speed and precision end point positioning of the flexible beam (which must be guaranteed in any condition variations such as payload) the boundary control is one of the major strategies in production and space applications.

Let ℓ be the length of the beam, ρ the uniform mass density per unit length, EI the uniform flexural rigidity and m be the mass of the tip body attached at the free end of the link, I_m the moment of inertia of the motor and J the moment of inertia associated with the tip body. Taking the motor's torque as the control input and neglecting rotary inertia and shear deformation effects and actuator dynamics, the total transversal displacement $y(x,t)$ at position x and time t

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can be described by the following coupled differential equation:

$$\begin{cases} \rho y_{tt}(x,t) + EIy_{xxxx}(x,t) = 0, \quad 0 < x < \ell, \quad t > 0, \\ y(0,t) = 0, \\ EIy_{xx}(0,t) - I_m y_{xtt}(0,t) + u(t) = 0, \\ EIy_{xxx}(\ell,t) - My_{tt}(\ell,t) = 0, \\ EIy_{xx}(\ell,t) + Jy_{xtt}(\ell,t) = 0. \end{cases} \tag{1}$$

Let the terminal state trajectory be $x\theta_d(t)$, where $\theta_d''(t) = 0$, i.e., the tracked state would be uniform motion or fixed in some direction for the flexible beam. Thus the difference displacement $e(x,t) = y(x,t) - x\theta_d(t)$ will satisfy the same equation (1). By this fact, we will, here and throughout this paper, use $y(x,t)$ to represent the error displacement as well as total displacement. It is obvious that the feedback control should make the energy of the dynamic system (1) be decreasing with time.

Let us briefly outline the content of this paper. In section 2, we design a dissipative nonlinear feedback control with angular velocity of motor and show the well-posedness of the closed loop system. It is also shown that the energy in this case will tend to zero as time goes to infinity. Section 3 is devoted to the uniform decay estimate of the closed loop system with the angular acceleration feedback.

2. Well-posedness of the Problem and the Asymptotic Stability

We design a nonlinear dissipative feedback control by

$$u(t) = -\alpha y_x(0,t) - f(y_{xt}(0,t))$$

(where $\alpha > 0$ is a positive constant) and study the following closed-loop system

$$\begin{cases} \rho y_{tt}(x,t) + EIy_{xxxx}(x,t) = 0, \quad 0 < x < \ell, \quad t > 0, \\ y(0,t) = 0, \\ EIy_{xx}(0,t) - I_m y_{xtt}(0,t) - \alpha y_x(0,t) - f(y_{xt}(0,t)) = 0, \\ EIy_{xxx}(\ell,t) - My_{tt}(\ell,t) = 0, \\ EIy_{xx}(\ell,t) + Jy_{xtt}(\ell,t) = 0, \end{cases} \tag{2}$$

where the feedback function f such that $f \in C^0(\mathbb{R})$ is increasing with

$$f(0) = 0 \text{ and } sf(s) > 0 \text{ for } s \neq 0. \tag{3}$$

Let $\mathcal{H} = H_E^2(0,\ell) \times L^2(0,\ell) \times \mathbb{R}^3$ be the underlying state space with the inner product

$$\langle (u(x), v(x), a_1, a_2, a_3), (\bar{u}(x), \bar{v}(x), \bar{a}_1, \bar{a}_2, \bar{a}_3) \rangle$$

$$= \frac{1}{2} \int_0^\ell [EIu''(x)\bar{u}''(x) + \rho v(x)\bar{v}(x)]dx + \frac{1}{2}\alpha u'(0)\bar{u}'(0) + \frac{1}{2}[I_m a_1 \bar{a}_1 + Ma_2\bar{a}_2 + Ja_3\bar{a}_3],$$

where $H_E^2(0,\ell) = \{u(x) \mid u \in H^2(0,\ell), u(0) = 0\}$. Define a nonlinear operator $\mathcal{A}: D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ by

$$\begin{aligned} &\mathcal{A}(u(x), v(x), v'(0), v(\ell), v'(\ell)) \\ &= (-v(x), \frac{EI}{\rho}u^{(4)}(x), -\frac{EI}{I_m}u''(0) + \frac{\alpha}{I_m}u'(0) + \frac{1}{I_m}f(v'(0)), \\ &\qquad -\frac{EI}{M}u'''(\ell), \frac{EI}{J}u''(\ell)) \end{aligned} \tag{4}$$

$$D(\mathcal{A}) = \{(u(x), v(x), v'(0), v(\ell), v'(\ell)), u \in H^4(0,\ell),$$
$$v \in H^2(0,\ell), u(0) = v(0) = 0\}.$$

