ASYMPTOTIC APPROXIMATIONS TO THE BAYES POSTERIOR RISK

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ABSTRACT

Suppose that, given $\omega = (\omega_1, \omega_2) \in \Re^2$, $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent random variables and their respective distribution functions G_{ω_1} and G_{ω_2} belong to a one parameter exponential family of distributions. We derive approximations to the posterior probabilities of ω lying in closed convex subsets of the parameter space under a general prior density. Using this, we then approximate the Bayes posterior risk for testing the hypotheses H_0: $\omega \in \Omega_1$ versus H_1: $\omega \in \Omega_2$ using a zero-one loss function, where Ω_1 and Ω_2 are disjoint closed convex subsets of the parameter space.

Key Words and Phrases: exponential families of distributions, Bayes risk, testing hypotheses, indifference zone.

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1. INTRODUCTION

Let Γ be a non-degenerate open interval of the real line $\Re$ and let G_γ, $\gamma \in \Gamma$, be a one parameter exponential family of probability distributions with natural parameter space Γ, that is,

$$G_\gamma\{dx\} = \exp\{\gamma x - \psi(\gamma)\}\mu(dx) \tag{1.1}$$

for $x \in \Re$ and $\gamma \in \Gamma$, where $\mu(\cdot)$ is a non-degenerate sigma-finite measure on $\Re$, $\exp\{\psi(\gamma)\}= \int \exp\{\gamma x\}\mu(dx)$ and $\Gamma=\{\gamma \in \Re : \int \exp\{\gamma x\}\mu(dx) < +\infty\}$. The function ψ is strictly convex and its second derivative is positive on Γ. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be independent random variables and suppose that the X's have common distribution function G_{ω_1}, for some unknown $\omega_1 \in \Gamma$, and the Y's have common distribution function G_{ω_2}, for an unknown $\omega_2 \in \Gamma$. Further, suppose that $\omega = (\omega_1, \omega_2)$ are jointly distributed on Γ^2 with a joint prior density Π.

Consider the problem of testing hypotheses of the form H_0: $\omega \in \Omega_1$ versus H_1: $\omega \in \Omega_2$ using a zero-one loss function, where Ω_1 and Ω_2 are disjoint closed convex subsets of Γ^2. There is a unit loss for making a wrong decision when $\omega \in \Omega_1 \cup \Omega_2$ and no loss when $\omega \in \Gamma^2 - (\Omega_1 \cup \Omega_2)$. In the literature this last subset is called an indifference zone. The concept of indifference zone was introduced by Schwarz (1962). A zero-one loss function for testing H_0 versus H_1 with the above indifference zone may be written $L(\omega, q) = qI_{\{\omega\in\Omega_1\}} + (1-q)I_{\{\omega\in\Omega_2\}}$, where $q = 0$ or 1 to indicate acceptance of H_0 or H_1, respectively, and $I_{\{\cdot\}}$ denotes the set indicator function.

Suppose we are to decide between H_0 and H_1 based on the observations $X_1, \ldots, X_{tN}$ and $Y_1, \ldots, Y_{(1-t)N}$, where N and Nt are fixed integer and $t \in (0,1)$. The posterior risk of any procedure (test) $q_N=q_N(X_1, \ldots, X_{tN}, Y_1, \ldots$

$, Y_{(1-t)N})$ for choosing between H_0 and H_1 is $\mathrm{E}[L(\omega, q_N)|\mathcal{F}_N] = q_N \mathrm{P}[\omega \in \Omega_1|\mathcal{F}_N] - (1-q_N)\mathrm{P}[\omega \in \Omega_2|\mathcal{F}_N]$, where $q_N = 0$ if H_0 is accepted and 1 otherwise and $\mathcal{F}_N$ is the sigma-algebra generated by $X_1, \ldots, X_{tN}, Y_1, \ldots,$ $Y_{(1-t)N}$. The posterior risk $\mathrm{E}[L(\omega, q_N)|\mathcal{F}_N]$ is minimized by the Bayes procedure: accept H_0 ($q_N = 0$) if $\mathrm{P}[\omega \in \Omega_1|\mathcal{F}_N] \geq \mathrm{P}[\omega \in \Omega_2|\mathcal{F}_N]$ and reject H_0 ($q_N = 1$) otherwise; is used. The posterior risk of the Bayes procedure, called the Bayes posterior risk, is

$$
\begin{aligned}
r(\Pi, \mathbf{z}) &= \min\left\{\mathrm{P}[\omega \in \Omega_1|\mathcal{F}_N],\ \mathrm{P}[\omega \in \Omega_2|\mathcal{F}_N]\right\} \\
&= \frac{\min_{i=1,2}\left\{\int_{\Omega_i} \exp[N\, l(\omega, \mathbf{z})]\Pi(\omega)d\omega\right\}}{\int_{\Gamma^2} \exp\{N\, l(\omega, \mathbf{z})\}\Pi(\omega)d\omega}
\end{aligned} \tag{1.2}
$$

where $l(\omega, \mathbf{z}) = t\omega_1\overline{X} - t\psi(\omega_1) + (1-t)\omega_2\overline{Y} - (1-t)\psi(\omega_2)$, $\mathbf{z} = (\overline{X}, \overline{Y})$ and $\overline{X}$ and $\overline{Y}$ denote the averages of $X_1, \ldots, X_{tN}$ and $Y_1 \ldots, Y_{(1-t)N}$ respectively.

The main goal of this paper is to derive approximations to the Bayes posterior risk $r(\Pi, \mathbf{z})$, which does not always have an explicit expression. Approximations to $r(\Pi, \mathbf{z})$ find applications in sequential analysis, where the experimenter may need to evaluate $r(\Pi, \mathbf{z})$ at each stage in order to decide whether to stop the experiment or not and/or whether to observe an X or a Y next. Since the Bayes posterior risk is the minimum of the posterior probabilities of Ω_1 and Ω_2 we will first consider the more general problem of approximating posterior probabilities of closed convex subsets of Γ^2 then approximations to $r(\Pi, \mathbf{z})$ will follow as a corollary. Bickel and Yahav (1969) studied the Bayes posterior risk of testing disjoint hypotheses using a zero-one loss function with indifference zone. They showed that for a certain class of distributions the nth root of the posterior risk converges to a quantity related to the Kullback-Liebler information numbers, as the sample size increases to infinity.

Let $P[\Omega|\mathbf{z}]$ denote the posterior probability of a subset Ω of Γ^2 given the sufficient statistics $\mathbf{z}$ and let

$$(1.3) \qquad P(\Omega;\ \mathbf{z}) = \int_{\Omega} \exp\{N\, l(\omega, \mathbf{z})\}\Pi(\omega)d\omega.$$

Hence $P(\Omega|\mathbf{z})=P(\Omega;\ \mathbf{z})/P(\Gamma^2;\ \mathbf{z})$. Let $\hat{\omega}(\mathbf{z})$ and $\tilde{\omega}(\mathbf{z})$ denote the maximum likelihood estimators (M.L.E.) of ω on Γ^2 and Ω, respectively. Recall, the M.L.E. of ω over a set Ω is the point $\tilde{\omega} = \tilde{\omega}(\mathbf{z}) \in \Omega$ such that $\sup_{\omega \in \Omega} l(\omega, \mathbf{z})= l(\tilde{\omega}, \mathbf{z})$. The subset $\psi'(\Gamma) \times \psi'(\Gamma)$ of $\Re^2$ is called the expectation space, where prime denotes derivative here. When $\mathbf{z} \in \psi'(\Gamma) \times \psi'(\Gamma)$ then by Lemma 2.1 the M.L.E.'s $\hat{\omega}(\mathbf{z})$ and $\tilde{\omega}(\mathbf{z})$ exist and are unique.

