BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH STOCHASTIC MONOTONE COEFFICIENTS

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We prove an existence and uniqueness result for backward stochastic differential equations whose coefficients satisfy a stochastic monotonicity condition. In this setting, we deal with both constant and random terminal times. In the random case, the terminal time is allowed to take infinite values. But in a Markovian framework, that is coupled with a forward SDE, our result provides a probabilistic interpretation of solutions to nonlinear PDEs.

1. Introduction

Backward stochastic differential equations (BSDEs), introduced by Pardoux and Peng [10], have been intensively studied in the last years. This class of equations is a powerful tool to give probabilistic formulas for solutions of semilinear partial differential equations (PDEs). We refer the reader to [8, 9] for a good presentation of BSDEs and their connections to PDEs. These equations have found a broad area of applications, namely, in stochastic optimal control (see [7]), mathematical finance (see [6]). Many existence and uniqueness results have been proved in relaxing the uniform Lipschitz condition on the coefficient. Among others, we refer to those with monotonicity condition (see [1, 3, 4]). In this setting (in relaxing the Lipschitz condition), Bender and Kohlmann [2] recently considered the so-called stochastic Lipschitz condition introduced by El Karoui and Huang [5] and dealt with BSDEs with random terminal time. Indeed, the Lipschitz coefficient is allowed to be an $\mathcal{F}_t$-adapted process. Doing so, one must reinforce the integrability conditions on the data as well as on the solutions. The interest in this type of extension of the classical existence and uniqueness result comes from the fact that, in many applications, the usual Lipschitz condition cannot be satisfied. For example, the pricing of a European claim is equivalent to solving the linear BSDE

$$\begin{gathered} -dY_t = \left[r(t)Y(t) + \theta(t)Z(t)\right]dt - Z(t)dW_t, \\ Y_T = \xi, \end{gathered} \tag{1.1}$$

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where ξ is the contingent claim, $r(t)$ is the interest rate, $\theta(t)$ is the risk premium vector, and T is the terminal time. Both $r(t)$ and $\theta(t)$ are not bounded in general. Therefore, the generator satisfies the so-called stochastic Lipschitz condition which means that the Lipschitz constant is a stochastic process.

In this paper, we continue this study by considering BSDEs with stochastic monotone coefficients. For example, our result treats generators of the following type: $f(t,y,z) = \mu(t)g(t,y)+h(t,0,z)$, $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$, where $\mu(t)$ is a nonnegative $\mathcal{F}_t$-adapted process, g satisfies a monotonicity condition in y and h is stochastic Lipschitzian in z. It is not possible to apply the results of [2, 4]. Our aim is to prove the existence and uniqueness of solutions for both constant and random terminal times. When the terminal time is random, it is allowed to take values in $[0,+\infty]$.

The paper is organized as follows. In Section 2, we give some notations, state the assumptions, and define the BSDEs we are concerned with. Section 3 treats the nonrandom terminal time case, and Section 4 deals with the random one.

2. Notations, assumptions, and definitions

2.1. Notations. Let $W = \{W_t, \mathcal{F}_t,\ t \geq 0\}$ be an n-dimensional Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. $\{\mathcal{F}_t,\ t\geq 0\}$ stands for the natural filtration of W, augmented with the $\mathbb{P}$-nul sets of $\mathcal{F}$. The inner product of $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$ and the Euclidean norm by $|\cdot|$. The norm of $\mathbb{R}^{d\times n}$ is denoted by $|Z|^2 = \operatorname{tr}(ZZ^*)$.

Let $\beta>0$, τ be a positive real-valued random variable and a a nonnegative $\mathcal{F}_t$-adapted process. We define the increasing process $A(t) = \int_0^t a^2(s)ds$ and consider the spaces:

$$
\begin{aligned}
L^2(\beta,a,\tau,\mathbb{R}^d) &= \left\{\begin{matrix}\xi;\mathbb{R}^d\text{-valued, } \mathcal{F}_\tau\text{-measurable random variables}\\ \text{such that } \|\xi\|_\beta^2 = \mathbb{E}(e^{\beta A(\tau)}|\xi|^2) < +\infty\end{matrix}\right\},\\
L^2(\beta,a,[0,\tau],\mathbb{R}^d) &= \left\{\begin{matrix}Y;\mathbb{R}^d\text{-valued, } \mathcal{F}_t\text{-adapted processes such that}\\ \|Y\|_\beta^2 = \mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|Y(s)|^2ds\right) < +\infty\end{matrix}\right\},\\
L^{2,a}(\beta,a,[0,\tau],\mathbb{R}^d) &= \left\{\begin{matrix}Y;\mathbb{R}^d\text{-valued, } \mathcal{F}_t\text{-adapted processes such that}\\ \|aY\|_\beta^2 = \mathbb{E}\left(\int_0^\tau e^{\beta A(s)}a^2(s)|Y(s)|^2ds\right) < +\infty\end{matrix}\right\},\\
L^{2,c}(\beta,a,[0,\tau],\mathbb{R}^d) &= \left\{\begin{matrix}Y;\mathbb{R}^d\text{-valued, continuous } \mathcal{F}_t\text{-adapted processes}\\ \text{such that } \|Y\|_{\beta,c}^2 = \mathbb{E}\left(\sup_{0\leq s\leq\tau} e^{\beta A(s)}|Y(s)|^2\right) < +\infty\end{matrix}\right\}.
\end{aligned}
\tag{2.1}
$$

$L^2(\beta,a,[0,\tau],\mathbb{R}^d)$ is a Banach space with the norm $\|\cdot\|_\beta$. Consequently,

$$
\mathcal{M}(\beta,a,\tau)=L^{2,a}(\beta,a,[0,\tau],\mathbb{R}^d)\times L^2(\beta,a,[0,\tau],\mathbb{R}^{d\times n})
\tag{2.2}
$$

is a Banach space with the norm $\|(Y,Z)\|_\beta^2 = \|aY\|_\beta^2 + \|Z\|_\beta^2$. We denote by $\mathcal{M}^c(\beta,a,\tau)$ the subspace of $\mathcal{M}(\beta,a,\tau)$ defined as follows:

$$
\mathcal{M}^c(\beta,a,\tau) = (L^{2,a}(\beta,a,[0,\tau],\mathbb{R}^d)\cap L^{2,c}(\beta,a,[0,\tau],\mathbb{R}^d))\times L^2(\beta,a,[0,\tau],\mathbb{R}^{d\times n});
\tag{2.3}
$$

and consider the norm $\|(Y,Z)\|_{\beta,c}^2 = \|Y\|_{\beta,c}^2 + \|aY\|_\beta^2 + \|Z\|_\beta^2$ on $\mathcal{M}^c(\beta,a,\tau)$.

Remark 2.1. If a and b are two nonnegative $\mathcal{F}_t$-adapted processes such that $b > a$, then $L^2(\beta,b,[0,\tau],\mathbb{R}^d) \subset L^2(\beta,a,[0,\tau],\mathbb{R}^d)$. Consequently, $\mathcal{M}^c(\beta,b,\tau) \subset \mathcal{M}^c(\beta,a,\tau)$.

2.2. Assumptions and definitions. Let $f : \Omega \times [0,\tau] \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$ be a function such that for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$, $f(\cdot,\cdot,y,z)$ is progressively measurable, and let ξ be an $\mathbb{R}^d$-valued $\mathcal{F}_\tau$-measurable random variable.

For some $\beta > 0$, we assume that the triple (τ,ξ,f) satisfies the following conditions.

(H1) There exist a $\mathcal{F}_t$-adapted process $\theta(t)$ and a nonnegative $\mathcal{F}_t$-adapted process $\nu(t)$ such that for all $(y,y',z,z') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$,
(i) $\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq \theta(t)|y-y'|^2$,
(ii) $|f(t,y,z) - f(t,y,z')| \leq \nu(t)|z-z'|$,
(iii) $y \mapsto f(\cdot,\cdot,y,z)$ is continuous $dt \otimes d\mathbb{P}$ a.e.