Then equation (2) can be written as a nonlinear evolution equation on $\mathcal{H}$:

$$\begin{cases} \dfrac{dY(t)}{dt} + \mathcal{A}Y(t) = 0, \\ Y(0) = Y_0, \end{cases} \tag{5}$$

where $Y(t) = (y(x,t), y_t(x,t), y_{xt}(0,t), y_t(\ell,t),\ y_{xt}(\ell,t))^T$. Notice that the norm of state is just the energy of the system:

$$E(t) = \| Y(t) \|^2$$
$$= \frac{1}{2}\int_0^\ell [\rho y_t^2 + EIy_{xx}^2]dx + \frac{1}{2}\alpha y_x^2(0,t) + \frac{1}{2}I_m y_{xt}^2(0,t) + \frac{1}{2}My_t^2(\ell,t) + \frac{1}{2}Jy_{xt}^2(\ell,t),$$

and formally

$$\frac{dE(t)}{dt} = -y_{xt}(0,t)f(y_{xt}(0,t)) \le 0.$$

Lemma 1: *Under the assumption (3), the operator $\mathcal{A}$ defined by (4) is maximal monotone on $\mathcal{H}$ with the domain $D(\mathcal{A})$ that is dense in $\mathcal{H}$.*

Proof: Let $U, V \in D(\mathcal{A})$, where

$$U = (u(x), v(x), v'(0), v(\ell), v'(\ell)),$$
$$V = (\overline{u}(x), \overline{v}(x), \overline{v}'(0), \overline{v}(\ell), \overline{v}'(\ell)).$$

Then a simple calculation yields

$$\langle \mathcal{A}U - \mathcal{A}V, U - V\rangle = \frac{1}{2}[f(v'(0)) - f(\overline{v}'(0))][v'(0) - \overline{v}'(0)] \ge 0.$$

This means that $\mathcal{A}$ is monotone. To prove the maximal monotonicity of $\mathcal{A}$ it is sufficient to prove the range condition (see [1])

$$\mathcal{R}(I + \mathcal{A}) = \mathcal{H},$$

i.e., for any given $u_0(x), v_0(x), a_1, a_2, a_3) \in \mathcal{H}$, there exists $(u(x), v(x), v'(0), v(\ell), v'(\ell)) \in D(\mathcal{A})$ such that $v(x) = u(x) - u_0(x)$ and u satisfies

$$\begin{cases} \frac{EI}{\rho}u^{(4)}(x) + u(x) = u_0(x) + v_0(x), \quad u(0) = 0, \\ (1 + \frac{\alpha}{I_m})u'(0) - \frac{EI}{I_m}u''(0) + \frac{1}{I_m}f(u'(0) - u_0'(0)) = a_1 + u_0'(0), \\ u(\ell) - \frac{EI}{M}u'''(\ell) = a_2 + u_0(\ell), \\ u'(\ell) + \frac{EI}{J}u''(\ell) = a_3 + u_0'(\ell). \end{cases} \tag{7}$$

If $u(x) \in H^4(0,\ell)$ is a solution of (7), then multiplying by $\rho\phi(x) \in H_E^2(0,\ell)$ both sides of the first equation of (7) and integrating from 0 to ℓ with respect to x, we have

$$a(u,\phi) - F(\phi) + f(u'(0) - u_0'(0))\phi'(0) = 0, \quad \phi(x) \in H_E^2(0,\ell),$$

where the bilinear functional $a(\cdot,\cdot)$ is defined and coercive on $H_E^2(0,\ell)$ and a linear bounded functional $F(\cdot)$ is defined on $H_E^2(0,\ell)$ as follows:

$$a(\psi,\phi) = \int_0^\ell [EI\psi''(x)\phi''(x) + \rho\psi(x)\phi(x)]dx$$

$$+ M\psi(\ell)\phi(\ell) + J\psi'(\ell)\phi'(\ell) + (\alpha + I_m)\psi'(0)\phi'(0), \quad \forall\psi,\phi \in H_E^2(0,\ell),$$

$$F(\psi) = M[a_2 + u_0(\ell)]\psi(\ell) + J[a_3 + u_0'(\ell)]\psi'(\ell)$$

$$+ I_m[a_1 + u_0'(0)]\psi'(0) + \rho\int_0^\ell [u_0(x) + v_0(x)]\psi(x)dx, \quad \forall\psi \in H_E^2(0,\ell).$$