The main results of this paper are given in Section 5. Proposition 5.1 gives approximations to posterior probabilities of closed convex subsets of Γ^2. The Bayes posterior risk $r(\Pi, \mathbf{z})$ is approximated in Corollary 5.1. Sections 2 through 4 contain technical results. In Section 2 we give some properties of exponential families of distributions. In Section 3, we derive approximations to $P(\Omega;\ \mathbf{z})$ when the M.L.E. $\hat{\omega}(\mathbf{z}) \notin \Omega$ and in Section 4 we approximate $P(\Omega;\ \mathbf{z})$ when $\hat{\omega}(\mathbf{z}) \in \Omega^0$, the interior of Ω.

2. PRELIMINARY RESULTS

In this section we present some results of exponential families of that will be needed in subsequent sections. For the proofs of most of these results the reader is refered to the monograph by Lawrence Brown on *Fundamentals of Exponential Statistical Families*. The first result states that when the natural parameter space Γ is open then the expectation space $\psi'(\Gamma) \times \psi'(\Gamma)$ is also open and both the maximum likelihood estimator (M.L.E.) over Γ^2 and the M.L.E. over any closed convex subset Ω of Γ^2 exist and are unique.

Lemma 2.1: *If* Γ *is open then* $\psi'(\Gamma)\times\psi'(\Gamma)$ *is open and for any* $\mathbf{z}\in\psi'(\Gamma)\times\psi'(\Gamma)$ *the Maximum Likelihood Estimator of* ω*, denoted by* $\hat{\omega}=\hat{\omega}(\mathbf{z})$*, exists and is unique.*

If Ω *is a closed convex region of* Γ^2*, then for every fixed* $\mathbf{z}\in\psi'(\Gamma)\times\psi'(\Gamma)$ *there exists a unique point* $\tilde{\omega}=\tilde{\omega}(\mathbf{z})\in\Omega$ *such that* $\sup_{\omega\in\Omega} l(\omega,\mathbf{z}) = l(\tilde{\omega},\mathbf{z})$*. Moreover if the M.L.E.* $\hat{\omega}\notin\Omega$ *then* $\tilde{\omega}$ *lies on the boundary of* Ω*.*

For a proof of this Lemma the reader is refered to Brown (1986) (Theorem 5.5, page 148 and Theorem 5.8, page 154).

Lemma 2.2: *Let* $C_\delta(\omega)$ *denote a closed ball with center* ω *and radius* δ*. Let* $A_\epsilon(\mathbf{z})=\{\omega\in\Omega : l(\tilde{\omega},\mathbf{z})-l(\omega,\mathbf{z})\le\epsilon\}$ *and let* F *be a compact subset of* $\psi'(\Gamma)\times\psi'(\Gamma)$*. Then* $\forall\delta>0$, $\exists\epsilon>0$ *such that* $A_\epsilon(\mathbf{z})\subset C_\delta\big(\tilde{\omega}(\mathbf{z})\big)\quad\forall\mathbf{z}\in F$*.*

Proof of Lemma 2.2: First we will show that $A_\epsilon(\mathbf{z})$ is bounded uniformly in $\mathbf{z}$, $\forall\mathbf{z}\in F$, that is, there exists a compact subset B_ϵ of Ω such that $\forall\mathbf{z}\in F$, $A_\epsilon(\mathbf{z})\subset B_\epsilon$. We will use a proof by contradiction to prove this claim.

Suppose that $A_\epsilon(\mathbf{z})$ is not uniformly bounded, then there exist a sequence of real numbers $(\delta_m)_m$ with $\lim\delta_m=+\infty$ as $m\to+\infty$, a sequence $(\mathbf{z}_m)_m\subset F$ and a sequence $(\omega_m)_m$ such that

$$(2.1)\qquad \begin{cases}\omega_m\in A_\epsilon(\mathbf{z}_m)\\ \|\omega_m\|>\delta_m \qquad \forall m\ge 1.\end{cases}$$

Since $(\mathbf{z}_m)_m$ is in a compact set then there exist a subsequence $(\mathbf{z}_{m_1})_{m_1}$ of $(\mathbf{z}_m)_m$ and a point $\mathbf{z}_1\in F$ such that $\lim\mathbf{z}_{m_1}=\mathbf{z}_1$ as $m_1\to+\infty$. Moreover by (2.1)

$$(2.2)\qquad 0<l(\tilde{\omega}(\mathbf{z}_{m_1}),\mathbf{z}_{m_1})-l(\omega_{m_1},\mathbf{z}_{m_1})\le\epsilon\quad\forall m_1\ge 1$$

and

$$(2.3)\qquad \lim_{m\to+\infty}\|\omega_{m_1}\|\ge\lim_{m_1\to+\infty}\delta_{m_1}=+\infty.$$

By letting $m_1 \to +\infty$ in (2.2) we get a contradiction. Since, by the strict concavity of l in ω, (2.3) implies that $\lim l(\omega_{m_1}, \mathbf{z}_{m_1}) = -\infty$ as $m_1 \to +\infty$ and the continuity of l and $\tilde{\omega}$ with respect to $(\omega, \mathbf{z})$ and $\mathbf{z}$ imply that $\lim l(\tilde{\omega}(\mathbf{z}_{m_1}), \mathbf{z}_{m_1}) = l(\tilde{\omega}(\mathbf{z}_1), \mathbf{z}_1) < +\infty$ as $m_1 \to +\infty$. Hence $\forall \epsilon > 0\ \exists B_\epsilon$ compact subset of Ω such that $\forall \mathbf{z} \in F\ A_\epsilon(\mathbf{z}) \subset B_\epsilon$.

Again, using a proof by contradiction, suppose the Lemma were not true. Then there exist a $\delta > 0$, a decreasing sequence $(\epsilon_m)_m \subset \Re$, with $\epsilon_m > 0$ and $\epsilon_m \to 0$ as $m \to +\infty$, a sequence $(\mathbf{z}_m)_m \subset F$ and a sequence $(\omega_m)_m$ such that

$$(2.4) \qquad \begin{cases} \omega_m \in A_{\epsilon_m}(\mathbf{z}_m) \\ \omega_m \notin C_\delta\Big(\tilde{\omega}(\mathbf{z}_m)\Big). \end{cases}$$