(H2) There exists $\varepsilon > 0$ such that $a^2(t) \triangleq |\theta(t)| + \nu^2(t) > \varepsilon$.

If (H2) is not fulfilled, replace $\nu(t)$ by $\nu(t) + \sqrt{\varepsilon}$.

(H3) (i) $\xi \in L^2(\beta,a,\tau,\mathbb{R}^d)$,
(ii) $f(\cdot,0,0)/a \in L^2(\beta,a,[0,\tau],\mathbb{R}^d)$,
(iii) $\mathbb{E}(e^{\beta A(\tau)}) < +\infty$.

(H4) There exists a positive $\mathcal{F}_t$-adapted process $\eta(t)$ and a positive constant K such that
(i) $\eta \in L^2(\beta,a,[0,\tau],\mathbb{R}^d)$,
(ii) $|f(t,y,z)| \leq |f(t,0,z)| + \eta(t) + K(1+|y|)$.

We end this section by specifying what we call a solution of our BSDE.

Definition 2.2. If τ is constant, then a solution of the BSDE with data (τ,ξ,f) is a pair of $\mathcal{F}_t$-adapted processes $\{(Y_t,Z_t);\ t \geq 0\}$ with values in $\mathbb{R}^d \times \mathbb{R}^{d\times n}$ such that

(J1) $(Y,Z) \in \mathcal{M}^c(\beta,a,\tau)$, that is,

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{\beta A(t)}\left|Y_t\right|^2 + \int_0^\tau e^{\beta A(s)}a^2(s)\left|Y_s\right|^2 ds + \int_0^\tau e^{\beta A(s)}\left|Z_s\right|^2 ds\right) < +\infty, \tag{2.4}$$

(J2) $Y_t = \xi + \int_t^\tau f(s,Y_s,Z_s)ds - \int_t^\tau Z_s dW_s$.

Definition 2.3. If τ is a random time, then a solution of the BSDE with data (τ,ξ,f) a pair of $\mathcal{F}_t$-adapted processes $\{(Y_t,Z_t),\ t \geq 0\}$, taking values in $\mathbb{R}^d \times \mathbb{R}^{d\times n}$ such that

(J3) $(Y,Z) \in \mathcal{M}^c(\beta,a,\tau)$,

(J4) for all $T \geq t \geq 0$,

$$Y_{t\wedge\tau} = Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s,Z_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s, \tag{2.5}$$

(J5) $Y_t = \xi$, on the set $\{t \geq \tau\}$.

3. Existence and uniqueness on fixed time interval

Throughout this section, τ is a fixed positive real number and C will denote a positive constant which may vary from line to line.

3.1. Uniqueness. We first state a priori estimates on the solutions.

Proposition 3.1. *Under (H1), (H2), (H3), and (H4), let* (Y,Z) *(resp.,* (Y',Z')*) be a solution of the BSDE with data* (τ,ξ,f) *(resp.,* (τ,ξ',f')*). Put* $\Delta f(t)=f(t,Y'_t,Z'_t)-f'(t,Y'_t,Z'_t)$*,* $\Delta Y_t = Y_t - Y'_t$*,* $\Delta Z_t = Z_t - Z'_t$*,* $\Delta\xi = \xi - \xi'$*, and*

$$\Gamma = \mathbb{E}\left(e^{\beta A(\tau)}|\Delta\xi|^2 + \frac{2}{\beta}\int_0^\tau e^{\beta A(s)}\frac{|\Delta f(s)|^2}{a^2(s)}ds\right). \tag{3.1}$$

Then, for β *sufficiently large, the following holds:*

(i) $\mathbb{E}(\int_0^\tau e^{\beta A(s)}(|\Delta f(s)|^2/a^2(s))ds) < +\infty$,
(ii) (a) $\mathbb{E}(\int_0^\tau e^{\beta A(s)}|\Delta Z_s|^2 ds) \le 2\Gamma$,
(b) $\mathbb{E}(\int_0^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2 ds) \le (2/(\beta-4))\Gamma$,
(iii) $\mathbb{E}(\sup_{0\le t\le\tau} e^{\beta A(t)}|\Delta Y_t|^2) \le C(\beta)\Gamma$,

where $C(\beta)$ *is a constant which depends on* β*.*

Proof. We assume without loss of generality that the coefficients $\theta(s)$ and $\nu(s)$ are the same for f and f'. Then, (i) follows from (H1)(ii), (H3), and (H4).

By virtue of Itô's formula, we have

$$\begin{aligned} e^{\beta A(t)}|\Delta Y_t|^2 + \beta\int_t^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2 ds + \int_t^\tau e^{\beta A(s)}|\Delta Z_s|^2 ds \\ = e^{\beta A(\tau)}|\Delta\xi|^2 + 2\int_t^\tau e^{\beta A(s)}\langle \Delta Y_s, f(s,Y_s,Z_s) - f'(s,Y'_s,Z'_s)\rangle ds \\ - 2\int_t^\tau e^{\beta A(s)}\langle \Delta Y_s, \Delta Z_s dW_s\rangle. \end{aligned} \tag{3.2}$$

Therefore, (H1)(i), (H1)(ii), and Young's inequality, $2uv \le (\alpha/2)u^2 + (2/\alpha)v^2$ for $\alpha > 0$, lead to

$$\begin{aligned} &2\langle \Delta Y_s, f(s,Y_s,Z_s) - f'(s,Y'_s,Z'_s)\rangle \\ &\quad\le 2\theta(s)|\Delta Y_s|^2 + 2|\Delta Y_s|\left[\nu(s)|\Delta Z_s| + |\Delta f(s)|\right] \\ &\quad\le \left(2+\frac{\beta}{2}\right)a^2(s)|\Delta Y_s|^2 + \frac{1}{2}|\Delta Z_s|^2 + \frac{2}{\beta}\frac{|\Delta f(s)|^2}{a^2(s)}. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} e^{\beta A(t)}|\Delta Y_t|^2 + \left(\frac{\beta}{2}-2\right)\int_t^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2 ds + \frac{1}{2}\int_t^\tau e^{\beta A(s)}|\Delta Z_s|^2 ds \\ \le e^{\beta A(\tau)}|\Delta\xi|^2 + \frac{2}{\beta}\int_t^\tau e^{\beta A(s)}\frac{|\Delta f(s)|^2}{a^2(s)}ds - 2\int_t^\tau e^{\beta A(s)}\langle \Delta Y_s, \Delta Z_s dW_s\rangle. \end{aligned} \tag{3.4}$$

In view of (3.4), we deduce that

$$\begin{aligned}\left(\frac{\beta}{2}-2\right)\mathbb{E}\left(\int_t^\tau e^{\beta A(s)}a^2(s)\left|\Delta Y_s\right|^2 ds\right)+\frac{1}{2}\mathbb{E}\left(\int_t^\tau e^{\beta A(s)}\left|\Delta Z_s\right|^2 ds\right)\\ \leq \mathbb{E}\left(e^{\beta A(\tau)}\left|\Delta\xi\right|^2+\frac{2}{\beta}\int_t^\tau e^{\beta A(s)}\frac{\left|\Delta f(s)\right|^2}{a^2(s)}ds\right),\end{aligned}\tag{3.5}$$

which leads to (ii).

Now, taking $\sup_{0\leq t\leq\tau}(\cdot)$ in (3.4), applying Burkhölder-Davis-Gundy's inequality and using (ii)(a), we obtain (iii). □

Corollary 3.2. *Under (H1), (H2), (H3), and (H4), the BSDEs (J1) and (J2) have at most one solution.*

Proof. It is an immediate consequence of Proposition 3.1. □

3.2. Existence. To reach our goal, we need first to establish the following technical result.

Proposition 3.3. *Under (H1), (H2), (H3), and (H4), let $\{V_t : 0 \leq t \leq \tau\}$ be an $\mathcal{F}_t$-adapted process satisfying $\mathbb{E}(\int_0^\tau e^{\beta A(s)}|V_s|^2 ds) < +\infty$. Assume moreover that there exists $\delta > 0$ such that*

(H5) $\mathbb{E}[e^{(1+\delta)\beta A(\tau)}(1+|\xi|^{2(1+\delta)})+(\int_0^\tau e^{\beta A(s)}\eta^2(s)ds)^{(1+\delta)}] < +\infty$.