On the other hand, let

$$J(\psi) = \frac{1}{2}a(\psi,\psi) - F(\psi) + \int_0^{\psi'(0)-u_0'(0)} f(s)ds, \quad \forall\psi \in H_E^2(0,\ell).$$

Since $\int_0^{\psi'(0)-u_0'(0)} f(s)ds \geq 0$, $J(\cdot)$ is convex, coercive and strongly continuous on $H_E^2(0,\ell)$, there exists a unique function $u \in H_E^2(0,\ell)$ such that

$$J(u) = \inf_{\psi \in H_E^2(0,\ell)} J(\psi).$$

This means that for all $\phi \in H_E^2(0,\ell)$,

$$a(u,\phi) - F(\phi) + f(u'(0) - u_0'(0))\phi'(0) = 0,$$

or $u(x) \in H_E^2(0,\ell)$ satisfies equation (7) in the sense of distribution.

Next, since equation (7) is a regular elliptic boundary value problem, from classical elliptic theory [3],

$$u \in H^4(0,\ell).$$

Because $u_0(x) \in H^2(0,\ell)$, we see that $v(x) = u(x) - u_0(x) \in H^2(0,\ell)$, and

$$(I + \mathcal{A})(u(x), v(x), v'(0), v(\ell), v'(\ell)) = (u_0(x), v_0(x), a_1, a_2, a_3).$$

So, $\mathcal{R}(I+\mathcal{A}) = \mathcal{H}$. Finally, if there is $U \in \mathcal{H}$ such that

$$\langle U, V\rangle = 0, \quad \forall V \in D(\mathcal{A}).$$

Assume that $U = (I+\mathcal{A})U_0$ for some $U_0 \in \mathcal{H}$. Then, $\langle U_0, U_0\rangle \le \langle U, U_0\rangle = 0$, which implies that $U_0 = U = 0$. Thus $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Since the operator $\mathcal{A}$ is maximal monotone with the dense domain $D(\mathcal{A})$ in the energy space $\mathcal{H}$, by applying a method developed in [2] to the evolution equation (2), we obtain the following existence result.

Theorem 1: *Assume that validity of* (3). *Then we have*

(*i*) *For* $Y_0 = (y_0(x), y_1(x), y_1'(0), y_1(\ell)),\ y_1'(\ell)^T \in D(\mathcal{A})$, *equation* (2) (*and* (5)) *has a unique strong solution* $Y(t)$ *with* $Y(0) = Y_0$, *such that*

$$Y(t) = (y(x,t), y_t(x,t), y_{xt}(0,t), y_t(\ell,t), y_{xt}(\ell,t))^T \in D(\mathcal{A}), \ \forall t \ge 0,$$

$$y(x,t) \in W^{1,\infty}(\mathbb{R}^+; H^2(0,\ell)) \cap L^\infty(\mathbb{R}^+; H^4(0,\ell)),$$

$$(y_{xt}(0,t), y_t(\ell,t), y_{xt}(\ell,t)) \in (W^{1,\infty}(\mathbb{R}^+, \mathbb{R}))^3,$$

$$\| Y(t) \| \le \| Y_0 \| .$$

(*ii*) *For any initial data* $Y_0 = (y_0(x), y_1(x), a_1, a_2, a_3)^T \in \mathcal{H}$, *equation* (2) (*and* (5)) *has a unique weak solution, with*

$$Y(t) = (y(x,t), y_t(x,t), y_{xt}(0,t), y_t(\ell,t), y_{xt}(\ell,t))^T = S(t)Y_0, \ \ \forall t \ge 0,$$

such that

$$y(x,t) \in C^0(\mathbb{R}^+, H^2(0,\ell)) \cap C^1(\mathbb{R}^+, L^2(0,\ell)),$$

$$(y_{xt}(0,t), y_t(\ell,t), y_{xt}(\ell,t)) \in (C^1(\mathbb{R}^+, \mathbb{R}))^3,$$

where $\{S(t)\}_{t \ge 0}$ *denotes the strongly continuous semigroup of contractions on* $\mathcal{H}$ *generated by maximal monotone operator* $\mathcal{A}$.