Since F is compact then there exists a subsequence $(\mathbf{z}_{m_1})_{m_1}$ of $(\mathbf{z}_m)_m$ and a $\mathbf{z}_2 \in F$ such that $\lim \mathbf{z}_{m_1} = \mathbf{z}_2$ as $m_1 \to +\infty$. Observe that for $\mathbf{z}$ fixed the set $A_\epsilon(\mathbf{z})$ are closed and increasing in ϵ, that is $A_\epsilon(\mathbf{z}) \subset A_{\epsilon'}(\mathbf{z})$ for $\epsilon \leq \epsilon'$. Since, as shown above, $A_{\epsilon_1}(\mathbf{z})$ is bounded uniformly in $\mathbf{z}$ then $A_{\epsilon_m}(\mathbf{z}_m) \subset A_{\epsilon_1}(\mathbf{z}_m) \subset B_{\epsilon_1}\ \forall m \geq 1$, where B_{ϵ_1} is a compact subset of Ω containing $(\omega_m)_m$. Hence there exists a subsequence $(\omega_{m_2})_{m_2}$ of $(\omega_{m_1})_{m_1}$ and a $\omega_* \in \Omega$ such that $\lim \omega_{m_2} = \omega_*$ as $m_2 \to +\infty$. By (2.4) we have

$$\begin{aligned} l(\omega_*, \mathbf{z}_2) = \lim_{m_2 \to +\infty} [l(\omega_{m_2}, \mathbf{z}_{m_2}) + \epsilon_{m_2}] &\geq \lim_{m_2 \to +\infty} l(\tilde{\omega}(\mathbf{z}_{m_2}), \mathbf{z}_{m_2}) \\ &= l(\tilde{\omega}(\mathbf{z}_2), \mathbf{z}_2). \end{aligned}$$

and $\omega_{m_2} \notin C_\delta\Big(\tilde{\omega}(\mathbf{z}_{m_2})\Big)$. Hence ω_* is a M.L.E. over Ω and $\omega_* \neq \tilde{\omega}(\mathbf{z}_2)$ which contradicts the unicity of $\tilde{\omega}(\mathbf{z}_2)$ as stated in Lemma 2.1.

A function ϕ of a complex variable is analytic on a domain U if $\phi(u)$ can be represented as a power series in a neighborhood of every point $u_0 \in U$.

Lemma 2.3: *Let ψ be the cumulant generating function of the G_γ, as defined in (1.1). The function $exp\{-\psi(u)\}$ is analytic on $\{u \in C : Re(u) \in \Gamma\}$, where C is the set set of complex numbers and $Re(u)$ denotes the real part of u.*

For a proof of this Lemma see Brown (1986) (Theorem 2.7, page 39). Let E^d be the set of all limit points of a set E. The set E is said to be connected if and only if for any partition E_1, E_2 of E, $E_1^d \cap E_2 \neq \emptyset$ or $E_1 \cap E_2^d \neq \emptyset$, where $\emptyset$ denotes the empty set.

Lemma 2.4: *Let U and V be two connected subsets of the plan $\Re^2$. Let $\phi(u,v)$ be an analytic function on $U \times V$. Let $C_1 \subset U$ and $C_2 \subset V$ be two circles with radiuses r_1 and r_2 and let $Int(C_1)$ and $Int(C_2)$ denote their interiors respectively, then*

$$\left|\frac{\partial^{i+j}\phi(u,v)}{\partial u^i \partial v^j}\right| \leq \frac{i!j!M(r_1,r_2)}{r_1^i r_2^j}$$

for every $(u,v) \in Int(C_1) \times Int(C_2)$, where $M(r_1,r_2) = \sup_{C_1 \times C_2} \mid \phi(\xi,\eta) \mid$ and ∂ denotes the partial derivative operator.

For a proof of this Lemma see Markushevich (1965), (Theorem 3.8, page 105, volume 2). In the sequel of this paper Ω will denote a closed convex subset of Γ^2 and $||\cdot||$ the euclidean norm on $\Re^2$.

3. APPROXIMATING $\mathbf{P}(\Omega;\mathbf{z})$: M.L.E. $\notin \Omega$

Let Ω be a closed convex subset of Γ^2. Recall that if $\hat{\omega}(\mathbf{z}) \notin \Omega$ then, by Lemma 2.1, $\tilde{\omega}(\mathbf{z})$ lies on the boundary of Ω. Next, for each $\mathbf{z} \in \psi'(\Gamma) \times \psi'(\Gamma)$ such that $\hat{\omega}(\mathbf{z}) \notin \Omega$ consider the following reparametrization of $l(\omega,\mathbf{z})$: To each $\omega = (\omega_1,\omega_2) \in \Gamma^2$ associate its coordinates, say $\theta = (\theta_\perp,\theta_p)$, in the set of axises obtained by rotating clockwise the ω-axises such that the ω_2-axis becomes parallel to the tangent to the boundary of Ω at $\tilde{\omega}(\mathbf{z})$. We will call the new ω_2-axis the θ_p-axis and the new ω_1-axis the $\theta_\perp$-axis.

Hence

$$\begin{cases} \omega_1 = \omega_1(\theta) = \theta_\perp \sin\lambda + \theta_p \cos\lambda \\ \omega_2 = \omega_2(\theta) = \theta_\perp \cos\lambda - \theta_p \sin\lambda \end{cases} \tag{3.1}$$

where $\lambda = \lambda(\mathbf{z})$ denotes the angle of the rotation. Let Θ denote the parameter space Γ^2 in the θ-axises and let $h(\theta, \mathbf{z})$ denote the reparametrized function $l(\omega(\theta), \mathbf{z})$, that is, $\forall \mathbf{z} \in \psi'(\Gamma) \times \psi'(\Gamma)$ with $\hat{\omega}(\mathbf{z}) \notin \Omega$ let $h(\theta, \mathbf{z}) = l(\omega(\theta), \mathbf{z})$, $\forall \theta \in \Theta$. For simplicity of notations we will use Ω to denote the set Ω before and after reparametrization. Also Π will denote both the prior density of ω and θ. Let $\hat{\theta}(\mathbf{z})$ and $\tilde{\theta}(\mathbf{z})$ be the Maximum Likelihood Estimators of θ over Θ and Ω. Observe that $\forall \mathbf{z} \in \psi'(\Gamma) \times \psi'(\Gamma)$ such that $\hat{\omega}(\mathbf{z}) \notin \Omega$

$$\frac{\partial h}{\partial \theta_p}(\tilde{\theta}(\mathbf{z}), \mathbf{z}) = 0, \tag{3.2}$$

$$\frac{\partial^2 h}{\partial \theta_p^2}(\tilde{\theta}, \mathbf{z}) = -\psi''(\tilde{\omega}_1) t \cos^2\lambda - \psi''(\tilde{\omega}_2)(1-t)\sin^2\lambda < 0 \tag{3.3}$$

and

$$[\theta_\perp - \tilde{\theta}_\perp(\mathbf{z})] \frac{\partial h}{\partial \theta_\perp}(\tilde{\theta}(\mathbf{z}), \mathbf{z}) < 0 \quad \forall \theta \in \Omega. \tag{3.4}$$

The main result of this section, that is the approximation of $P(\Omega; \mathbf{z})$, is given in Proposition 3.1. Next we introduce three lemmas that will be needed in the proof of Proposition 3.1. The proofs of the following Lemmas are extensions of Johnson (1967) to the two sample case. In this paper, Johnson gives an asymptotic expansion for posterior distributions when the observations come from a distribution belonging to a one-parameter exponential family of distributions.

Lemma 3.1: *Let* $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. *Assume that* Π *is bounded, twice continuously differentiable in a neighborhood of* $\tilde{\theta} = \tilde{\theta}(\mathbf{z}_0)$ *and* $\Pi(\tilde{\theta}(\mathbf{z}_0)) > 0$.