Then, there exists an $\mathcal{F}_t$-adapted processes (Y,Z) with values in $\mathbb{R}^d\times\mathbb{R}^{d\times n}$ such that

(J6) $(Y,Z)\in\mathcal{M}^c(\beta,a,\tau)$,

(J7) $Y_t = \xi+\int_t^\tau f(s,Y_s,V_s)ds-\int_t^\tau Z_s dW_s$.

Proof. In what follows, we put $h(s,y)\triangleq f(s,y,V_s)$ for every $s\in[0,\tau]$ and we split the proof in two parts.

Part I. We set $\overline{\xi}=e^{(\beta/2)A(\tau)}|\xi|$ and assume that

$$|\overline{\xi}|^2+\sup_{0\leq t\leq\tau}\left|h(t,0)\right|^2\leq C.\tag{3.6}$$

Let φ_q be a smooth function from $\mathbb{R}^d$ to $\mathbb{R}_+$ such that $0\leq\varphi_q\leq 1$ and $\varphi_q(x)=1$ if $|x|\leq q$, $\varphi_q(x)=0$ as soon as $|x|\geq q+1$. We set $q(n)=[(C+(6\tau/\beta\varepsilon)(nC+n^3+4K^2n))^{1/2}]$, where $[r]$ is the integer part of r. Define

$$h_n(t,y)=1_{\{\eta(t)+e^{\beta A(t)}\leq n\}}\int_{\mathbb{R}^d}\varphi_{q(n)+2}(y-u)h(t,y-u)\rho_n(u)du,\tag{3.7}$$

where $\rho_n:\mathbb{R}^d\to\mathbb{R}_+$ is a sequence of smooth functions with compact support in the ball $B(0,1)$ which approximate the Dirac measure at 0 and satisfy $\int_{\mathbb{R}^d}\rho_n(u)du=1$. Clearly, $h_n(t,\cdot)$ is a sequence of smooth functions with compact support satisfying the following:

(a) $h_n(t,\cdot)$ converges towards $h(t,\cdot)$ on compact sets,

(b) $h_n(t,\cdot)$ is globally stochastic Lipschitz with the coefficient $K(n,t)=a^2(t)+C_n$, where $C_n=(1/4)\alpha_n^2C+\alpha_n(n+K(2+q(n)+3))$ and $\alpha_n=\int_{\mathbb{R}^d}|\nabla\rho_n(u)|du$,

(c) for $|y|,|y'| \leq q(n)+1, \langle y-y', h_n(t,y)-h_n(t,y')\rangle \leq 1_{\{\eta(t)+e^{\beta A(t)}\leq n\}}\theta(t)|y-y'|^2$,
(d) $|h_n(t,y)| \leq 1_{\{\eta(t)+e^{\beta A(t)}\leq n\}}[|h(t,0)|+\eta(t)+K(2+|y|)]$.

Now, set $a_n^2(t) \triangleq K(n,t)$ and $A_n(t) \triangleq \int_0^t a_n^2(s)ds = A(t)+C_nt$. One can easily check that

$$\mathbb{E}\left(e^{\beta A_n(\tau)}|\xi|^2\right) < +\infty, \qquad \mathbb{E}\left(\int_0^\tau e^{\beta A_n(s)}\frac{|h_n(s,0)|^2}{a_n^2(s)}ds\right) < +\infty. \tag{3.8}$$

Thus, in light of El Karoui and Huang [5], the equation

$$Y_t^n = \xi + \int_t^\tau h_n(s,Y_s^n)ds - \int_t^\tau Z_s^n dW_s \tag{3.9}$$

has a unique solution (Y^n, Z^n) which belongs to the space $\mathcal{M}^c(\beta, a_n, \tau)$. But, in view of Remark 2.1, one has $(Y^n,Z^n) \in \mathcal{M}^c(\beta,a,\tau)$.

Now, we note that for all y, $2\langle y, h_n(t,y)\rangle \leq (2+\beta/2)a^2(t)|y|^2 + (2/\beta)(|h_n(t,0)|^2/a^2(t))$. Consequently, for a fixed $t \in [0,\tau]$, by applying Itô's formula to $e^{\beta[A(s)-A(t)]}|Y_s^n|^2$ for every $s \in [t,\tau]$, taking conditional expectation $\mathbb{E}(\cdot \mid \mathcal{F}_t)$ and choosing β large enough, we obtain

$$|Y_t^n|^2 \leq \mathbb{E}\left(e^{\beta[A(\tau)-A(t)]}|\xi|^2 + \frac{2}{\beta\varepsilon}\int_t^\tau e^{\beta[A(u)-A(t)]}|h_n(u,0)|^2 du \mid \mathcal{F}_t\right). \tag{3.10}$$

But, in view of (3.6) and (d), we have

$$\int_t^\tau e^{\beta[A(s)-A(t)]}|h_n(s,0)|^2 ds \leq (3\tau Cn + n^3 + 4K^2 n). \tag{3.11}$$

It follows that

$$\forall t \in [0,\tau] \quad |Y_t^n|^2 \leq C + \frac{6\tau}{\beta\varepsilon}(nC + n^3 + 4K^2 n), \tag{3.12}$$

which justifies the choice of the integer $q(n)$. The rest of this part is based on the following two lemmas.

Lemma 3.4. *Under (H1), (H2), (H3), (H4), and (H5), for β sufficiently large, the following holds:*

(i) $\sup_{n\in\mathbb{N}^*}\mathbb{E}(\int_0^\tau e^{\beta A(s)}a^2(s)|Y_s^n|^2 ds + \int_0^\tau e^{\beta A(s)}|Z_s^n|^2 ds) < +\infty$,
(ii) $\sup_{n\in\mathbb{N}^*}\mathbb{E}(\int_0^\tau e^{\beta A(s)}|h_n(s,Y_s^n)|^2 ds) < +\infty$,
(iii) $\sup_{n\in\mathbb{N}^*}\mathbb{E}(\sup_{0\leq t\leq\tau} e^{(1+\delta)\beta A(t)}|Y_t^n|^{2(1+\delta)}) < +\infty$.

Proof. By virtue of Itô's formula, we have

$$\begin{aligned} &e^{\beta A(t)}|Y_t^n|^2 + \beta\int_t^\tau e^{\beta A(s)}a^2(s)|Y_s^n|^2 ds + \int_t^\tau e^{\beta A(s)}|Z_s^n|^2 ds \\ &\quad\leq e^{\beta A(\tau)}|\xi|^2 + 2\int_t^\tau e^{\beta A(s)}|Y_s^n||h_n(s,Y_s^n)|ds - 2\int_t^\tau e^{\beta A(s)}\langle Y_s^n, Z_s^n dW_s\rangle. \end{aligned} \tag{3.13}$$

By using Young's inequality and (d), one can check that

$$2|Y_s^n||h_n(s,Y_s^n)| \le \left(\frac{\beta}{2}+2+\frac{K}{\varepsilon}\right)a^2(s)|Y_s^n|^2+\frac{2}{\beta}\frac{|h(s,0)|^2}{a^2(s)}+\frac{\eta^2(s)+4K^2}{a^2(s)}. \tag{3.14}$$

If we take β sufficiently large such that $\beta/2-2-K/\varepsilon>0$, (3.13) becomes

$$\begin{aligned} e^{\beta A(t)}|Y_t^n|^2&+\left(\frac{\beta}{2}-2-\frac{K}{\varepsilon}\right)\int_t^\tau e^{\beta A(s)}a^2(s)|Y_s^n|^2ds+\int_t^\tau e^{\beta A(s)}|Z_s^n|^2ds\\ &\le e^{\beta A(\tau)}|\xi|^2+\frac{2}{\beta}\int_t^\tau e^{\beta A(s)}\frac{|h(s,0)|^2}{a^2(s)}ds+\frac{1}{\varepsilon}\int_t^\tau e^{\beta A(s)}\eta(s)^2ds\\ &\quad+\frac{4K^2}{\varepsilon}\int_t^\tau e^{\beta A(s)}ds-2\int_t^\tau e^{\beta A(s)}\langle Y_s^n,Z_s^n dW_s\rangle. \end{aligned} \tag{3.15}$$

Therefore, by using (H3), (H4), and (3.6), we derive (i).