Lemma 2: *The following holds true:*

$$0 \in \mathcal{R}(\mathcal{A}),\ (I+\mathcal{A})^{-1} \textit{ is compact.}$$

Proof: $0 \in \mathcal{R}(\mathcal{A})$ immediately follows from the definition. Thus, we only consider the second condition. Let $\{V_n\} \subseteq \mathcal{H}$, $\| V_n \| \le C$, be a bounded series and let $\{U_n\}$ satisfy $(I+\mathcal{A})U_n = V_n$. Then, by the monotonicity of $\mathcal{A}$, $\| AU_n \| \le \| V_n \| \le C$ and $\| U_n \| \le \| V_n \| \le C$. These imply that

$$\| u_n \|_{H^4} \le C_1, \ \| v_n \|_{H^2} \le C_1, \ \ | v_n'(0) | \le C_1, \ | v_n(\ell) | \le C_1, \ | v_n'(\ell) | \le C_1$$

for some uniform constant C_1, provided that

$$U_n = (u_n(x), v_n(x), v_n'(0), v_n(\ell), v_n'(\ell))^T.$$

By the Sobolev embedding theorem, there is a subsequence of U_n, still indexed by n for notational simplicity, and $U_0 \in H^2(0,\ell) \times L^2(0,\ell) \times \mathbb{R}^2$, such that

$$U_n \to U_0 \text{ in the topology of } \mathcal{H}.$$

$u_0(0) = 0$ is hence an obvious fact. The proof is complete.

Lemma 3: *Let $S(t)$ be the semigroup defined by Theorem 1. If for some $Y_0 = (y_0(x), y_1(x), y_1'(0), y_1(\ell), y'(\ell))^T \in D(\mathcal{A})$,*

$$\frac{dE(t)}{dt} = \frac{d}{dt} \| S(t)Y_0 | \equiv 0, \quad t \geq 0,$$

then $Y_0 = 0$.

Proof: Let $Y(t) = (y(x,t), y_t(x,t), y_{xt}(0,t),\ y_t(\ell,t),\ y_{xt}(\ell,t))^T$ be a solution of equation (2) with the initial condition $Y(0) = Y_0$. By (6) and assumption (3), $\frac{d}{dt} \| S(t)Y_0 \| \equiv 0$ means that

$$\begin{cases} \rho y_{tt}(x,t) + EIy_{xxxx}(x,t) = 0, \quad 0 < x < \ell, \quad t > 0, \\ y(0,t) = y_{xt}(0,t) = 0, \\ EIy_{xx}(0,t) - \alpha y_x(0,t) = 0, \\ EIy_{xxx}(\ell,t) - My_{tt}(\ell,t) = 0, \\ EIy_{xx}(\ell,t) + Jy_{xtt}(\ell,t) = 0. \end{cases} \tag{8}$$

Multiplying by x on both sides of the first equation of (8) and integrating it from 0 to ℓ with respect to x, we have

$$\rho \int_0^\ell zy_{tt}(z,t)dz + M\ell y_{tt}(\ell,t) + Jy_{xtt}(\ell,t) + EIy_{xx}(0,t) = 0. \tag{9}$$

Then, integrating (9) from 0 to $T > 0$ with respect to t one gets

$$EI \int_0^T y_{xx}(0,t)dt$$
$$= \rho \int_0^\ell z[y_t(z,T) - y_t(z,0)]dz - M\ell[y_t(\ell,T) - y_t(\ell,0)] - J[y_{xt}(\ell,T) - y_{xt}(\ell,0)],$$

which, along with the fact that $E(t) = E(0)$, imply that

$$| \int_0^T y_{xx}(0,t)dt | \leq Const.$$

Furthermore, noticing that $y_x(0,t)$ is constant, from the boundary condition at $x = 0$, we obtain

$$\alpha y_x(0,t)T = \int_0^T EIy_{xx}(0,t)dt, \quad \forall T > 0,$$

which implies that

$$\alpha y_x(0,t) = y_{xx}(0,t) = y_{xtt}(0,t) = 0, \quad \forall t \geq 0.$$

Next, multiplying by $y(x,t)$ and $2(x - \ell - \epsilon)y_x(x,t)$ both sides of the first equation of (8) and then integrating with respect to x and t, respectively, where $\epsilon > 0$ is constant (to be determined), we obtain

$$-\int_0^T\int_0^\ell [\rho y_{tt}(x,t)y(t,x) + EIy_{xxxx}(x,t)y(x,t)]dxdt$$

$$= -EI\int_0^T\int_0^\ell y_{xx}^2 dxdt + \rho\int_0^T\int_0^\ell y_t^2 dxdt + M\int_0^T y_t^2(\ell,t)dt$$