If $\hat{\theta}(\mathbf{z}_0) \notin \Omega$ then there exists a compact subset F_1, $\mathbf{z}_0 \in F_1$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ and three positive constants δ_1, M, and N_1 such that for $N \geq N_1$, $|\xi_\perp| \leq N^{-2/3}$, $|\xi_p| \leq N^{-1/3}$, $\|\xi\| \leq \delta_1$ and $\forall \mathbf{z} \in F_1$

$$(3.5)\quad \begin{aligned} &\left|\Pi(\xi+\tilde{\theta})\exp\Big\{N\left[h(\xi+\tilde{\theta},\mathbf{z})-h(\tilde{\theta},\mathbf{z})\right]\Big\}-S(\xi,\mathbf{z})\right| \leq M\Big[\|\xi\|^2 \\ &+\|(N^{2/3}\xi_\perp, N^{1/3}\xi_p)\|^2\Big]\exp\left\{N\xi_\perp\frac{\partial h}{\partial \xi_\perp}(\tilde{\theta},\mathbf{z})+\frac{N}{2}\xi_p^2\frac{\partial^2 h}{\partial \xi_p^2}(\tilde{\theta},\mathbf{z})\right\}\end{aligned}$$

where

$$(3.6)\quad \begin{aligned} S(\xi,\mathbf{z}) = &\Big[\Pi(\tilde{\theta})\Big(1+N^{2/3}\xi_\perp+N^{1/3}\xi_p\Big)+\xi.\nabla\Pi(\tilde{\theta})'\Big] \\ &\cdot\exp\left\{N\xi_\perp\frac{\partial h}{\partial \xi_\perp}(\tilde{\theta},\mathbf{z})+\frac{N}{2}\xi_p^2\frac{\partial^2 h}{\partial \xi_p^2}(\tilde{\theta},\mathbf{z})\right\},\end{aligned}$$

$\nabla\Pi(\tilde{\theta})$ is the gradient of Π at $\tilde{\theta}$, $\xi = (\xi_\perp, \xi_p)$ and prime denotes transpose.

<u>Proof of Lemma 3.1:</u> Let $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. let $\delta_{01} > 0$ be such that Π is positive bounded and twice continuously differentiable on $\{\theta \in \Theta : \|\theta - \tilde{\theta}(\mathbf{z}_0)\| < \delta_{01}\}$. Since $\hat{\theta}(\mathbf{z})$ and $\tilde{\theta}(\mathbf{z})$ are continuous functions of $\mathbf{z}$, $\forall \mathbf{z} \in \psi'(\Gamma) \times \psi'(\Gamma)$, then there exists a compact subset F_1 of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that Π is positive, bounded and twice continuously differentiable on $\{\theta \in \Theta : \|\theta - \tilde{\theta}(\mathbf{z})\| \leq \delta_{01}/2\}$ $\forall \mathbf{z} \in F_{01}$ and $\hat{\theta}(\mathbf{z}) \notin \Omega$, $\forall \mathbf{z} \in F_1$.

By (3.1) and Lemma 2.3 the function h is a composition of two analytic functions in Θ, hence it is analytic in Θ. Therefore, for fixed $\mathbf{z}$ in F_1, we can expand $h(\theta, \mathbf{z})$ in a power series in a neighborhood of $\tilde{\theta}(\mathbf{z})$. Let $\mathbf{z}_1 \in F_1$ then $\forall \xi \in \Re^2$ such that $\|\xi\| < \delta_{01}/2$

$$(3.7)\quad \begin{aligned} h(\xi+\tilde{\theta},\mathbf{z}) = &h(\tilde{\theta},\mathbf{z}) + \xi_\perp\frac{\partial h}{\partial \theta_\perp}(\tilde{\theta},\mathbf{z}) + \frac{1}{2}\xi_p^2\frac{\partial^2 h}{\partial \theta^2}(\tilde{\theta},\mathbf{z}) \\ &+\sum_{k\geq 3}\sum_{2i+j=k} D_{ij}(\mathbf{z})\xi_\perp^i\xi_p^j\end{aligned}$$

where $D_{ij}(\mathbf{z}) = (1/i!j!)\, \partial^{i+j} h(\tilde{\theta},\mathbf{z})/\partial\theta_{\perp}^{i}\theta_{p}^{j}$. Hence

$$\Pi(\xi+\tilde{\theta})\exp\Big\{N\left[h(\xi+\tilde{\theta},\mathbf{z})-h(\tilde{\theta},\mathbf{z})\right]\Big\}$$

$$(3.8)\qquad =\exp\left\{N\xi_{\perp}\frac{\partial h}{\partial\xi_{\perp}}(\tilde{\theta},\mathbf{z})+\frac{N}{2}\xi_p^2\frac{\partial^2 h}{\partial\xi_p^2}(\tilde{\theta},\mathbf{z})\right\}.Q(\xi,\mathbf{Z},\mathbf{z})$$

where

$$(3.9)\qquad Q(\xi,\mathbf{Z},\mathbf{z})=\Pi(\xi+\tilde{\theta})\exp\Big\{\sum_{k\geq 3}N^{1-k/3}\sum_{2i+j=k}D_{ij}(\mathbf{z})Z_{\perp}^{i}Z_{p}^{j}\Big\}$$

and $\mathbf{Z}=(Z_{\perp},Z_p)=(N^{2/3}\xi_{\perp},\ N^{1/3}\xi_p)$. By Lemma 2.4, there exists $M_1>0$, such that

$$(3.10)\qquad \left|D_{ij}(\mathbf{z})\right|\leq\sup_{\substack{\|\xi\|\leq\delta_{01}/2\\ \mathbf{z}\in F_1}}\frac{|\,h(\xi+\tilde{\theta},\mathbf{z})\,|}{\delta_{01}^{i+j}2^{-(i+j)}}\leq\frac{M_1}{\delta_{01}^{i+j}2^{-(i+j)}}\qquad\forall\mathbf{z}\in F_1.$$

Let $\delta_{02}=\delta_{01}/2$. By (3.10), for $N\geq\max(\delta_{02}^{-3},\ 1)=N_1$, $\exists M_2>0$ such that $|N^{1-k/3}D_{ij}(\mathbf{z})|\leq M_2$, $\forall k\geq 3$ and $\forall\mathbf{z}\in F_1$. Hence for $N\geq N_1$ the series $\sum_{k\geq 3}\sum_{2i+j=k}N^{1-k/3}D_{ij}(\mathbf{z})\ Z_{\perp}^{i}Z_{p}^{j}$ is uniformly convergent on the interior of the unit circle $\{\mathbf{Z}\in\Re^2:\|\,\mathbf{Z}\,\|<1\}$, $\forall\mathbf{z}\in F_1$, hence it is analytic inside the unit circle $\forall\mathbf{z}\in F_1$. Since Π is twice continuously differentiable on $\{\theta\in\Theta:\|\,\theta-\tilde{\theta}(\mathbf{z})\,\|<\delta_{02}\}$, $\forall\mathbf{z}\in F_1$, therefore for $N\geq N_1$, $\|\,\xi\,\|<\delta_{02}$, $\|\,\mathbf{Z}\,\|<1$ and $\mathbf{z}\in F_1$

$$Q(\xi,\mathbf{Z},\mathbf{z})=Q(0,0,\mathbf{z})+\left[\nabla Q(0,0,\mathbf{z})\right]\cdot(\xi,\mathbf{Z})'$$

$$(3.11)\qquad +\frac{1}{2}(\xi,\mathbf{Z})\cdot\nabla^2 Q(\tilde{\xi},\tilde{\mathbf{Z}},\mathbf{z})\cdot(\xi,\mathbf{Z})'$$

where $\nabla Q(0,0,\mathbf{z})$ is the gradient of Q with respect to $(\xi,\mathbf{Z})$ at $(0,0,\mathbf{z})$ and $\nabla^2 Q(\tilde{\xi},\tilde{\mathbf{Z}},\mathbf{z})$ is the hessian of Q with respect to $(\xi,\mathbf{Z})$ at the intermediary point $(\tilde{\xi},\tilde{\mathbf{Z}},\mathbf{z})$.