In view of (d), (3.6), and (i), (ii) is obvious.

By taking the conditional expectation in (3.15) and using (3.6), one obtains that

$$e^{\beta A(t)}|Y_t^n|^2 \le \mathbb{E}\left\{e^{\beta A(\tau)}(C_1+|\xi|^2)+\frac{1}{\varepsilon}\int_0^\tau e^{\beta A(s)}\eta(s)^2ds \mid \mathcal{F}_t\right\}, \tag{3.16}$$

where $C_1=2C/\beta\varepsilon^2+4K^2/\varepsilon^2$. Then, by Doob's maximal inequality (setting $p=1+\delta$), we derive that

$$\mathbb{E}\left(\sup_{0\le t\le\tau}e^{p\beta A(t)}|Y_t^n|^{2p}\right)\le C_p\mathbb{E}\left[e^{p\beta A(\tau)}(1+|\xi|^{2p})+\frac{1}{\varepsilon^p}\left(\int_0^\tau e^{\beta A(s)}\eta(s)^2ds\right)^p\right], \tag{3.17}$$

which ensures (iii). □

Lemma 3.5. *Under (H1), (H2), (H3), (H4), and (H5),* (Y^n,Z^n) *is a Cauchy sequence in* $\mathcal{M}^c(\beta,a,\tau)$.

Proof. Let $m\ge n$, and put $\Delta Y_t=Y_t^m-Y_t^n$, $\Delta Z_t=Z_t^m-Z_t^n$. By Itô's formula, we have

$$\begin{aligned} e^{\beta A(t)}|\Delta Y_t|^2&+\beta\int_t^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2ds+\int_t^\tau e^{\beta A(s)}|\Delta Z_s|^2ds\\ &=2\int_t^\tau e^{\beta A(s)}\langle\Delta Y_s,h_m(s,Y_s^m)-h_n(s,Y_s^n)\rangle ds-2\int_t^\tau e^{\beta A(s)}\langle\Delta Y_s,\Delta Z_s dW_s\rangle. \end{aligned} \tag{3.18}$$

We have

$$\begin{aligned} 2\langle\Delta Y_s,&h_m(s,Y_s^m)-h_n(s,Y_s^n)\rangle\\ &=2\langle\Delta Y_s,h_m(s,Y_s^m)-h_m(s,Y_s^n)\rangle+2\langle\Delta Y_s,h_m(s,Y_s^n)-h_n(s,Y_s^n)\rangle. \end{aligned} \tag{3.19}$$

But in view of (3.12), $|Y_s^m|\le q(m)+1$, $|Y_s^n|\le q(n)+1\le q(m)+1$. Therefore,

$$\langle\Delta Y_s,h_m(s,Y_s^m)-h_m(s,Y_s^n)\rangle\le 1_{\{\eta(s)+e^{\beta A(s)}\le n\}}\theta(s)|\Delta Y_s|^2\le 2a^2(s)|\Delta Y_s|^2. \tag{3.20}$$

It follows that

$$\begin{aligned} &e^{\beta A(t)}|\Delta Y_t|^2+(\beta-2)\int_t^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2ds+\int_t^\tau e^{\beta A(s)}|\Delta Z_s|^2ds \\ &\quad\le 2\int_t^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,ds-2\int_t^\tau e^{\beta A(s)}\langle \Delta Y_s,\Delta Z_s dW_s\rangle. \end{aligned} \tag{3.21}$$

Since β is chosen sufficiently large, we deduce that

$$\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Z_s|^2ds\right)\le 2\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,ds\right). \tag{3.22}$$

On the other hand, Burkhölder-Davis-Gundy inequality leads to

$$\begin{aligned} &2\mathbb{E}\left(\sup_{0\le t\le\tau}e^{\beta A(t)}\left|\int_t^\tau e^{\beta A(s)}\langle\Delta Y_s,\Delta Z_s dW_s\rangle\right|\right) \\ &\quad\le\frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le\tau}e^{\beta A(t)}|\Delta Y_t|^2\right)+2C^2\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Z_s|^2ds\right). \end{aligned} \tag{3.23}$$

Therefore, taking $\sup_{0\le t\le\tau}(\cdot)$ in (3.21), in view of (3.22) and (3.23), we obtain

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\le t\le\tau}e^{\beta A(t)}|\Delta Y_t|^2\right)+(2\beta-4)\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2ds\right)+\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Z_s|^2ds\right) \\ &\quad\le C\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,ds\right). \end{aligned} \tag{3.24}$$

Now, to reach our goal, it remains to show that

$$I_{m,n}=\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,ds\right) \tag{3.25}$$

tends to zero as $m,n\to+\infty$.

For every $M>0$, set $B_{m,n}^M=\{(s,\omega):|Y_s^m|+|Y_s^n|>M\}$ and $\overline{B}_{m,n}^M=\Omega\setminus B_{m,n}^M$. Let $C(\delta,\tau)$ denote a positive constant which may vary from line to line. We have

$$I_{m,n}=I_{m,n}^1+I_{m,n}^2, \tag{3.26}$$

where

$$\begin{aligned} I_{m,n}^1&=\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,1_{\overline{B}_{m,n}^M}ds\right), \\ I_{m,n}^2&=\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,1_{B_{m,n}^M}ds\right). \end{aligned} \tag{3.27}$$

We first estimate $I_{m,n}^2$. We use Holder's inequality and Young's inequality to obtain

$$
\begin{aligned}
I_{m,n}^2 &\le \frac{1}{M^\delta}\mathbb{E}\left\{\int_0^\tau e^{\beta A(s)}|\Delta Y_s|\,|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,(|Y_s^m|+|Y_s^n|)^\delta ds\right\}\\
&\le \frac{1}{M^\delta}\mathbb{E}\left\{\int_0^\tau e^{\beta A(s)}(|Y_s^m|+|Y_s^n|)^{1+\delta}|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,ds\right\}\\
&\le \frac{1}{M^\delta}\mathbb{E}\left\{\left(\int_0^\tau e^{\beta A(s)}(|Y_s^m|+|Y_s^n|)^{2+2\delta}ds\right)^{1/2}\right.\\
&\qquad\qquad \left.\times\left(\int_0^\tau e^{\beta A(s)}|h_m(s,Y_s^n)-h_n(s,Y_s^n)|^2ds\right)^{1/2}\right\}\\
&\le \frac{1}{2M^\delta}\mathbb{E}\left\{\int_0^\tau e^{\beta A(s)}(|Y_s^m|+|Y_s^n|)^{2+2\delta}ds+\int_0^\tau e^{\beta A(s)}|h_m(s,Y_s^n)-h_n(s,Y_s^n)|^2ds\right\}\\
&\le \frac{C(\tau,\delta)}{M^\delta}\mathbb{E}\left\{\sup_{0\le t\le\tau}e^{\beta A(t)}|Y_t^n|^{2+2\delta}+\int_0^\tau e^{\beta A(s)}|h_m(s,Y_s^n)-h_n(s,Y_s^n)|^2ds\right\}\\
&\le \frac{C(\tau,\delta)}{M^\delta}\sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\sup_{0\le t\le\tau}e^{\beta A(t)}|Y_t^n|^{2(1+\delta)}\right)\\
&\quad+\frac{C(\tau,\delta)}{M^\delta}\sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|h_m(s,Y_s^n)-h_n(s,Y_s^n)|^2ds\right).
\end{aligned}
\tag{3.28}
$$

Now, in view of (d), (ii), (iii) in Lemma 3.4, we deduce that

$$
I_{m,n}^2 \le \frac{C(\tau,\delta)}{M^\delta}. \tag{3.29}
$$

Since M is arbitrary, $I_{m,n}^2$ can be made arbitrarily small by choosing M large enough.