$$+ J\int_0^T y_{xt}^2(\ell,t)dt - My_ty(\ell,t)\,|_0^T + Jy_{xt}y_x(\ell,t)\,|_0^T - \rho\int_0^\ell y_ty(x,t)\,|_0^T dx = 0, \tag{10}$$

and

$$2\int_0^T\int_0^\ell (x-\ell-\epsilon)y_x(x,t)[\rho y_{tt}(x,t) + EIy_{xxxxx}(x,t)]dxdt$$

$$= \int_0^T\int_0^\ell [\rho y_t^2 + 3EIy_{xx}^2]dxdt + \rho\epsilon\int_0^T y_t^2(\ell,t)dt + 2M\epsilon\int_0^T y_{xt}(\ell,t)y_t(\ell,t)dt$$

$$-2J\int_0^T y_{xt}^2(\ell,t)dt + 2\rho\int_0^\ell (x-\ell-\epsilon)y_ty_x\,|_0^T dx - 2M\epsilon y_x(\ell,t)y_t(\ell,t)\,|_0^T$$

$$+ 2Jy_x(\ell,t)y_{xt}(\ell,t)\,|_0^T + EI\epsilon\int_0^T y_{xx}^2(\ell,t)dt = 0. \tag{11}$$

Computing $(2+\delta)\times(10)+(11)$ for $\delta = 2M/J\epsilon$, we have

$$\int_0^T\int_0^\ell [\rho(3+\delta)y_t^2 + EI(1-\delta)y_{xx}^2]dxdt + [(2+\delta)M + \rho\epsilon]\int_0^T y_t^2(\ell,t)dt$$

$$+ J\delta\int_0^T y_{xt}^2(\ell,t)dt + 2M\epsilon\int y_{xt}(\ell,t)y_t(\ell,t)dt + EI\epsilon\int_0^T y_{xx}^2(\ell,t)dt + C(t) = 0,$$

where

$$C(t) = (2+\delta)[-My_t(\ell,t)y(\ell,t)\,|_0^T + Jy_{xt}(\ell,t)y_x(\ell,t)\,|_0^T + \rho\int_0^\ell yy_t\,|_0^T dx]$$

$$-2\rho\int_0^\ell (x-\ell-\epsilon)y_ty_x\,|_0^T dx - 2M\epsilon y_x(\ell,t)y_t(\ell,t)\,|_0^T + 2Jy_x(\ell,t)y_{xt}(\ell,t)\,|_0^T.$$

Obviously, $|C(t)| \le Const\cdot E(t)$. So,

$$\int_0^T\int_0^\ell [\rho(3+\delta)y_t^2 + EI(1-\delta)y_{xx}]dxdt$$

$$\leq -[(2+\delta)M+\rho\epsilon - M\epsilon]\int_0^T y_t^2(\ell,t)dt - [J\delta - M\epsilon]\int_0^T y_{xt}^2(\ell,t)dt - C(t). \tag{12}$$

$$\leq -(2-\epsilon)M\int_0^T y_t^2(\ell,t)dt - M\epsilon\int_0^T y_{xt}^2(\ell,t)dt - C(t).$$

Taking $0 < \epsilon < \min\{1, J/(2M)\}$, (12) implies that

$$\int_0^T E(t)dt \leq Const.$$

Thus $E(t) \equiv 0$ by the arbitrariness of T and $E(t) = E(0)$. The proof is complete.

By Theorem 1, Lemmas 2 and 3, and using LaSalle's invariance principle [1], we have immediately:

Theorem 2: *Let $E(t)$ be the energy of the hybrid system (2). Then,*

$$\lim_{t\to\infty} E(t) = 0.$$

3. The Uniform Decay

To get the uniform decay, we design, in this section, a boundary feedback control as

$$u(t) = I_m y_{xtt}(0,t) - \alpha y_x(0,t) - f(y_{xt}(0,t)). \tag{13}$$

Then the closed loop system would be

$$\begin{cases} \rho y_{tt}(x,t) + EIy_{xxxx}(x,t) = 0, \quad 0 < x < \ell, \quad t > 0, \\ y(0,t) = 0, \\ EIy_{xx}(0,t) - \alpha y_x(0,t) - f(y_{xt}(0,t)) = 0, \\ EIy_{xxx}(\ell,t) - My_{tt}(\ell,t) = 0, \\ EIy_{xx}(\ell,t) + Jy_{xtt}(\ell,t) = 0. \end{cases} \tag{14}$$