Moreover by computing $\nabla^2(\xi, \mathbf{Z}, \mathbf{z})$ it can be easily checked that all the second derivatives of Q with respect to $(\xi, \mathbf{Z})$ are bounded whenever $N \geq 1$, $\| \xi \| < \delta_{02}$, $\| \mathbf{Z} \| < 1$ and $\mathbf{z} \in F_1$. Hence $\exists M > 0$ (independent of N) such that for $\| \xi \| \leq \delta_{02}$, $\| \mathbf{Z} \| < 1$, $N \geq N_1$ and $\forall \mathbf{z} \in F_1$

$$\Big| \Pi(\xi + \tilde{\theta}) \exp\Big\{ N\big[h(\xi + \tilde{\theta}, \mathbf{z}) - h(\tilde{\theta}, \mathbf{z})\big]\Big\} - \Big[Q(0,0,\mathbf{z}) + [\nabla Q(0,0,\mathbf{z})] \cdot (\xi, \mathbf{Z})'\Big] \exp\Big\{ N\xi_\perp \frac{\partial h}{\partial \xi_\perp}(\tilde{\theta}, \mathbf{z}) + \frac{N}{2}\xi_p^2 \frac{\partial^2 h}{\partial \xi_p^2}(\tilde{\theta}, \mathbf{z})\Big\}\Big|$$
$$\leq M\left[\| \xi \|^2 + \| \mathbf{Z} \|^2\right] \exp\Big\{ N\xi_\perp \frac{\partial h}{\partial \xi_\perp}(\tilde{\theta}, \mathbf{z}) + \frac{N}{2}\xi_p^2 \frac{\partial^2 h}{\partial \xi_p^2}(\tilde{\theta}, \mathbf{z})\Big\}$$

which proves the lemma.

Lemma 3.2: *Let $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. If $\hat{\theta}(\mathbf{z}_0) \notin \Omega$ then there exists a compact subset F_2, $\mathbf{z}_0 \in F_2$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ and a constant $\delta_2 > 0$ such that for $\| \xi \| \leq \delta_2$ and $\forall \mathbf{z} \in F_2$*

$$h(\xi + \tilde{\theta}, \mathbf{z}) - h(\tilde{\theta}, \mathbf{z}) \leq \frac{3}{4}\xi_\perp \frac{\partial h}{\partial \theta_\perp}(\tilde{\theta}, \mathbf{z}) + \frac{1}{4}\xi_p^2 \frac{\partial^2 h}{\partial \theta_p^2}(\tilde{\theta}, \mathbf{z}). \tag{3.12}$$

Proof of Lemma 3.2: By the same argument leading to (3.7), there exists a compact subset F_1 of $\psi'(\Gamma) \times \psi'(\Gamma)$ and a constant $\delta_{02} > 0$ such that $\forall \mathbf{z} \in F_1$ and $\| \xi \| < \delta_{02}$

$$h(\xi + \tilde{\theta}, \mathbf{z}) = h(\tilde{\theta}, \mathbf{z}) + \xi_\perp \frac{\partial h}{\partial \theta_\perp}(\tilde{\theta}, \mathbf{z}) + \frac{1}{2}\xi_p^2 \frac{\partial^2 h}{\partial \theta_p^2}(\tilde{\theta}, \mathbf{z}) + \sum_{k \geq 3} \sum_{2i+j=k} D_{ij}(\mathbf{z}) \xi_\perp^i \xi_p^j \tag{3.13}$$

where $D_{ij}(\mathbf{z})$ is defined in (3.7). Moreover, by (3.10), $| D_{ij}(\mathbf{z}) | \leq M_1/\delta_{02}^{i+j}$, $\forall \mathbf{z} \in F_1$. Hence $\forall \mathbf{z} \in F_1$,

$$\Big| \sum_{k \geq 3} \sum_{2i+j=k} D_{ij}(\mathbf{z}) \xi_\perp^i \xi_p^j \Big| \leq M_1 \frac{\rho^2}{(1-\rho)^2} \tag{3.14}$$

where $\rho = \max(|\xi_\perp|, |\xi_p|)/\delta_{02}$. Observe that for $0 < \|\xi\| \leq \delta_{02}/2$, $\rho^2/(1-\rho)^2 \leq 4\rho^2$. Hence, by (3.2), (3.3) and the continuity of $\partial h(\tilde{\theta}, \mathbf{z})/\partial\theta_\perp$ and $\partial^2 h(\tilde{\theta}, \mathbf{z})/\partial\theta_p^2$ with respect to $\mathbf{z}$, $\exists\delta_2 > 0$, $\delta_2 \leq \delta_{02}/2$, such that $\forall \mathbf{z} \in F_1$ and $\|\xi\| \leq \delta_2$

$$\frac{1}{4}\xi_\perp \frac{\partial h}{\partial\theta_\perp}(\tilde{\theta}, \mathbf{z}) + \frac{1}{4}\xi_p^2 \frac{\partial^2 h}{\partial\theta_p^2}(\tilde{\theta}, \mathbf{z}) + 4\rho^2 M_1 < 0 \quad \forall \mathbf{z} \in F_1. \tag{3.15}$$

The lemma follows by (3.13)-(3.15).

Remark 3.1: Observe that Lemma 2.2 can be restated in the θ-axises. That is, let F be a compact subset of $\psi'(\Gamma) \times \psi'(\Gamma)$ then $\forall\delta > 0\ \exists\epsilon > 0$ such that $B_\epsilon(\mathbf{z}) \subset C_\delta\big(\tilde{\theta}(\mathbf{z})\big)$, $\forall \mathbf{z} \in F$, where $C_\delta(\tilde{\theta}(\mathbf{z}))$ is as defined in Lemma 2.2 and $B_\epsilon(\mathbf{z}) = \{\theta \in \Omega : h(\tilde{\theta}(\mathbf{z}), \mathbf{z}) - h(\theta, \mathbf{z}) \leq \epsilon\}$.