Now, we estimate $I_{m,n}^1$:

$$
\begin{aligned}
I_{m,n}^1 &\le M\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|h_m(s,Y_s^n)-h_n(s,Y_s^n)|\,1_{\{|Y_s^n|\le M\}}ds\right)\\
&\le M\mathbb{E}\left(\int_0^\tau \sup_{|y|\le M}e^{\beta A(s)}|h_m(s,y)-h_n(s,y)|\,ds\right).
\end{aligned}
\tag{3.30}
$$

Since $h_n(t,\cdot)$ converges towards $h(t,\cdot)$ on compact sets and $\sup_{|y|\le M}e^{\beta A(s)}|h_n(s,y)|\le \{|h(s,0)|+\eta(s)+K(2+M)\}e^{\beta A(s)}$, Lebesgue's dominated convergence theorem ensures that the right-hand side of (3.30) tends to zero as $m,n\to+\infty$.

Hence, in view of (3.24), we conclude that

$$
\mathbb{E}\left(\sup_{0\le t\le\tau}e^{\beta A(t)}|\Delta Y_t|^2\right)+(2\beta-4)\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}a^2(s)|\Delta Y_s|^2ds\right)+\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|\Delta Z_s|^2ds\right)
\tag{3.31}
$$

tends to zero as $m,n\to+\infty$. □

Now, set $Y = \lim_n Y^n$, $Z = \lim_n Z^n$; we will end this part by showing that (Y,Z) is a solution of the BSDE with data (τ,ξ,h). In view of the definition of the space $\mathcal{M}^c(\beta,a,\tau)$, we have

$$\lim_{n\to+\infty} \mathbb{E}\left(\sup_{0\le t\le\tau} e^{\beta A(t)} \left| Y_t^n - Y_t \right|^2\right) = 0, \tag{3.32}$$

$$\lim_{n\to+\infty} \mathbb{E}\left(\int_0^\tau e^{\beta A(s)} \left| Z_s^n - Z_s \right|^2 ds\right) = 0. \tag{3.33}$$

By virtue of (3.33), we have for all $t \in [0,\tau]$, $\int_t^\tau Z_s^n dW_s \to \int_t^\tau Z_s dW_s$ in $\mathbb{P}$-probability. Thus, to reach our purpose, we only need to show that

$$\forall t \in [0,\tau], \quad \int_t^\tau h_n(s,Y_s^n)ds \longrightarrow \int_t^\tau h(s,Y_s)ds \quad \text{(in probability)}. \tag{3.34}$$

From (3.32), we deduce that there exists a subsequence (Y^{n_k}) such that

$$\forall t \in [0,\tau], \quad Y_t^{n_k} \longrightarrow Y_t, \quad \mathbb{P}\text{-a.s.} \tag{3.35}$$

For simplicity, we assume that (3.35) holds without extracting a subsequence.
We have

$$\begin{aligned} &\mathbb{E}\left(\left|\int_t^\tau h_n(s,Y_s^n)ds - \int_t^\tau h(s,Y_s)ds\right|\right) \\ &\quad\le \mathbb{E}\left(\int_t^\tau \left|h_n(s,Y_s^n) - h(s,Y_s^n)\right| ds\right) + \mathbb{E}\left(\int_t^\tau \left|h(s,Y_s^n) - h(s,Y_s)\right| ds\right) \\ &\quad= I_1 + I_2. \end{aligned} \tag{3.36}$$

The fact that I_1 tends to zero is obtained by a similar argument as in the proof of Lemma 3.5.

Let $X_s^n = |h(s,Y_s^n) - h(s,Y_s)|$. We have

$$\begin{aligned} I_2 &= \mathbb{E}\left(\int_t^\tau X_s^n ds\right) \\ &\le \mathbb{E}\left(\int_0^\tau e^{\beta A(s)} X_s^n ds\right) \\ &\le \mathbb{E}\left(\int_0^\tau e^{\beta A(s)} X_s^n 1_{\{X_s^n \le r\}} ds\right) + \mathbb{E}\left(\int_0^\tau e^{\beta A(s)} X_s^n 1_{\{X_s^n > r\}} ds\right). \end{aligned} \tag{3.37}$$

By virtue of Fubini's theorem and Chebychev's inequality, we have

$$I_2 \le \int_0^\tau \mathbb{E}\left(e^{\beta A(s)} X_s^n(s) 1_{\{X_s^n \le r\}}\right) ds + \frac{\mathbb{E}\left(\int_0^\tau e^{\beta A(s)} (X_s^n)^2 ds\right)}{r}. \tag{3.38}$$

Moreover, in view of (H4) and Lemma 3.4, it is clear that

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\int_0^\tau e^{\beta A(s)} (X_s^n)^2 ds\right) < +\infty. \tag{3.39}$$

Therefore, the second term on the right-hand side of (3.38) can be made arbitrarily small by choosing r large enough. Now, since $y \mapsto h(\cdot, y)$ is continuous, we deduce from (3.35) that for fixed s, $X_s^n \to 0$ almost surely as $n \to \infty$. So, it follows from (H2), (H3)(ii), Fubini's theorem, and Lebesgue dominated convergence theorem that the first term of (3.38) goes to zero as $n \to \infty$.

Hence, $\int_\cdot^\tau h_n(s, Y_s^n)ds \to \int_\cdot^\tau h(s, Y_s)ds$ (in probability), which leads to the conclusion of this part.

Part II. Let

$$\xi_n = \frac{\inf\left(n, e^{\beta A(\tau)/2}|\xi|\right)}{e^{\beta A(\tau)/2}|\xi|}\xi,$$
$$h_n(t,y) = \begin{cases} h(t,y) - h(t,0) + \dfrac{\inf\left(n, |h(t,0)|\right)}{|h(t,0)|}h(t,0), & \text{if } h(t,0) \neq 0, \\ h(t,y), & \text{if } h(t,0) = 0. \end{cases} \tag{3.40}$$

We have $\mathbb{E}(e^{\beta A(\tau)}|\xi_n - \xi|^2) \to 0$, $\mathbb{E}(\int_0^\tau e^{\beta A(s)}(|h_n(s,0) - h(s,0)|^2/a^2(s))ds) \to 0$, as $n \to +\infty$, and (ξ_n, h_n) satisfies (3.6). Hence, for each $n \in \mathbb{N}^*$, there exists (Y^n, Z^n) which satisfies (J3) and

$$Y_t^n = \xi_n + \int_t^\tau h_n(s, Y_s^n)ds - \int_t^\tau Z_s^n dW_s, \quad 0 \leq t \leq \tau. \tag{3.41}$$

One can easily prove that for every $n, m \in \mathbb{N}^*$,

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{\beta A(t)}|Y_t^n - Y_t^m|^2 + \left(\frac{\beta}{2} - 2\right)\int_0^\tau e^{\beta A(s)}a^2(s)|Y_s^n - Y_s^m|^2 ds\right) \\ &\qquad + \mathbb{E}\left(\int_0^\tau e^{\beta A(s)}|Z_s^n - Z_s^m|^2 ds\right) \\ &\quad \leq C\mathbb{E}\left(e^{\beta A(\tau)}|\xi_n - \xi_m|^2 + \int_0^\tau e^{\beta A(s)}\frac{|h_n(s,0) - h_m(s,0)|^2}{a^2(s)}ds\right). \end{aligned} \tag{3.42}$$

The right-hand side tends to zero as $n, m \to +\infty$. Hence, there exists (Y, Z) a pair of $\mathcal{F}_t$-adapted processes such that

$$\lim_n \|(Y^n, Z^n) - (Y, Z)\|_{\beta,c}^2 = 0, \tag{3.43}$$

which satisfies (J3) and (J4). The proof of Proposition 3.3 is complete. □

Now, we state the main result of this section.