In this case, the energy takes the form

$$E(t) = \frac{1}{2}\int_0^\ell [\rho y_t^2 + EIy_{xx}^2]dx + \tfrac{1}{2}\alpha y_x^2(0,t) + \tfrac{1}{2}My_t^2(\ell,t) + \tfrac{1}{2}Jy_{xt}^2(\ell,t). \tag{15}$$

Consequently, the underlying state space becomes $\mathcal{H} = H_E^2(0,\ell) \times L^2(0,\ell) \times \mathbb{R}^2$ with the inner product

$$\langle (u(x), v(x), a_1, a_2), (\overline{u}(x), \overline{v}(x), \overline{a}_1, \overline{a}_2)\rangle$$

$$= \frac{1}{2} \int_0^\ell [EIu''(x)\bar{u}''(x) + \rho v(x)\bar{v}(x)]dx + \frac{1}{2}\alpha u'(0)\bar{u}'(0) + \frac{1}{2}[Ma_1\bar{a}_1 + Ja_2\bar{a}_2],$$

where $H_E^2(0,\ell)$ is defined in section 2. Equation (14) can also be written as a nonlinear evolution equation in $\mathcal{H}$:

$$\begin{cases} \frac{dY(t)}{dt} + \mathcal{A}Y(t) = 0, \\ Y(0) = Y_0, \end{cases} \tag{16}$$

but instead of (4), here $\mathcal{A}: D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ is defined by

$$\mathcal{A}(u(x), v(x), v(\ell), v'(\ell)) = (-v(x), \frac{EI}{\rho}u^{(4)}(x), -\frac{EI}{M}u'''(\ell), \frac{EI}{J}u''(\ell)) \tag{17}$$

$$D(\mathcal{A}) = \{(u(x), v(x), v(\ell), v'(\ell)), u \in H^4(0,\ell),$$

$$v \in H^2(0,\ell), u(0) = v(0) = EIu''(0) - \alpha u'(0) - f(v'(0)) = 0\}$$

and state variable $Y(t) = (y(x,t), y_t(x,t), y_t(\ell,t), y_{xt}(\ell,t))^T$.

Following the same line as that of section 2, we also have

Theorem 3: *The operator $\mathcal{A}$ defined by (17) is maximal monotone with the dense domain in the space $\mathcal{H}$ and hence generates an asymptotically stable nonlinear semigroup of contractions on $\mathcal{H}$.*

In the sequel, we always assume that the initial data belongs to the domain of operator $\mathcal{A}$ and hence the solution of equation (14) has the regularity properties expressed by (*ii*) of Theorem 1. Let $\delta > 0$ and $\phi(x) = ax - a\ell - 1$, where a is a constant to be determined. Define

$$\begin{aligned} \beta(t) = {} & 2\int_0^\ell \phi(x) y_t(x,t) y_x(x,t)dx \\ & - 2\phi(0)y_x(0,t)[\int_0^\ell y_t dx + \frac{\alpha}{EI}\int_0^\ell xy_t dx + \frac{M}{\rho}(1 + \frac{\alpha\ell}{EI})y_t(\ell,t) + \frac{\alpha J}{EI\rho}y_{xt}(\ell,t)] \\ & + \frac{2}{\rho}y_x(\ell,t)[M\phi(\ell)y_t(\ell,t) + J\phi'(\ell)y_{xt}(\ell,t)] \\ & + (2+\delta)\phi'[\int_0^\ell y_t y dx + \frac{M}{\rho}y(\ell,t)y_t(\ell,t) + \frac{J}{\rho}y_x(\ell,t)y_{xt}(\ell,t)] \end{aligned} \tag{18}$$

Lemma 4: *Let $\beta(t)$ be defined by (18). Then there exist a $\delta > 0$ and positive constants C_0, C_1, C_2, and C_3 such that*

$$|\beta(t)| \le C_0 E(t), \tag{19}$$

$$B'(t) \le -C_1 E(t) + C_2 y_{xt}^2(0,t) + C_3 f^2(y_{xt}(0,t)). \tag{20}$$

Proof: By the defined form of $\beta(t)$ and Sobolev's embedding theorem, we can always find a constant C_0 such that $|\beta(t)| \le C_0 E(t)$ once the constants a and δ are determined. To prove the second condition, we find the derivative $\beta'(t)$ directly.