Proposition 3.1: *Let $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. Assume that Π is bounded, twice continuously differentiable in a neighborhood of $\tilde{\theta}(\mathbf{z}_0)$ and $\Pi(\tilde{\theta}(\mathbf{z}_0)) > 0$. If $\hat{\theta}(\mathbf{z}_0) \notin \Omega$ then there exists a compact subset F_3, $\mathbf{z}_0 \in F_3$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that*

$$\left|\exp\{-Nh(\tilde{\theta}, \mathbf{z})\}\mathrm{P}(\Omega; \mathbf{z}) - \frac{\sqrt{2\pi}\Pi(\tilde{\theta})}{N^{4/3} \cdot E \cdot \sqrt{F}}\left(1 + \frac{1}{N^{1/3}\, E}\right)\right| \leq O(N^{-11/6}) \tag{3.16}$$

as $N \to +\infty$, $\forall \mathbf{z} \in F_3$, where $E = \partial h(\tilde{\theta}, \mathbf{z})/\partial\theta_\perp$ and $F = -\partial^2 h(\tilde{\theta}, \mathbf{z})/\partial\theta_p^2$.

Proof of Proposition 3.1: By Lemma 3.1 and Lemma 3.2 there exist two compact subsets F_1 and F_2 of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that (3.5) and (3.12) hold, respectively. Let $F_3 = F_1 \cap F_2$, the compact set F_3 is non-empty since $\mathbf{z}_0 \in F_1 \cap F_2$. So, by Lemma 3.2, $\exists\delta_2 > 0$ such that for $\|\xi\| \leq \delta_2$, (3.12) holds $\forall \mathbf{z} \in F_3$. Now choose δ in Remark 3.1 such that $\delta \leq \delta_2$. Then $\exists\epsilon > 0$ such that $B_\epsilon(\mathbf{z}) \subset C_\delta(\tilde{\theta}(\mathbf{z}))\ \forall \mathbf{z} \in F_3$. Hence, $\forall \mathbf{z} \in F_3$,

$$\int_{B_\epsilon(\mathbf{z})} \exp\Big\{N\big[h(\theta, \mathbf{z}) - h(\tilde{\theta}, \mathbf{z})\big]\Big\}\Pi(\theta)\, d\theta \leq \int_{-\xi}^{0}\int_{-\delta}^{\delta} T_1\, d\xi, \tag{3.17}$$

where $T_1 = T_1(\xi, \mathbf{z}) = \exp\left\{N\left[h(\xi+\tilde{\theta}, \mathbf{z}) - h(\tilde{\theta}, \mathbf{z})\right]\right\}\Pi(\xi+\tilde{\theta})$. Let $\delta_3 = \min(\delta_1, \delta)$ where δ_1 is as defined in Lemma 3.1. Now note that for $N \geq \max(\delta_3^{-3}, 1)$ the r.h.s. of (3.17) can be written as

$$\int_{-\delta}^{0}\int_{-\delta}^{\delta} T_1 d\xi = \int_{-\delta}^{0}\left[\int_{-\delta}^{-N^{-1/3}} + \int_{N^{-1/3}}^{\delta}\right] T_1\, d\xi + \int_{-\delta}^{-N^{-2/3}}\int_{-N^{-1/3}}^{N^{-1/3}} T_1\, d\xi$$

$$+\int_{-N^{-2/3}}^{0}\int_{-N^{-1/3}}^{N^{-1/3}} T_1 d\xi \quad \forall \mathbf{z} \in F_3. \tag{3.18}$$

By Lemma 3.1, $\forall \mathbf{z} \in F_3$

$$\left|\int_{-N^{-2/3}}^{0}\int_{-N^{-1/3}}^{N^{-1/3}} \left[T_1 - S\right] d\xi\right| \leq \int_{-N^{-2/3}}^{0}\int_{-N^{-1/3}}^{N^{-1/3}} (\text{the right side of (3.5)})\, d\xi. \tag{3.19}$$

By (3.2), (3.3) and the continuity of h and $\tilde{\theta}$ in $\mathbf{z}$, the r.h.s. of (3.19) can be found, using ordinary calculus, to be $O(N^{-11/6})$, as $N \to +\infty$, $\forall \mathbf{z} \in F_3$. So, $\forall \mathbf{z} \in F_3$,

$$\left|\int_{-N^{-2/3}}^{0}\int_{-N^{-1/3}}^{N^{-1/3}} \left[T_1(\xi, \mathbf{z}) - S(\xi, \mathbf{z})\right] d\xi\right| \leq O(N^{-11/6}). \tag{3.20}$$

Similarly, by (3.2), (3.3) and Lemma 3.2, $\forall \mathbf{z} \in F_3$,

$$\int_{-\delta}^{0}\left[\int_{-\delta}^{-N^{-1/3}} + \int_{N^{-1/3}}^{\delta}\right] T_1(\xi, \mathbf{z}) d\xi \leq \exp\left\{\frac{1}{4}N^{1/3}\frac{\partial^2 h}{\partial \theta_p^2}(\tilde{\theta}, \mathbf{z})\right\} = O(N^{-11/6}) \tag{3.21}$$

and

$$\int_{-\delta}^{-N^{-2/3}}\int_{-N^{-1/3}}^{N^{-1/3}} T_1(\xi, \mathbf{z}) d\xi \leq \exp\left\{-\frac{3}{4}N^{1/3}\frac{\partial h}{\partial \theta_\perp}(\tilde{\theta}, \mathbf{z})\right\} = O(N^{-11/6}). \tag{3.22}$$

So by (3.17), (3.18), (3.20)-(3.22) and the definition of $B_\epsilon(\mathbf{z})$,

$$\left|\int_\Omega T_1(\theta - \tilde{\theta}, \mathbf{z}) d\theta - \int_{-N^{-2/3}}^{0}\int_{-N^{-1/3}}^{N^{-1/3}} S(\xi, \mathbf{z}) d\xi\right| \leq 2\left|\int_{\overline{B}_\epsilon(\mathbf{z})} T_1(\theta - \tilde{\theta}, \mathbf{z}) d\theta\right|$$

$$+\left|\int_{-\delta}^{0}\int_{-\delta}^{\delta} T_1 d\xi - \int_{-N^{-2/3}}^{0}\int_{-N^{-1/3}}^{N^{-1/3}} S d\xi\right| \leq O(N^{-11/6}). \tag{3.23}$$

The proposition follows by evaluating the integral of $S(\xi, \mathbf{z})$ in the left side of (3.23).

4. APPROXIMATING $\mathrm{P}(\Omega; \mathbf{z})$: M.L.E. $\in \Omega^0$

Let Ω be a closed convex subset of Γ^2 and let Ω^0 denote its interior. In this section we approximate $\mathrm{P}(\Omega; \mathbf{z})$ when the Maximum Likelihood Estimator $\hat{\omega}(\mathbf{z}) \in \Omega^0$. The approximation of $\mathrm{P}(\Omega; \mathbf{z})$ is given in Proposition 4.1. Next we introduce some lemmas that be will needed in the proof of Proposition 4.1.