THEOREM 3.6. *Under (H1), (H2), (H3), (H4), and (H5), for β sufficiently large, the BSDEs (J1) and (J2) have a unique solution.*

Proof. For a fixed $(U,V)\in\mathcal{M}(\beta,a,\tau)$, thanks to Proposition 3.3 and Corollary 3.2, the BSDE

$$Y_t=\xi+\int_t^\tau f(s,Y_s,V_s)ds-\int_t^\tau Z_s dW_s \tag{3.44}$$

has a unique solution. So, we can define the mapping

$$\begin{aligned}\Pi:\mathcal{M}(\beta,a,\tau)&\longrightarrow\mathcal{M}(\beta,a,\tau),\\ (U,V)&\longmapsto\Pi(U,V)\end{aligned} \tag{3.45}$$

such that $\Pi(U,V)$ is the unique solution of the corresponding BSDE. Let $(U,V,U',V')\in\mathcal{M}(\beta,a,\tau)\times\mathcal{M}(\beta,a,\tau)$ and $\Pi(U,V)=(Y,Z)$, $\Pi(U',V')=(Y',Z')$. We combine (ii)(a) and (ii)(b) of Proposition 3.1 to obtain that

$$\begin{aligned}\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}a^2(s)\,|\,Y_s-Y'_s\,|^2ds\right)+\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}\,|\,Z_s-Z'_s\,|^2ds\right)&\\ \le\left(\frac{4}{\beta}+\frac{4}{\beta^2-4\beta}\right)\mathbb{E}\left(\int_0^\tau e^{\beta A(s)}\,|\,V_s-V'_s\,|^2ds\right).&\end{aligned} \tag{3.46}$$

In others words,

$$||(Y,Z)-(Y',Z')||_\beta^2\le\left(\frac{4}{\beta}+\frac{4}{\beta^2-4\beta}\right)||(U,V)-(U',V')||_\beta^2. \tag{3.47}$$

Hence, if β is sufficiently large, Π is a contracting mapping and its unique fixed point solves our BSDE. □

4. Random terminal time

In the sequel, we assume that (H1) to (H5) hold with τ being a random terminal time, which is allowed to take values in $[0,+\infty]$.

The following lemma is an important result for both the construction and the convergence of the approximation scheme.

Lemma 4.1. *Let ξ satisfy (H3)(i). Then,*

(i) *there exists $\{\gamma_t,\ t\ge 0\}$, an L_2-integrable process such that $\xi=\mathbb{E}(\xi)+\int_0^\tau\gamma(s)dW_s$,*

(ii) *the process $\{\xi_t,\ t\ge 0\}$ defined by setting $\xi_t=\mathbb{E}(\xi/\mathcal{F}_t)$ is such that $(\xi_t,\gamma_t)\in\mathcal{M}^c(\beta,a,\tau)$.*

Proof. (i) Since $L^2(\beta,a,\tau)\subset L^2(0,0,\tau)$, γ is given by Itô's representation theorem.

(ii) Since $e^{(\beta/2)A(t\wedge\tau)}|\xi_{t\wedge\tau}|\le\mathbb{E}(e^{(\beta/2)A(t\wedge\tau)}|\xi|/\mathcal{F}_{t\wedge\tau})$, Doob's inequality and Jensen's inequality yield

$$\mathbb{E}\left(\sup_{0\le t\le\tau}e^{\beta A(t)}\,|\,\xi_t\,|^2\right)\le 4\mathbb{E}\left(e^{\beta A(\tau)}|\xi|^2\right). \tag{4.1}$$

Now, in view of Itô's formula, Burkhölder-Davis-Gundy inequality, and Young's inequality, $2ab \le \alpha^2 a^2 + b^2/\alpha^2$, one obtains that for any $T \ge t \ge 0$,

$$\begin{aligned}
&\beta\mathbb{E}\left(\int_0^{T\wedge\tau} e^{\beta A(s)} a^2(s)\,|\xi_s|^2 ds\right) + \mathbb{E}\left(\int_0^{T\wedge\tau} e^{\beta A(s)}\,|\gamma_s|^2 ds\right) \\
&\quad\le \mathbb{E}\left(e^{\beta A(T\wedge\tau)}\,|\xi_{T\wedge\tau}|^2\right) + \alpha^2\mathbb{E}\left(\sup_{0\le t\le T\wedge\tau} e^{\beta A(t)}\,|\xi_t|^2\right) + \frac{C^2}{\alpha^2}\mathbb{E}\left(\int_0^{T\wedge\tau} e^{\beta A(s)}\,|\gamma_s|^2 ds\right).
\end{aligned} \tag{4.2}$$

Then, taking $\alpha^2 > 2C^2$, letting $T \to +\infty$, and using Fatou's lemma and Lebesgue's dominated convergence theorem, we derive that

$$\begin{aligned}
&2\beta\mathbb{E}\left(\int_0^{T\wedge\tau} e^{\beta A(s)} a^2(s)\,|\xi_s|^2 ds\right) + \mathbb{E}\left(\int_0^{T\wedge\tau} e^{\beta A(s)}\,|\gamma_s|^2 ds\right) \\
&\qquad\le 2\mathbb{E}\left(e^{\beta A(\tau)}|\xi|^2\right) + \alpha^2\mathbb{E}\left(\sup_{0\le t\le \tau} e^{\beta A(t)}\,|\xi_t|^2\right).
\end{aligned} \tag{4.3}$$

Therefore, (4.1) and (H3)(i) lead to (ii). □

The main result of this section is the following.

Theorem 4.2. *Under (H1), (H2), (H3), (H4), and (H5), the BSDEs (J3), (J4), and (J5) have a unique solution.*

The existence result is based on the following sequence.

For each $n \in \mathbb{N}^*$, we know by Theorem 3.6 that the BSDE with data $(n, \xi_n, 1_{[0,\tau]} f)$ has a unique solution (Y^n, Z^n) on $[0,n]$. We have

$$Y_t^n = \xi_n + \int_{t\wedge\tau}^{n\wedge\tau} f(s, Y_s^n, Z_s^n)\,ds - \int_{t\wedge\tau}^{n\wedge\tau} Z_s^n\,dW_s, \quad \text{for } 0 \le t \le n. \tag{4.4}$$

We extend the sequence (Y^n, Z^n) by setting $Y_t^n = \xi_t$; $Z_t^n = \gamma_t$, for $t > n$.

Hence, (Y^n, Z^n) solves the BSDE

$$Y_t^n = \xi + \int_{t\wedge\tau}^{\tau} 1_{[0,n]}(s) f(s, Y_s^n, Z_s^n)\,ds - \int_{t\wedge\tau}^{\tau} Z_s^n\,dW_s, \quad t \ge 0. \tag{4.5}$$

We turn to convergence of the sequence $\{(Y^n, Z^n) : n \ge 0\}$. To this end, we need the following lemmas.