Let

$$\beta_1(t) = 2\int_0^\ell \phi(x)y_t(x,t)y_x(x,t)dx$$

$$\beta_2(t) = -2\phi(0)y_x(0,t)[\int_0^\ell y_t dx + \frac{\alpha}{EI}\int_0^\ell xy_t dx + \frac{M}{\rho}(1+\frac{\alpha\ell}{EI})y_t(\ell,t) + \frac{\alpha J}{EI\rho}y_{xt}(\ell,t)]$$

$$\beta_3(t) = \frac{2}{\rho}y_x(\ell,t)[M\phi(\ell)y_t(\ell,t) + J\phi'(\ell)y_{xt}(\ell,t)]$$

$$\beta_4(t) = (2+\delta)\phi'[\int_0^\ell y_t y dx + \frac{M}{\rho}y(\ell,t)y_t(\ell,t) + \frac{J}{\rho}y_x(\ell,t)y_{xt}(\ell,t)].$$

After a tedious but very straightforward calculation we find that

$$\beta_1'(t) = -\int_0^\ell \phi' y_t^2 dx - \frac{3EI}{\rho}\int_0^\ell \phi' y_{xx}^2 dx + \phi(\ell)y_t^2(\ell,t)$$

$$-\frac{2EI}{\rho}[\phi y_{xxx}y_x\,|_0^\ell - \phi'(\ell)y_x(\ell,t)y_{xx}(\ell,t) + \frac{1}{2}\phi(\ell)y_{xx}^2(\ell,t)]$$

$$-\frac{2\alpha}{\rho}\phi(0)y_{xx}(0,t)y_x(0,t) + \frac{\alpha}{\rho}(\frac{\alpha\phi(0)}{EI} - 2\phi'(0)y_x^2(0,t)$$

$$+\frac{2\phi'(0)}{\rho}y_x(0,t)f(y_{xt}(0,t)) - \frac{\phi(0)}{\rho EI}f^2(y_{xt}(0,t))$$

$$\beta_2'(t) = -2\phi(0)y_{xt}(0,t)[\int_0^\ell y_t dx + \frac{2}{EI}\int_0^\ell xy_t dx + \frac{M}{\rho}(1+\frac{\alpha\ell}{EI})y_t(\ell,t) + \frac{\alpha J}{EI\rho}y_{xt}(\ell,t)]$$

$$-2\phi(0)y_x(0,t)[-\frac{EI}{\rho}y_{xxx}(0,t) - \frac{\alpha}{\rho}y_{xx}(0,t)]$$

$$\beta_3'(t) = \frac{2}{\rho}y_{xt}(\ell,t)[M\phi(\ell)y_t(\ell,t) + J\phi'(\ell)y_{xt}(\ell,t)]$$

$$+\frac{2}{\rho}y_x(\ell,t)[M\phi(\ell)y_{tt}(\ell,t) + J\phi'(\ell)y_{xtt}(\ell,t)]$$

$$\beta_4'(t) = -(2+\delta)\phi'[\int_0^\ell y_t^2 dx - \frac{EI}{\rho}\int_0^\ell y_{xx}^2 dx + \frac{M}{\rho}y_t^2(\ell,t)$$

$$+\frac{J}{\rho}y_{xt}^2(\ell,t) - \frac{\alpha}{\rho}y_x^2(0,t) - \frac{1}{\rho}y_x^2(0,t)f(y_{xt}(0,t))].$$

Therefore,

$$\beta'(t) = \beta_1'(t) + \beta_2'(t) + \beta_3'(t) + \beta_4'(t)$$

$$= -a\int_0^\ell [(3+\delta)y_t^2 + (1-\delta)\frac{EI}{\rho}y_{xx}^2]dx - \frac{\alpha}{\rho}[(\frac{\alpha\ell}{EI} - \delta)a + \frac{\alpha}{EI}]y_x^2(0,t)$$