Lemma 4.1: *Let $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. Assume that Π is bounded, thrice continuously differentiable in a neighborhood of $\hat{\omega} = \hat{\omega}(\mathbf{z}_0)$ and $\Pi(\hat{\omega}(\mathbf{z}_0)) > 0$. Then there exists a compact subset F_4, $\mathbf{z}_0 \in F_4$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ and three positive constants δ_4, M_4 and N_2 such that for $N \geq N_2$, $\| \xi \| \leq \delta_4$ and $\mathbf{z} \in F_4$*

$$\left|\Pi(\hat{\omega} + \xi) \exp\Big\{N\big[l(\xi + \hat{\omega}, \mathbf{z}) - l(\hat{\omega}, \mathbf{z})\big]\Big\} - R(\xi, \mathbf{z})\right| \leq M_4\Big[\big(| \xi_1 |^3 + | \xi_2 |^3\big)(1 + N)\Big] \exp\left\{\frac{N}{2}\left[\xi_1^2 \frac{\partial^2 l}{\partial \omega_1^2}(\hat{\omega}, \mathbf{z}) + \xi_2^2 \frac{\partial^2 l}{\partial \omega_2^2}(\hat{\omega}, \mathbf{z})\right]\right\} \tag{4.1}$$

where

$$R(\xi, \mathbf{z}) = \Bigg[\Pi(\hat{\omega}) + \Big[\nabla \Pi(\hat{\omega}) + N^{1/3}\Pi(\hat{\omega}) \cdot \underline{1}\Big] \cdot \xi' + \frac{1}{2}\xi \cdot \Big[\nabla^2 \Pi(\hat{\omega}) + 2N^{1/3}\underline{1}' \cdot \nabla \Pi(\hat{\omega}) + N^{2/3}\Pi(\hat{\omega})\underline{1}' \cdot \underline{1}\Big] \cdot \xi'\Bigg] \cdot \exp\left\{\frac{N}{2}\left[\xi_1^2 \frac{\partial^2 l}{\partial \omega_1^2}(\hat{\omega}, \mathbf{z}) + \xi_2^2 \frac{\partial^2 l}{\partial \omega_2^2}(\hat{\omega}, \mathbf{z})\right]\right\},$$

$\underline{1} = (1, 1)$ *and prime denotes transpose.*

Proof of Lemma 4.1: By a similar argument as in the proof of Lemma 3.1, there exists a compact subset F_4 of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that $\forall \mathbf{z} \in F_4$, Π

is positive bounded and thrice continuously differentiable on $\{\omega \in \Gamma^2 : \| \omega - \hat{\omega}(\mathbf{z}) \| \leq \delta_{04}/2\}$. By Lemma 2.3, the function $l(\cdot, \mathbf{z})$ is analytic on Γ^2. Let $\mathbf{z} \in F_4$ then $\forall \xi \in \Re^2$ such that $\| \xi \| \leq \delta_{04}/2$

$$l(\hat{\omega} + \xi, bz) = l(\hat{\omega}, \mathbf{z}) + \frac{1}{2}\xi_1^2 \frac{\partial^2 l}{\partial \omega_1^2}(\hat{\omega}, \mathbf{z}) + \frac{1}{2}\xi_2^2 \frac{\partial^2 l}{\partial \omega_2^2}(\hat{\omega}, \mathbf{z}) + \sum_{i+j\geq 3} D_{ij}(\mathbf{z})\xi_2^i \xi_2^j,$$

where $D_{ij}(\mathbf{z}) = \big(1/i!j!\big)\partial^{i+j} l(\hat{\omega}, \mathbf{z})/\partial \omega_1^i \partial \omega_2^j$. The rest of the proof follows as in the proof of Lemma 3.1.

Lemma 4.2: $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. *There exist a compact subset* F_5, $\mathbf{z}_0 \in F_5$, *of* $\psi'(\Gamma) \times \psi'(\Gamma)$ *and a positive constant* δ_5 *such that for* $\| \xi \| \leq \delta_5$, $\forall \mathbf{z} \in F_5$

$$l(\hat{\omega} + \xi, \mathbf{z}) - l(\hat{\omega}, \mathbf{z}) \leq \frac{1}{4}\xi_1^4 \frac{\partial^2 l}{\partial \omega_1^2}(\hat{\omega}, \mathbf{z}) + \frac{1}{4}\xi_2^2 \frac{\partial^2 l}{\partial \omega_2^2}(\hat{\omega}, \mathbf{z}). \tag{4.2}$$

The proof of this Lemma is similar to the proof of Lemma 3.2, so it will be omitted.

Proposition 4.1: *Let* $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. *Assume that* Π *is bounded, thrice continuously differentiable in a neighborhood of* $\hat{\omega}(\mathbf{z}_0)$ *and* $\Pi(\hat{\omega}(\mathbf{z}_0)) > 0$. *If* $\hat{\omega}(\mathbf{z}_0) \in \Omega^0$ *then there exists a compact subset* F_7, $\mathbf{z}_0 \in F_7$, *of* $\psi'(\Gamma) \times \psi'(\Gamma)$ *such that*

(4.3)
$$\left|\exp\{-Nl(\hat{\omega}, \mathbf{z})\}\mathrm{P}(\Omega; \mathbf{z}) - \frac{2\pi\Pi(\hat{\omega})}{N\sqrt{C \cdot D}}\Big[1 - \frac{1}{N^{1/3}C} - \frac{1}{N^{1/3}D}\Big]\right| \leq O(N^{-3/2})$$

as $N \to +\infty$, $\forall \mathbf{z} \in F_7$, *where* $C = \partial^2 l(\hat{\omega}, \mathbf{z})/\partial \omega_1^2$ *and* $D = \partial^2 l(\hat{\omega}, \mathbf{z})/\partial \omega_2^2$.

The proof of this proposition is similar to the proof of Proposition 3.1. It will be omitted.

5. APPROXIMATING POSTERIOR PROBABILITIES

Let Ω be a closed convex subset of Γ^2 and let Ω^0 denote its interior. In the next proposition we give an approximation to the posterior probability of Ω given the sufficient statistics $\mathbf{z} = (\overline{X}, \overline{Y})$, that is $\mathrm{P}(\Omega|\mathbf{z})$.

Proposition 5.1: *Let $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. Assume that Π is bounded, thrice continuously differentiable in the neighborhood of $\tilde{\omega}(\mathbf{z}_0)$ and $\hat{\omega}(\mathbf{z}_0)$. Also suppose that $\Pi(\tilde{\omega}(\mathbf{z}_0)) > 0$ and $\Pi(\hat{\omega}(\mathbf{z}_0)) > 0$.*

i) If $\hat{\omega}(\mathbf{z}_0) \notin \Omega$ then there exists a compact subset F_8, $\mathbf{z}_0 \in F_8$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that

$$\left| \exp\left\{N\left[l(\hat{\omega},\mathbf{z}) - l(\tilde{\omega},\mathbf{z})\right]\right\}\mathrm{P}\left[\Omega|\mathbf{z}\right] - \frac{\Pi(\tilde{\theta})\left[1 + N^{-1/3}E^{-1}\right]\sqrt{C\cdot D}}{\sqrt{2\pi}\Pi(\hat{\omega})N^{1/3}E\sqrt{F}\left[1 - N^{-1/3}(C^{-1} - D^{-1})\right]}\right| \leq O(N^{-5/6}) \tag{5.1}$$

as $N \to +\infty$, $\forall \mathbf{z} \in F_8$, where $E = \partial h(\tilde{\theta},\mathbf{z})/\partial\theta_\perp$, $F = -\partial^2 h(\tilde{\theta},\mathbf{z})/\partial\theta_p^2$, $C = \partial^2 l(\hat{\omega},\mathbf{z})/\partial\omega_1^2$ and $D = \partial^2 l(\hat{\omega},\mathbf{z})/\partial\omega_2^2$.

ii) If $\hat{\omega}(\mathbf{z}_0) \in \Omega^0$ then there exists a compact subset F_9, $\mathbf{z}_0 \in F_9$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that $\mathrm{P}\left[\Omega|\mathbf{z}\right] \geq 1 - O(N^{-1/2})$, $\forall \mathbf{z} \in F_9$, as $N \to +\infty$.