Lemma 4.3. *Put*

$$\Gamma = \mathbb{E}\left(e^{\beta A(\tau)}|\xi|^2 + \frac{2}{\beta}\int_0^{\tau} e^{\beta A(s)} \frac{|f(s,0,0)|^2}{a^2(s)}\,ds\right). \tag{4.6}$$

Then,

$$\sup_{n\in\mathbb{N}^*}\left[\frac{1}{2}\mathbb{E}\left(\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}|Z_s^n|^2 ds\right) + \left(\frac{\beta}{2}-2\right)\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}a^2(s)|Y_s^n|^2 ds\right] \le \Gamma, \tag{4.7}$$

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\sup_{0\le t\le n\wedge\tau} e^{\beta A(t\wedge\tau)}|Y_t^n|^2\right) \le 2\Gamma. \tag{4.8}$$

Proof. We apply Itô's formula to the process $e^{\beta A(t)}|Y_t^n|^2$ to obtain

$$\begin{aligned} &e^{\beta A(t\wedge\tau)}|Y_t^n|^2 + \frac{1}{2}\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}|Z_s^n|^2 ds + \left(\frac{\beta}{2}-2\right)\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}a^2(s)|Y_s^n|^2 ds \\ &\quad\le e^{\beta A(n\wedge\tau)}|Y_n^n|^2 + \frac{2}{\beta}\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}ds - 2\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}\langle Y_s^n, Z_s^n dW_s\rangle. \end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left(e^{\beta A(t\wedge\tau)}|Y_t^n|^2\right) + \frac{1}{2}\mathbb{E}\left\{\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}|Z_s^n|^2 ds + \left(\frac{\beta}{2}-2\right)\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}a^2(s)|Y_s^n|^2 ds\right\} \\ &\quad\le \mathbb{E}\left(e^{\beta A(n\wedge\tau)}|Y_n^n|^2\right) + \frac{2}{\beta}\mathbb{E}\left(\int_{t\wedge\tau}^{\tau\wedge n} e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}ds\right) \\ &\quad\le \mathbb{E}\left(e^{\beta A(\tau)}|\xi|^2\right) + \frac{2}{\beta}\mathbb{E}\left(\int_0^{\tau} e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}ds\right), \end{aligned} \tag{4.10}$$

which gives (4.7). To prove (4.8), it suffices to apply Burkhölder-Davis-Gundy inequality. □

Lemma 4.4. *Under (H1), (H2), (H3), (H4), and (H5), $(Y^n,Z^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathcal{M}^c(\beta,a,\tau)$.*

Proof. Let $(m,n)\in\mathbb{N}^{*2}$ such that $m>n$. We put

$$\Delta Y_t = Y_t^m - Y_t^n; \qquad \Delta Z_t = Z_t^m - Z_t^n. \tag{4.11}$$

(i) For $n\le t\le m$, one has

$$\begin{aligned} Y_t^n &= \xi_t = \mathbb{E}(\xi \mid \mathcal{F}_t) = \xi_m - \int_{t\wedge\tau}^{\tau\wedge m}\gamma(s)dW_s, \\ Y_t^m &= \xi_m + \int_{t\wedge\tau}^{\tau\wedge m} f(s,Y_s^m,Z_s^m)ds - \int_{t\wedge\tau}^{\tau\wedge m} Z_s^m dW_s. \end{aligned} \tag{4.12}$$

This leads to

$$\Delta Y_t = \int_{t\wedge\tau}^{\tau\wedge m} f(s,Y_s^m,Z_s^m)ds - \int_{t\wedge\tau}^{\tau\wedge m}\Delta Z_s dW_s. \tag{4.13}$$

We have

$$
\begin{aligned}
&|\Delta Y_t|^2 e^{\beta A(t\wedge\tau)} + \int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}|\Delta Z_s|^2 ds + \beta\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}|\Delta Y_s|^2 ds \\
&\quad = 2\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}\langle \Delta Y_s, f(s,Y_s^m,Z_s^m)\rangle ds - 2\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}\langle \Delta Y_s, \Delta Z_s dW_s\rangle.
\end{aligned}
\tag{4.14}
$$

Since

$$
\begin{aligned}
&2\langle \Delta Y_s, f(s,Y_s^m,Z_s^m)\rangle \\
&\quad = 2\langle \Delta Y_s, f(s,Y_s^m,Z_s^m) - f(s,\xi_s,Z_s^m)\rangle \\
&\qquad + 2\langle \Delta Y_s, f(s,\xi_s,Z_s^m) - f(s,\xi_s,\gamma_s)\rangle + 2\langle \Delta Y_s, f(s,\xi_s,\gamma_s)\rangle,
\end{aligned}
\tag{4.15}
$$

we get

$$
\begin{aligned}
&|\Delta Y_t|^2 e^{\beta A(t\wedge\tau)} + \frac{1}{2}\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}|\Delta Z_s|^2 ds + \left(\frac{\beta}{2}-2\right)\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)} a^2(s)|\Delta Y_s|^2 ds \\
&\quad \le \frac{2}{\beta}\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)} \frac{|f(s,\xi_s,\gamma_s)|^2}{a^2(s)} ds - 2\int_{t\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}\langle \Delta Y_s, \Delta Z_s dW_s\rangle.
\end{aligned}
\tag{4.16}
$$

Now, in view of Burkhölder-Davis-Gundy inequality,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{n\le t\le m}|\Delta Y_t|^2 e^{\beta A(t\wedge\tau)}\right) + \mathbb{E}\left(\int_{n\wedge\tau}^{\tau\wedge m} e^{\beta A(s)}|\Delta Z_s|^2 ds\right) + C\mathbb{E}\left(\int_{n\wedge\tau}^{\tau\wedge m} e^{\beta A(s)} a^2(s)|\Delta Y_s|^2 ds\right) \\
&\quad \le C\mathbb{E}\left(\int_{n\wedge\tau}^{\tau} e^{\beta A(s)} \frac{|f(s,\xi_s,\gamma_s)|^2}{a^2(s)} ds\right).
\end{aligned}
\tag{4.17}
$$

By virtue of Lemma 4.1, the last term of this inequality tends to 0 as n goes to infinity.

(ii) For $t \le n \le m$, one has

$$
\Delta Y_t = \Delta Y_n + \int_t^{\tau\wedge n} [f(s,Y_s^m,Z_s^m) - f(s,Y_s^n,Z_s^n)] ds - \int_t^{\tau\wedge n} \Delta Z_s dW_s. \tag{4.18}
$$

We repeat the same calculation as in the previous case and we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le\tau\wedge n}|\Delta Y_t|^2 e^{\beta A(t)}\right) + \mathbb{E}\left(\int_0^{\tau\wedge n} e^{\beta A(s)}|\Delta Z_s|^2 ds\right) + C\mathbb{E}\left(\int_0^{\tau\wedge n} e^{\beta A(s)} a^2(s)|\Delta Y_s|^2 ds\right) \\
&\quad \le \mathbb{E}\left(|\Delta Y_n|^2 e^{\beta A(n\wedge\tau)}\right).
\end{aligned}
\tag{4.19}
$$

In view of (4.17), we deduce that the right-hand side tends to zero as n goes to infinity. □

Proof of Theorem 4.2

Uniqueness. Let (Y,Z) and (Y',Z') be two solutions of our BSDE. We set $\Delta Y_t = Y_t - Y'_t$; $\Delta Z_t = Z_t - Z'_t$. One has

$$\Delta Y_{t\wedge\tau} = \Delta Y_{t\wedge T} + \int_{t\wedge\tau}^{T\wedge\tau} [f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s)]ds - \int_{t\wedge\tau}^{T\wedge\tau} \Delta Z_s dW_s, \quad T \geq t \geq 0. \tag{4.20}$$

Itô's formula yields

$$\begin{aligned} e^{\beta A(t\wedge\tau)}|\Delta Y_{t\wedge\tau}|^2 &+ \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}a^2(s)|\Delta Y_s|^2 ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}|\Delta Z_s|^2 ds \\ &= e^{\beta A(T\wedge\tau)}|\Delta Y_{T\wedge\tau}|^2 + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle \Delta Y_s, f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s)\rangle ds \\ &\quad - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle \Delta Y_s, \Delta Z_s dW_s\rangle. \end{aligned} \tag{4.21}$$

By using condition (H1) and taking the expectation, we get for β sufficiently large

$$\mathbb{E}\left(e^{\beta A(t\wedge\tau)}|\Delta Y_{t\wedge\tau}|^2\right) \leq \mathbb{E}\left(e^{\beta A(T\wedge\tau)}|\Delta Y_{T\wedge\tau}|^2\right). \tag{4.22}$$

Since $(\Delta Y,\Delta Z) \in \mathcal{M}^c(\beta,a,\tau)$, we obtain by letting $T \to +\infty$ and using Lebesgue's dominated convergence theorem that

$$\mathbb{E}\left(e^{\beta A(t\wedge\tau)}|\Delta Y_{t\wedge\tau}|^2\right) = 0. \tag{4.23}$$

Therefore,

$$\Delta Y_{t\wedge\tau} = 0, \qquad \Delta Z_{t\wedge\tau} = 0. \tag{4.24}$$

Existence. We denote the limit of (Y_t^n,Z_t^n) by (Y_t,Z_t) and prove that (Y_t,Z_t) solves the BSDE (τ,ξ,f).