$$
\begin{aligned}
&-[1+(2+\alpha)\frac{Ma}{\rho}]y_t^2(\ell,t)-\frac{J\delta a}{\rho}y_{xt}^2(\ell,t)\\
&-\frac{EI}{\rho}y_{xx}^2(\ell,t)-\frac{2M}{\rho}y_{xt}(\ell,t)y_t(\ell,t)-\beta_0(t) \qquad (21)\\
\le & -a\int_0^\ell [(3+\delta)y_t^2+(1-\delta)\frac{EI}{\rho}y_{xx}^2]dx-\frac{\alpha}{\rho}[(\frac{\alpha\ell}{EI}-\delta)a+\frac{\alpha}{EI}]y_x^2(0,t)\\
&-\left(1+\frac{M}{\rho}[(2+\delta)a-\frac{1}{\theta}]\right)\, y_t^2(\ell,t)-\frac{1}{\rho}(\delta Ja-M\theta)y_{xt}^2(\ell,t)-\beta_0(t),
\end{aligned}
$$

where $\theta>0$ is an arbitrary constant and

$$
\begin{aligned}
\beta_0(t)=&\,2\phi(0)y_{xt}(0,t)[\int_0^\ell y_t dx+\frac{\alpha}{EI}\int_0^\ell xy_t dx+\frac{M}{\rho}(1+\frac{\alpha\ell}{EI})y_t(\ell,t)+\frac{\alpha J}{EI\rho}y_{xt}(\ell,t)]\\
&+\frac{\phi(0)}{EI\rho}f^2(y_{xt}(0,t))-\frac{\phi'(0)\delta}{EI\rho}y_x(0,t)f(y_{xt}(0,t)).
\end{aligned}
$$

Take

$$
\delta=\min\{1/2,\frac{\alpha\ell}{2EI}\},\ a=\frac{M^2+\rho}{\rho J\delta},\ \theta=\frac{M}{\rho}. \qquad (22)
$$

With these defined constants,

$$
1-\delta\ge 1/2,\ \ (\frac{\alpha\ell}{EI}-\delta)a+\frac{\alpha}{EI}\ge\frac{\alpha}{EI},\ \delta Ja-M\theta=1,
$$

$$
1-\frac{M}{\rho\theta}+(2+\delta)\frac{Ma}{\rho}=\frac{(2+\delta)M(\rho+M^2)}{\rho^2J\delta}=\mu>0,
$$

and hence

$$
\beta'(t)\le -a\int_0^\ell [3y_t^2+\frac{EI}{2\rho}y_{xx}^2]dt-\frac{\alpha^2}{\rho EI}y_x^2(0,t)-\mu y_t^2(\ell,t)-\frac{1}{\rho}y_{xt}^2(\ell,t)-\beta_0(t). \qquad (23)
$$

Since $E(t)\le E(0)$, it is easily seen from (21) and the definition of $\beta_0(t)$ that there exist positive constants C_1, C_2, and C_3 such that

$$
\beta'(t)\le -C_1E(t)+C_2y_{xt}^2(0,t)+C_3f^2(y_{xt}(0,t)).
$$

The proof is complete.

Along the same line as that of [4], we have the following uniform decay estimate as that of [4] made for the hybrid string equation.

Theorem 4: *Let* (3) *hold true. Let y be the solution of equation* (14) *with initial data being in $D(\mathcal{A})$.*

(*i*) *If there exist positive constants L_1 and L_2 such that*

$$
L_1|s|\le|f(s)|\le L_2|s|,\quad \forall s\in\mathbb{R}, \qquad (24)
$$

then, given any $K>1$, there exists a constant $\omega>0$ such that

$$
E(t)\le KE(0)\,exp(-\omega t),\ \ \forall t\ge 0. \qquad (25)
$$

(ii) *If there exist positive constants* L_1, L_2 *and* $p > 1$ *such that*

$$L_1 \min\{|s|, |s|^p\} \leq |f(s)| \leq L_2 |s|, \ \forall s \in \mathbb{R}, \tag{26}$$

then, given any $K > 1$, *there exists a constant* $\omega > 0$ *depending continuously on* $E(0)$ *such that*

$$E(t) \leq KE(0)(1 + \omega t)^{-2/(p-1)}, \ \forall t \geq 0. \tag{27}$$

(iii) *If there exist positive constants* L_1 *and* L_2 *and* $0 < p < 1$ *such that*

$$L_1 |s| \leq |f(s)| \leq L_2 \max\{|s|, |s|^p\}, \ \forall s \in \mathbb{R}, \tag{28}$$

then, given any $K > 1$, *there exists a constant* $\omega > 0$ *depending continuously on* $E(0)$ *such that*

$$E(t) \leq KE(0)(1 + \omega t)^{-2p/(p-1)}, \ \forall t \geq 0. \tag{29}$$

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