Proof of Proposition 5.1: Let $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. First we will evaluate $\exp\{-Nl(\hat{\omega},\mathbf{z})\}\ \mathrm{P}\left[\Gamma^2;\mathbf{z}\right]$. Since Γ^2 is open and $\hat{\omega}(\mathbf{z}_0) \in \Gamma^2$ then there exists a closed convex subset Λ of Γ^2 such that $\hat{\omega}(\mathbf{z}_0) \in \Lambda^0$. By Proposition 4.1, there exists a compact subset F_7, $\mathbf{z}_0 \in F_7$, of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that

$$\begin{aligned} \left|\exp\{-Nl(\hat{\omega},\mathbf{z})\}\mathrm{P}\left[\Gamma^2;\mathbf{z}\right] - R_2(\mathbf{z})\right| &\leq \exp\{-Nl(\hat{\omega},\mathbf{z})\}\mathrm{P}\left[\overline{\Lambda};\mathbf{z}\right] + O(N^{-3/2}) \\ &\leq \exp\left\{-N\inf_{\mathbf{z}\in F_7}\{l(\hat{\omega},\mathbf{z}) - l(\omega,\mathbf{z})\}\right\} + O(N^{-3/2}) = O(N^{-3/2}), \end{aligned} \tag{5.3}$$

where $R_2(\mathbf{z})$ is the approximation of $\exp\{-Nl(\hat{\omega},\mathbf{z})\}\mathrm{P}\left[\Omega;\mathbf{z}\right]$ in Proposition 4.1. Similarly, let $R_1(\mathbf{z})$ denote the approximation to $\exp\{-Nh(\tilde{\theta},\mathbf{z})\}$

$P[\Omega;\mathbf{z}]$ in Proposition 3.1. If $\hat{\omega}(\mathbf{z}_0) \notin \Omega$ then by Proposition 3.1 and (5.3) there exists a compact subset F_8 of $\psi'(\Gamma) \times \psi'(\Gamma)$ such that

$$\frac{R_1(\mathbf{z}) - O(N^{-11/6})}{R_2(\mathbf{z}) + O(N^{-3/2})} \tag{5.4}$$
$$\leq \exp\Big\{N\big[l(\hat{\omega},\mathbf{z}) - l(\tilde{\omega},\mathbf{z})\big]\Big\}\mathrm{P}(\Omega|\mathbf{z}) \leq \frac{R_1(\mathbf{z}) + O(N^{-11/6})}{R_2(\mathbf{z}) - O(N^{-3/2})}.$$

Now observe that $R_1(\mathbf{z}) = O(N^{-4/3})$ and $R_2(\mathbf{z}) = O(N^{-1})$, $\forall \mathbf{z} \in F_8$. So the right side of (5.4) is equal to $R_1(\mathbf{z})/R_2(\mathbf{z}) + O(N^{-5/6})$. By the same argument we find that the left side of (5.4) is also a $R_1(\mathbf{z})/R_2(\mathbf{z})+O(N^{-5/6})$ and the first part of the proposition follows. The case where $\hat{\omega}(\mathbf{z}_0) \in \Omega^0$ is proved similarly.

Next we approximate the Bayes posterior risk $r(\Pi,\mathbf{z})$, as defined in (1.2), for testing H_0: $\omega \in \Omega_1$ versus H_1: $\omega \in \Omega_2$, using a zero-one loss function with indifference zone $\Gamma^2 - (\Omega_1 \cup \Omega_2)$. Let $\tilde{\omega}^{(i)}(\mathbf{z})$, $i = 1,2$, denote the M.L.E. of ω over Ω_i, $i = 1,2$, and $\tilde{\theta}^{(i)}(\mathbf{z})$ denote $\tilde{\omega}^{(i)}(\mathbf{z})$ after the reparametrization of section 3.

Corollary 5.1: *Let* $\mathbf{z}_0 \in \psi'(\Gamma) \times \psi'(\Gamma)$. *Assume that* Π *is bounded, thrice continuously differentiable in a neighborhood of* $\tilde{\omega}^{(i)}(\mathbf{z}_0)$, $i = 1,2$, *and* $\hat{\omega}(\mathbf{z}_0)$. *Also suppose that* $\Pi(\tilde{\omega}^{(i)}(\mathbf{z}_0)) > 0$, $i = 1,2$, *and* $\Pi(\hat{\omega}(\mathbf{z}_0)) > 0$.

i) If $\hat{\omega}(\mathbf{z}_0) \notin \Omega_1 \cup \Omega_2$ *then there exists a compact subset* F_{10}, $\mathbf{z}_0 \in F_{10}$, *of* $\psi'(\Gamma) \times \psi'(\Gamma)$ *such that*

$$\Big|r(\Pi,\mathbf{z}) - \min_{i=1,2}\Big\{U_i \cdot \exp\{N[l(\tilde{\omega}^{(i)},\mathbf{z}) - l(\hat{\omega},\mathbf{z})]\}\Big\}\Big|$$
$$\leq \max_{i=1,2}\Big\{\exp\{N[l(\tilde{\omega}^{(i)},\mathbf{z}) - l(\hat{\omega},\mathbf{z})]\}\Big\} \cdot O(N^{-5/6}),$$

as $N \to +\infty$, $\forall \mathbf{z} \in F_{10}$.

ii) If $\hat{\omega}(\mathbf{z}_0) \in \Omega_i^0$ *then there exists a compact subset* F_{11}, $\mathbf{z}_0 \in F_{11}$, *of* $\psi'(\Gamma) \times \psi'(\Gamma)$ *such that*

$$\min\Big\{1; U_j \cdot \exp\{N[l(\tilde{\omega}^{(j}, \mathbf{z}) - l(\hat{\omega}, \mathbf{z})]\}\Big\} - O(N^{-1/2}) \leq r(\Pi, \mathbf{z})$$
$$\leq \big(U_j + O(N^{-5/6})\big) \cdot \exp\{N[l(\tilde{\omega}^{(j)}, \mathbf{z}) - l(\hat{\omega}, \mathbf{z})]\}, \quad j \neq i$$

as $N \to +\infty$, $\forall \mathbf{z} \in F_{11}$, *where*

$$U_i = \frac{\Pi(\tilde{\theta}^{(i)})\left[1 + N^{-1/3}E_i^{-1}\right]\sqrt{C \cdot D}}{\sqrt{2\pi}\Pi(\hat{\omega})N^{1/3}E_i\sqrt{F_i}\Big[1 - N^{-1/3}\big(C^{-1} - D^{-1}\big)\Big]}, \quad i = 1, 2$$

$E_i = \partial h(\tilde{\theta}^{(i)}, \mathbf{z})/\partial\theta_\perp$, $F_i = -\partial^2 h(\tilde{\theta}^{(i)}, \mathbf{z})/\partial\theta_p^2$, $i = 1, 2$; $C = \partial^2 l(\hat{\omega}, \mathbf{z})/\partial\omega_1^2$ *and* $D = \partial^2 l(\hat{\omega}, \mathbf{z})/\partial\omega_2^2$.

The proof of the corollary follows directly from Proposition 5.1. It will be omitted.

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