By virtue of Theorem 3.6, for each $T > 0$, the BSDE $(T,Y_T,1_{[0,\tau]}f)$ has a unique solution $(\overline{Y}_t,\overline{Z}_t)$. For our purpose, it suffices to prove that $(Y_t,Z_t) = (\overline{Y}_t,\overline{Z}_t)$ for all $t \leq T$.

One has

$$\begin{aligned} \overline{Y}_t &= Y_T + \int_t^T 1_{[0,\tau]}(s)f(s,\overline{Y}_s,\overline{Z}_s)ds - \int_t^T \overline{Z}_s dW_s, \\ Y_t^n &= Y_T^n + \int_t^T 1_{[0,\tau]}(s)f(s,Y_s^n,Z_s^n)ds - \int_t^T Z_s^n dW_s. \end{aligned} \tag{4.25}$$

After extracting a subsequence still denoted by Y_t^n, it holds that

$$\mathbb{E}\left(e^{\beta A(t\wedge\tau)}|\overline{Y}_t - Y_t|^2\right) = \lim_{n\to\infty}\mathbb{E}\left(e^{\beta A(t\wedge\tau)}|\overline{Y}_t - Y_t^n|^2\right). \tag{4.26}$$

Therefore, we have to show that the right-hand side is zero. We denote $\widehat{\Delta Y}_t = \overline{Y}_t - Y_t^n$; $\widehat{\Delta Z}_t = \overline{Z}_t - Z_t^n$. One has

$$\widehat{\Delta Y}_t = \widehat{\Delta Y}_T + \int_t^T 1_{[0,\tau]}(s)[f(s,\overline{Y}_s,\overline{Z}_s) - f(Y_s^n,Z_s^n)]ds - \int_t^T \widehat{\Delta Z}_s dW_s. \tag{4.27}$$

By using an analogous argument as in the proof of uniqueness, we have

$$\mathbb{E}\Big(e^{\beta A(t\wedge\tau)}|\widehat{\Delta Y}_t|^2\Big) \le \mathbb{E}\Big(e^{\beta A(t\wedge\tau)}|\widehat{\Delta Y}_T|^2\Big). \tag{4.28}$$

Hence, $\mathbb{E}(e^{\beta A(t\wedge\tau)}|\overline{Y}_t - Y_t|^2) \le \lim_{n\to\infty}\mathbb{E}(e^{\beta A(t\wedge\tau)}|Y_T - Y_T^n|^2) = 0$.

Consequently, for all $t \le T$, we have $(Y_t,Z_t) = (\overline{Y}_t,\overline{Z}_t)$. □

We now state a comparison result.

Proposition 4.5 (a comparison theorem). *Assume $d=1$. Under (H1), (H2), (H3), (H4), and (H5), let (Y,Z) (resp., (Y',Z')) be a solution of the BSDE (τ,ξ,f) (resp., (τ,ξ',f')) such that $\xi \le \xi'$ a.s., $f(t,Y_t',Z_t') \le f'(t,Y_t',Z_t')dt\times d\mathbb{P}$ a.e. Then $Y_t \le Y_t'$ a.s. on the set $\{t\le\tau\}$.*

Proof. We put $\Delta Y_t^+ = (Y_t - Y_t')^+$, $\Delta Z_t = Z_t - Z_t'$. Itô's formula yields

$$\begin{aligned}
e^{\beta A(t\wedge\tau)}|\Delta Y_{t\wedge\tau}^+|^2 &+ \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}a^2(s)|\Delta Y_s^+|^2ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}|\Delta Z_s|^2ds \\
&= e^{\beta A(T\wedge\tau)}|\Delta Y_{T\wedge\tau}|^2 + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle \Delta Y_s^+, f(s,Y_s,Z_s) - f'(s,Y_s',Z_s')\rangle ds \\
&\quad - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle \Delta Y_s^+, \Delta Z_s dW_s\rangle, \quad \forall 0\le t\le T,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&\langle \Delta Y_s^+, f(s,Y_s,Z_s) - f'(s,Y_s',Z_s')\rangle \\
&\quad \le \Big(2+\frac{\beta}{2}\Big)a^2(s)|\Delta Y_s^+|^2 + \frac{1}{2}|\Delta Z_s|^2 + \langle \Delta Y_s^+, f(s,Y_s',Z_s') - f'(s,Y_s',Z_s')\rangle.
\end{aligned}$$

By analogous calculus as in the proof of uniqueness, one can prove that

$$\mathbb{E}\Big(e^{\beta A(t\wedge\tau)}|\Delta Y_{t\wedge\tau}^+|^2\Big) = 0, \tag{4.30}$$

which leads to $\Delta Y_{t\wedge\tau}^+ = 0$. □

Remark 4.6. When the uncertainty comes from a solution of a forward SDE, Theorem 4.2 provides a representation of the viscosity solution for elliptic PDE. More precisely, let

$$X_s^x = x + \int_0^s b(X_r^x)dr + \int_0^s \sigma(X_r^x)dW_r \tag{4.31}$$

be a diffusion process with the infinitesimal generator

$$\mathcal{L} = \frac{1}{2}(\sigma\sigma^t)_{i,j}(x)\frac{\partial^2}{\partial x_i\partial x_j} + b_i(x)\frac{\partial}{\partial x_i}. \tag{4.32}$$

Let (Y^x, Z^x) be the unique solution of the BSDE

$$Y_s^x = l(X_{\tau_x}^x) + \int_{s\wedge\tau_x}^{\tau_x} f(r, X_r^x, Y_r^x, Z_r^x)\,dr - \int_{s\wedge\tau_x}^{\tau_x} Z_r^x\,dW_r, \quad t \geq 0, \tag{4.33}$$

where $(y,z) \mapsto f(\cdot, X_\cdot^x, y, z)$ satisfies (H1)–(H5),

$$\sup_{x\in\overline{G}} \mathbb{E}\left(\int_0^{\tau_x} e^{\beta A(t)} \frac{|f(t, X_t^x, 0, 0)|^2}{a^2(t)}\,dt\right) < +\infty, \tag{4.34}$$

and $\sup_{x\in\overline{G}} \mathbb{E}(e^{\beta A(\tau_x)}) < +\infty$. Then, $u(x) = Y_0^x$ is continuous on $\overline{G}$ and is a viscosity solution of the semilinear system

$$\begin{aligned} \mathscr{L}u_i + f_i(x, u(x), (\nabla u_i \sigma)(x)) &= 0, \quad x \in G,\ i = 1,\dots,d, \\ u_i(x) &= l_i(x), \quad x \in \partial G,\ i = 1,\dots,d, \end{aligned} \tag{4.35}$$

where G is an open bounded subset of $\mathbb{R}^d$, whose boundary ∂G is of class C^1, $l \in C(\overline{G}, \mathbb{R}^d)$ and $\tau_x = \inf\{t \geq 0,\ X_t^x \notin \overline{G}\}$ finite $\mathbb{P}$-a.s.

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